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ON THE ABSOLUTE SUMMABILITY FACTORS AND ABSOLUTE SUMMABILITY METHODS

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ABSTRACT. In this paper we have proved two theorems on the absolute Cesàro and weighted mean summability methods. These theorems include some known results.

1. Introduction. Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by w_n^α and t_n^α the nth Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, i.e.,

$$w_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1.1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.2}$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = O(n^\alpha), \qquad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \tag{1.3}$$

for $n > 0$. The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, $\alpha > -1$, if (see [**10**])

$$\sum_{n=1}^{\infty} n^{k-1} |w_n^\alpha - w_{n-1}^\alpha|^k < \infty \tag{1.4}$$

and it is said to be summable $|C,\alpha;\delta|_k$, $k \geq 1$, $\alpha > -1$ and $\delta \geq 0$, if (see [**11**])

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |w_n^\alpha - w_{n-1}^\alpha|^k < \infty. \tag{1.5}$$

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But since $t_n^\alpha = n(w_n^\alpha - w_{n-1}^\alpha)$ (see [**12**]) the condition (1.5) can also be written as

$$\sum_{n=1}^{\infty} n^{\delta k-1}|t_n^\alpha|^k < \infty. \tag{1.6}$$

Let (φ_n) be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi - |C,\alpha|_k$, $k \geq 1$, if (see [**1**])

$$\sum_{n=1}^{\infty} n^{-k}|\varphi_n t_n^\alpha|^k < \infty. \tag{1.7}$$

In the special case when $\varphi_n = n^{1-1/k}$, respectively $\varphi_n = n^{\delta+1-1/k}$, $\varphi - |C,\alpha|_k$ summability is the same as $|C,\alpha|_k$, respectively $|C,\alpha;\delta|_k$, summability.

Let (p_n) be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty \quad \text{as } n \longrightarrow \infty, \qquad P_{-i} = p_{-i} = 0, \quad i \geq 1. \tag{1.8}$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1.9}$$

defines the sequence (T_n) of the $(\overline{N},p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|\overline{N},p_n|_k$, $k \geq 1$, if (see [**2**])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1}|\Delta T_{n-1}|^k < \infty \tag{1.10}$$

and it is said to be summable $|\overline{N},p_n;\delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [**4**])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k+k-1}|\Delta T_{n-1}|^k < \infty, \tag{1.11}$$

where

$$\Delta T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \qquad n \geq 1. \tag{1.12}$$

2. Preliminary results. Bor [**6**] proved the following theorem for $\varphi - |C,1|_k$ summability methods.

Theorem A. *Let* (X_n) *be a positive monotonic nondecreasing sequence, and let* (λ_n) *be a seqeunce such that*

$$X_n \lambda_n = O(1) \quad \textit{as } n \longrightarrow \infty \tag{2.1}$$

$$\sum_{v=1}^{n} v X_v |\Delta^2 \lambda_v| = O(1) \quad \textit{as } n \to \infty. \tag{2.2}$$

If there exists an $\varepsilon > 0$ *such that the sequence* $(n^{\varepsilon - k} |\varphi_n|^k)$ *is nonincreasing and*

$$\sum_{v=1}^{n} v^{-k} |\varphi_v t_v^1|^k = O(X_n) \quad \textit{as } n \longrightarrow \infty, \tag{2.3}$$

then the series $\sum a_n \lambda_n$ *is summable* $\varphi - |C,1|_k$, $k \geq 1$.

It should be noted that, under the conditions of Theorem A, we have that

$$\Delta \lambda_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.4}$$

It is known that the summability $|C,\alpha;\delta|_k$ and summability $|\overline{N},p_n;\delta|_k$ are, in general, independent of each other. For $\alpha = 1$ Bor [**5**] has established a relation between the $|C,1;\delta|_k$ and $|\overline{N},p_n;\delta|_k$ summability methods by proving the following theorem.

