ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 4, Fall 1995

ARITHMETICAL CONSEQUENCES OF A SEXTUPLE PRODUCT IDENTITY

JOHN A. EWELL

ABSTRACT. The author derives a sixfold infinite-product identity in three complex variables. From this identity two formulas, one giving the number of representations of a given natural number by sums of four triangular numbers and the other giving the number of representations by sums of eight triangular numbers, are then deduced.

1. Introduction. The main result of this paper is the identity of the following theorem.

Theorem 1.1. *For each triple of complex numbers* a, b, x, *with* $a \neq 0$, $b \neq 0$ *and* $|x| < 1$,

$$\begin{aligned}(1.1)\quad \prod_1^\infty (1-x^{2n})^2(1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1})&\\ (1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1})&\\ = \sum_{-\infty}^{\infty} x^{2m^2}a^{2m}\sum_{-\infty}^{\infty} x^{2n^2}b^{2n}&\\ + x\sum_{-\infty}^{\infty} x^{2m(m+1)}a^{2m+1}\sum_{-\infty}^{\infty} x^{2n(n+1)}b^{2n+1}&.\end{aligned}$$

Section 2 supplies a proof of this theorem. In Section 3 we apply the theorem to derive formulas for the numbers of representations of a given natural number by sums of four triangular numbers and by sums of eight triangular numbers. Our concluding remarks compare our formulas with two formulas of Jacobi for representations of numbers by sums of squares.

Received by the editors on August 23, 1993.
1991 *Mathematics Subject Classification.* Primary 11E25, Secondary 11N99.
Key words and phrases. Sums of triangular numbers, sextuple product identity.

2. Proof of Theorem 1.1. Our proof is predicated on the celebrated Gauss-Jacobi triple-product identity

$$(2.1)\qquad \prod_1^\infty (1-x^{2n})(1+tx^{2n-1})(1+t^{-1}x^{2n-1}) = \sum_{-\infty}^{\infty} x^{n^2}t^n,$$

which is valid for each pair of complex numbers t, x such that $t \neq 0$ and $|x| < 1$. We also require the following entirely transparent lemma.

Lemma 2.1. *The function* $F : \mathbf{Z}^2 \to \mathbf{Z}^2$*, defined by* $F(i,j) := (i+j, i-j)$*, is one-to-one from* $\mathbf{Z}^2$ *onto the set*

$$E := \{(r,s) \in \mathbf{Z}^2 \mid r \text{ and } s \text{ have the same parity}\}.$$

(*Of course,* $\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$.)

Now, for an arbitrary triple $a, b, x \in \mathbf{C}$, with $a \neq 0$, $b \neq 0$ and $|x| < 1$, we appeal to (2.1), first letting $t \to ab$, then letting $t \to ab^{-1}$, and multiplying the resulting two identities to get

$$\begin{aligned}
\prod_1^\infty &(1-x^{2n})^2(1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1}) \\
&\qquad (1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1}) \\
&\qquad = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} x^{i^2+j^2}a^ib^ia^jb^{-j} \\
&\qquad = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} x^{[(i+j)^2+(i-j)^2]/2}a^{i+j}b^{i-j} \\
&\qquad = \sum_{(r,s)\in E} x^{[r^2+s^2]/2}a^rb^s \\
&\qquad = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x^{2m^2+2n^2}a^{2m}b^{2n} \\
&\qquad\quad + x\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x^{2m(m+1)+2n(n+1)}a^{2m+1}b^{2n+1}
\end{aligned}$$

$$= \sum_{-\infty}^{\infty} x^{2m^2} a^{2m} \sum_{-\infty}^{\infty} x^{2n^2} b^{2n}$$
$$+ x \sum_{-\infty} x^{2m(m+1)} a^{2m+1} \sum_{-\infty}^{\infty} x^{2n(n+1)} b^{2n+1}.$$

3. Applications. First of all, we collect all of the arithmetical functions which we shall eventually require.

Definition 3.1. For $\mathbf{N} := \{0, 1, 2, \dots\}$, put $\mathbf{P} := \mathbf{N} \backslash \{0\}$. Then, for each $k \in \mathbf{P}$ and each $n \in \mathbf{N}$,

$$t_k(n) := \left| \left\{ (x_1, \dots, x_k) \in \mathbf{N}^k \;\middle|\; n = x_1 \frac{x_1 + 1}{2} + \cdots + x_k \frac{x_k + 1}{2} \right\} \right|,$$
$$r_k(n) := |\{(x_1, \dots, x_k) \in \mathbf{Z}^k \mid n = x_1^2 + \cdots + x_k^2\}|.$$

For each $n \in \mathbf{P}$, $b(n) :=$ the exponent of the exact power of 2 dividing n, and then $Od(n) := n2^{-b(n)}$ is the odd part of n. For each $k \in \mathbf{N}$ and each $n \in \mathbf{P}$, $\sigma_k(n) :=$ the sum of the kth powers of all of the positive divisors of n. (For simplicity, $\sigma(n) := \sigma_1(n)$.)

Secondly, we shall also require the following two well-known identities due respectively to Euler and Gauss.

$$\prod_{1}^{\infty} (1 + x^n)(1 - x^{2n-1}) = 1, \tag{3.1}$$

$$\prod_{1}^{\infty} \frac{1 - x^{2n}}{1 - x^{2n-1}} = \sum_{0}^{\infty} x^{n(n+1)/2}. \tag{3.2}$$

These identities are valid for each $x \in \mathbf{C}$ such that $|x| < 1$. In passing we note that the kth power of the right side of (3.2) generates the sequence $t_k(n)$.

Corollary 3.2. *For each* $n \in \mathbf{N}$, $t_4(n) = \sigma(2n + 1)$.

Proof. By appeal to (2.1) we express each series on the right side of (1.1) as an infinite product.

$$(3.3)\quad \prod_1^\infty (1-x^{2n})^2(1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1})(1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1})$$
$$= \prod_1^\infty (1-x^{4n})^2(1+a^2x^{4n-2})(1+a^{-2}x^{4n-2})(1+b^2x^{4n-2})(1+b^{-2}x^{4n-2})$$
$$+ x(a+a^{-1})(b+b^{-1})\prod_1^\infty (1-x^{4n})^2(1+a^2x^{4n})(1+a^{-2}x^{4n})(1+b^2x^{4n})(1+b^{-2}x^{4n}).$$

Next, in (3.3) we let $a = b$ and subsequently let $a \to ia$ to get

$$(3.4)\quad \prod_1^\infty (1-x^{2n})^2(1+x^{2n-1})^2(1-a^2x^{2n-1})(1-a^{-2}x^{2n-1})$$
$$= \prod_1^\infty (1-x^{4n})^2(1-a^2x^{4n-2})^2(1-a^{-2}x^{4n-2})^2$$
$$- (a-a^{-1})^2\cdot x\prod_1^\infty (1-x^{4n})^2(1-a^2x^{4n})^2(1-a^{-2}x^{4n})^2.$$

Now, with D_a denoting derivation with respect to a, we (i) operate on both sides of (3.4) with $(aD_a)^2$, (ii) put $a = 1$, (iii) divide each side of the resulting identity by the product

$$\prod_1^\infty (1-x^{2n})^2(1-x^{4n-2})^2,$$

and, (iv) simplify by appeal to (3.1) to get

$$\begin{aligned}\sum_0^\infty t_4(n)x^{2n+1} &= x\prod_1^\infty \frac{(1-x^{4n})^4}{(1-x^{4n-2})^4}\\ &= \sum_1^\infty \frac{kx^k}{1-x^{2k}} - 2\sum_1^\infty \frac{kx^{2k}}{1-x^{4k}}\\ &= \sum_0^\infty \frac{(2k+1)x^{2k+1}}{1-x^{4k+2}} \qquad (3.5)\\ &= \sum_{k=0}^\infty\sum_{j=0}^\infty (2k+1)x^{(2j+1)(2k+1)}\\ &= \sum_0^\infty \sigma(2n+1)x^{2n+1}.\end{aligned}$$

Equating coefficients of like powers of x, we thus prove our corollary. □

Corollary 3.3. *For each* $n \in \mathbf{P}$, $t_8(n-1) = 2^{3b(n)}\sigma_3(Od(n))$.

Proof. In (3.4) let $x \to -x$, multiply the resulting identity and (3.4), and then let $x \to x^{1/2}$ to get

$$(3.6)\quad \prod_1^\infty (1-x^n)^4(1-x^{2n-1})^2(1-a^4x^{2n-1})(1-a^{-4}x^{2n-1})$$
$$= \prod_1^\infty (1-x^{2n})^4(1-a^2x^{2n-1})^4(1-a^{-2}x^{2n-1})^4$$
$$- (a-a^{-1})^4\cdot x\prod_1^\infty (1-x^{2n})^4(1-a^2x^{2n})^4(1-a^{-2}x^{2n})^4.$$

Now we (i) operate on both sides of (3.6) with $(aD_a)^4$, (ii) put $a=1$, (iii) divide each side of the resulting identity by the product

$$\prod_1^\infty (1-x^n)^4(1-x^{2n-1})^4,$$

and, (iv) simplify to get

$$\begin{aligned}\sum_1^\infty t_8(n-1)x^n &= \sum_0^\infty t_8(n)x^{n+1} = x\prod_1^\infty \frac{(1-x^{2n})^8}{(1-x^{2n-1})^8}\\ &= \sum_{k=1}^\infty \frac{k^3x^k}{1-x^{2k}} = \sum_{k=1}^\infty\sum_{j=0}^\infty k^3x^{(2j+1)k}\\ &= \sum_{n=1}^\infty x^n \sum_{d|Od(n)} (n/d)^3\\ &= \sum_{n=1}^\infty 2^{3b(n)}x^n \sum_{d|Od(n)} (Od(n)/d)^3\\ &= \sum_{n=1}^\infty 2^{3b(n)}\sigma_3(Od(n))x^n.\end{aligned} \tag{3.7}$$

Finally, equating coefficients of like powers of x, we obtain the desired result. □

Concluding remarks. Our real point of departure for the foregoing discussion is part of a famous comment of Fermat [**1**, page 6], "...every number is either triangular or the sum of 2 or 3 triangular numbers,... ." Gauss [**1**, page 17] first settled this part of Fermat's comment. In fact, he proved the following theorem.

Theorem. *Every natural number is the sum of 3 or fewer triangular numbers; that is, for each* $n \in \mathbf{N}$, $t_3(n) > 0$.

Recently, Ewell [**2**] gave a characterization of the positive integers n for which $t_2(n) > 0$.

Finally, we observe that Corollaries 3.2 and 3.3 parallel well-known results of Jacobi for representations of positive integers by sums of 4 squares and by sums of 8 squares. These are: $r_4(n) = 8(2 + (-1)^n)\sigma(Od(n))$ and $r_8(n) = 16(-1)^n \sum_{d|n}(-1)^d d^3$, each of which is valid for all positive integers. For example, see [**3**, page 314].

REFERENCES

1. L.E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952.

2. J.A. Ewell, *On sums of triangular numbers and sums of squares*, Amer. Math. Monthly **99** (1992), 752–757.

3. G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, Fourth edition, Clarendon Press, Oxford, 1960.

Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115