Theorem B. *Let* $k \geq 1$ *and* $0 \leq \delta_k < 1$. *Let* (p_n) *be a sequence of positive real constants such that as* $n \to \infty$,

$$P_n = O(np_n) \quad \textit{and} \quad np_n = O(P_n). \tag{2.5}$$

If the series $\sum a_n$ *is summable* $|\overline{N}, p_n; \delta|_k$*, then it is also summable* $|C, 1; \delta|_k$*.*

If we take $\delta = 0$ in this theorem, then we get a result due to Bor [**3**].

3. The main results. The aim of this paper is to generalize the above theorems in the form of the following theorems.

Theorem 1. *Let* $k \geq 1$ *and* $0 < \alpha \leq 1$*. Let the sequences* (X_n) *and* (λ_n) *be such that conditions* (2.1)–(2.2) *of Theorem* A *are satisfied. If there exists an* $\varepsilon > 0$ *such that the sequence* $(n^{\varepsilon-k}|\varphi_n|^k)$ *is nonincreasing and if the sequence* (u_n^α)*, defined by*

$$u_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1 \\ \max_{1 \leq v \leq n} |t_v^\alpha|, & 0 < \alpha < 1, \end{cases} \tag{3.1}$$

satisfies the condition

$$\sum_{v=1}^{n} v^{-k}(u_v^\alpha|\varphi_v|)^k = O(X_n) \quad \text{as } n \longrightarrow \infty, \tag{3.2}$$

then the series $\sum a_n \lambda_n$ *is summable* $\varphi - |C, \alpha|_k$*.*

If we take $\alpha = 1$ in this theorem, then we get Theorem A. Furthermore, if we take $\alpha = 1$ and $\varphi_n = n^{1-1/k}$, then we get a theorem due to Mazhar [**13**] concerning the $|C, 1|_k$ summability factors.

Theorem 2. *Let* $k \geq 1$*,* $0 < \alpha \leq 1$ *and* $0 \leq \delta_k < 1$*. Let* (p_n) *be a sequence of positive real constants such that condition* (2.5) *of Theorem* B *is satisfied and let* (T_n) *be the* $(\overline{N}, p_n)$ *mean of the series* $\sum a_n$*. If*

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + (2-\alpha)k - 1} |\Delta T_{n-1}|^k < \infty, \tag{3.3}$$

then the series $\sum a_n$ *is summable* $|C, \alpha; \delta|_k$*.*

It should be noted that, if we take $\alpha = 1$ in this theorem, then we get Theorem B. In fact, in this case condition (3.3) reduces to the condition

(1.11). Also, if we take $\delta = 0$ in this theorem, then we get a result due to Bor [**8**].

4. Needed lemmas. We need the following lemmas for the proof of our theorems.

Lemma 1 [**9**]. *If* $0 < \alpha \leq 1$ *and* $1 \leq v \leq n$, *then*

$$\left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} a_p\right| \leq \max_{1\leq m\leq v}\left|\sum_{p=1}^{m} A_{m-p}^{\alpha-1} a_p\right|, \tag{4.1}$$

where A_n^α *is as in* (1.3).

Lemma 2 [**7**]. *If the conditions* (2.1)–(2.2) *of Theorem* A *are satisfied, then*

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty \tag{4.2}$$

$$nX_n|\Delta\lambda_n| = O(1) \quad \textit{as } n \longrightarrow \infty. \tag{4.3}$$

Lemma 3 [**14**]. *If* $\sigma > \beta > 0$, *then*

$$\sum_{n=v+1}^{\infty} \frac{(n-v)^{\beta-1}}{n^\sigma} = O(v^{\beta-\sigma}). \tag{4.4}$$

5. Proof of the theorems.

Proof of Theorem 1. Let T_n^α be the nth (C, α) means of the sequence $(na_n\lambda_n)$, with $0 < \alpha \leq 1$. Then, by (1.2), we have that

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{5.1}$$

Using Abel's transformation, we get

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v,$$

so that making use of Lemma 1, we have

$$\begin{aligned}|T_n^\alpha| &\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\Delta\lambda_v|\left|\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p\right| + \frac{|\lambda_n|}{A_n^\alpha}\left|\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v\right| \\ &\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}A_v^\alpha u_v^\alpha|\Delta\lambda_v| + |\lambda_n|u_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha.\end{aligned}$$

By Minkowski's inequality for $k > 1$, to complete the proof it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k}|\varphi_n T_{n,r}^\alpha|^k < \infty \quad \text{for } r = 1,2, \quad \text{by (1.7).} \tag{5.2}$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $1/k + 1/k' = 1$, we get that

$$\begin{aligned}\sum_{n=2}^{m+1} n^{-k}|\varphi_n T_{n,1}^\alpha|^k &= \sum_{n=2}^{m+1} n^{-k}(A_n^\alpha)^{-k}|\varphi_n|^k\left\{\sum_{v=1}^{n-1}A_v^\alpha u_v^\alpha|\Delta\lambda_v|\right\}^k \\ &\le \sum_{n=2}^{m+1} n^{-k}(A_n^\alpha)^{-k}|\varphi_n|^k \\ &\qquad \left\{\sum_{v=1}^{n-1}(A_v^\alpha)^k(u_v^\alpha)^k|\Delta\lambda_v|\right\} \times \left\{\sum_{v=1}^{n-1}|\Delta\lambda_v|\right\}^{k-1} \\ &= O(1)\sum_{n=2}^{m+1} n^{-k}n^{-\alpha k}|\varphi_n|^k\sum_{v=1}^{n-1}v^{\alpha k}(u_v^\alpha)^k|\Delta\lambda_v|\end{aligned}$$

$$\begin{aligned}
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(u_v^{\alpha})^k|\Delta\lambda_v|\sum_{n=v+1}^{m+1}\frac{|\varphi_n|^k}{n^{\alpha k+k}}\\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(u_v^{\alpha})^k|\Delta\lambda_v|\sum_{n=v+1}^{m+1}\frac{n^{\varepsilon}|\varphi_n|^k}{n^{\alpha k+k+\varepsilon}}\\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(u_v^{\alpha})^k|\Delta\lambda_v|\sum_{n=v+1}^{m+1}\frac{n^{\varepsilon-k}|\varphi_n|^k}{n^{\alpha k+\varepsilon}}\\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(u_v^{\alpha})^k|\Delta\lambda_v|v^{\varepsilon-k}|\varphi_v|^k\sum_{n=v+1}^{m+1}\frac{1}{n^{\alpha k+\varepsilon}}\\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(u_v^{\alpha}|\varphi_v|)^k|\Delta\lambda_v|v^{\varepsilon-k}\int_v^{\infty}\frac{dx}{x^{\alpha k+\varepsilon}}\\
&= O(1)\sum_{v=1}^{m} v|\Delta\lambda_v|v^{-k}(u_v^{\alpha}|\varphi_v|)^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v} r^{-k}(u_r^{\alpha}|\varphi_r|)^k\\
&\quad + O(1)m|\Delta\lambda_m|\sum_{v=1}^{m} v^{-k}(u_v^{\alpha}|\varphi_v|)^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v|\Delta\lambda_v|)|X_v + O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1} vX_v|\Delta^2\lambda_v|\\
&\quad + O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v + O(1)m|\Delta\lambda_m|X_m\\
&= O(1) \quad \text{as } m \longrightarrow \infty,
\end{aligned}$$

by virtue of (3.2), (4.2) and (4.3). Since $\lambda_n = O(1/X_n) = O(1)$, by (2.1) we have that

$$\begin{aligned}\sum_{n=1}^{m} n^{-k}|\varphi_n T_{n,2}^{\alpha}|^k &= \sum_{n=1}^{m} |\lambda_n|\,|\lambda_n|^{k-1} n^{-k}(u_n^{\alpha}|\varphi_n|)^k \\ &= O(1)\sum_{n=1}^{m} |\lambda_n| n^{-k}(u_n^{\alpha}|\varphi_n|)^k \\ &= O(1)\sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} v^{-k}(u_v^{\alpha}|\varphi_v|)^k \\ &\quad + O(1)|\lambda_m| \sum_{n=1}^{m} n^{-k}(u_n^{\alpha}|\varphi_n|)^k \\ &= O(1)\sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m \\ &= O(1) \quad \text{as } m \longrightarrow \infty,\end{aligned}$$

by virtue of (2.1), (3.2) and (4.2). Therefore, we get that

$$\sum_{n=1}^{m} n^{-k}|\varphi_n T_{n,r}^{\alpha}|^k = O(1) \quad \text{as } m \longrightarrow \infty \quad \text{for } r = 1, 2.$$

This completes the proof of Theorem 1. □

If we take $\varepsilon = 1$ and $\varphi_n = n^{\delta+1-1/k}$ in this theorem, then we get a result for $|C, \alpha; \delta|_k$ summability factors.

Proof of Theorem 2. Let t_n^{α} be the nth (C, α) means of the sequence (na_n), with $0 < \alpha \leq 1$. By (1.12), we have that

$$a_n = -\frac{P_n}{p_n}\Delta T_{n-1} + \frac{P_{n-2}}{p_{n-1}}\Delta T_{n-2}. \tag{5.3}$$

If we put (5.3) in (1.2), then we have that

$$\begin{aligned}
t_n^\alpha &= \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v\left(-\frac{P_v}{p_v}\Delta T_{v-1} + \frac{P_{v-2}}{p_{v-1}}\Delta T_{v-2}\right) \\
&= -\frac{nP_n}{p_n A_n^\alpha}\Delta T_{n-1} + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}(-v)A_{n-v}^{\alpha-1}\frac{P_v}{p_v}\Delta T_{v-1} \\
&\quad + \frac{1}{A_n^\alpha}\sum_{v=1}^{n} vA_{n-v}^{\alpha-1}\frac{P_{v-2}}{p_{v-1}}\Delta T_{v-2} \\
&= -\frac{nP_n}{p_n A_n^\alpha}\Delta T_{n-1} + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}(-v)A_{n-v}^{\alpha-1}\frac{P_v}{p_v}\Delta T_{v-1} \\
&\quad + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}(v+1)A_{n-v-1}^{\alpha-1}\frac{P_{v-1}}{p_v}\Delta T_{v-1} \\
&= -\frac{nP_n}{p_n A_n^\alpha}\Delta T_{n-1} + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\frac{1}{p_v}\Delta T_{v-1} \\
&\qquad \{-vP_vA_{n-v}^{\alpha-1} + (v+1)A_{n-v-1}^{\alpha-1}P_{v-1}\}.
\end{aligned}$$

Since

$$\begin{aligned}
&-vP_vA_{n-v}^{\alpha-1} + (v+1)P_{v-1}A_{n-v-1}^{\alpha-1} \\
&\qquad = -vP_v\Delta_v A_{n-v}^{\alpha-1} - vp_vA_{n-v-1}^{\alpha-1} + P_{v-1}A_{n-v-1}^{\alpha-1}
\end{aligned}$$

we have

$$\begin{aligned}
t_n^\alpha &= -\frac{nP_n}{p_n A_n^\alpha}\Delta T_{n-1} - \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} v\frac{P_v}{p_v}\Delta_\nu A_{n-v}^{\alpha-1}\Delta T_{v-1} \\
&\quad - \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} vA_{n-v-1}^{\alpha-1}\Delta T_{v-1} \\
&\quad + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\frac{P_{v-1}}{p_v}A_{n-v-1}^{\alpha-1}\Delta T_{v-1} \\
&= t_{n,1}^\alpha + t_{n,2}^\alpha + t_{n,3}^\alpha + t_{n,4}^\alpha.
\end{aligned}$$

By Minkowski's inequality for $k > 1$, to complete the proof it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\delta k-1}|t_{n,r}^\alpha|^k < \infty, \quad \text{for } r = 1,2,3,4, \quad \text{by (1.6).} \tag{5.4}$$

First we have that

$$\sum_{n=1}^{m} n^{\delta k-1}|t_{n,1}^{\alpha}|k = \sum_{n=1}^{m} n^{\delta k-1}\left|\frac{nP_n}{p_n A_n^{\alpha}}\Delta T_{n-1}\right|^k$$

$$= O(1)\sum_{n=1}^{m} n^{\delta k+k-1}(P_n/p_n)^k n^{-\alpha k}|\Delta T_{n-1}|^k$$

$$= O(1)\sum_{n=1}^{m}(P_n/p_n)^{\delta k+(2-\alpha)k-1}|\Delta T_{n-1}|^k = O(1)$$

$$\text{as } m \longrightarrow \infty,$$

by virtue of the hypotheses of Theorem 2.

If $\alpha = 1$, $\Delta A_{n-v}^{\alpha-1} = 0$, hence $t_{n,2}^{\alpha} = 0$. If $0 < \alpha < 1$ we have, since $k > 1$, by Hölder's inequality,

$$\sum_{n=2}^{m+1} n^{\delta k-1}|t_{n,2}^{\alpha}|^k$$

$$\leq \sum_{n=2}^{m+1} n^{\delta k-1}\frac{1}{(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} v(P_v/p_v)|\Delta A_{n-v}^{\alpha-1}|\,|\Delta T_{v-1}|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k-\alpha k-1}\left\{\sum_{v=1}^{n-1} v^k(P_v/p_v)^k(n-v)^{\alpha-2}|\Delta T_{v-1}|^k\right\}$$

$$\times\left\{\sum_{v=1}^{n-1}(n-v)^{\alpha-2}\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+2} n^{\delta k-\alpha k-1}\left\{\sum_{v=1}^{n-1} v^k(P_v/p_v)^k(n-v)^{\alpha-2}|\Delta T_{v-1}|^k\right\}$$

$$= O(1)\sum_{v=1}^{m} v^k(P_v/p_v)^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-2}}{n^{1+\alpha k-\delta k}}$$

$$= O(1)\sum_{v=1}^{m} v^k(P_v/p_v)^k|\Delta T_{v-1}|^k v^{\delta k-\alpha k+\alpha-2}$$

$$= O(1)\sum_{v=1}^{m} v^{\delta k+k-1}(P_v/p_v)^k v^{-\alpha k}|\Delta T_{v-1}|^k v^{\alpha-1}.$$

Since $v^{\alpha-1} = O(1)$ when $0 < \alpha < 1$. Hence

$$\begin{aligned}\sum_{n=2}^{m+1} n^{\delta k-1}|t_{n,2}^{\alpha}|^k &= O(1)\sum_{v=1}^{m}(P_v/p_v)^{k+(2-\alpha)k-1}|\Delta T_{v-1}|^k \\ &= O(1) \quad \text{as } m \longrightarrow \infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma 3. Also, we have that

$$\begin{aligned}\sum_{n=2}^{m+1} n^{\delta k-1}|t_{n,3}^{\alpha}|^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1}\frac{1}{(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} vA_{n-v-1}^{\alpha-1}|\Delta T_{v-1}|\right\}^k \\ &= O(1)\sum_{n=2}^{m+1} n^{\delta k-1}\frac{1}{(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} vA_{n-v}^{\alpha-1}|\Delta T_{v-1}|\right\}^k \\ &= O(1)\sum_{n=2}^{m+1} n^{\delta k-1}\frac{1}{A_n^{\alpha}}\left\{\sum_{v=1}^{n-1} v^k A_{n-v}^{\alpha-1}|\Delta T_{v-1}|^k\right\} \\ &\qquad\qquad \times\left\{\frac{1}{A_n^{\alpha}}\sum_{v=1}^{n-1} A_{n-v}^{\alpha-1}\right\}^{k-1} \\ &= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{\alpha+1-\delta k}}\left\{\sum_{v=1}^{n-1} v^k(n-v)^{\alpha-1}|\Delta T_{v-1}|^k\right\} \\ &= O(1)\sum_{v=1}^{m} v^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-1}}{n^{\alpha+1-\delta k}} \\ &= O(1)\sum_{v=1}^{m} v^k|\Delta T_{v-1}|^k v^{\delta k-1}.\end{aligned}$$

Since $1-\alpha > 0$ and $k \geq 1$, we have that $1 \leq v^{(1-\alpha)k}$. Thus,

$$\begin{aligned}\sum_{n=2}^{m+1} n^{\delta k-1}|t_{n,3}^{\alpha}|^k &= O(1)\sum_{v=1}^{n} v^k|\Delta T_{v-1}|^k v^{\delta k-1}v^{(1-\alpha)k} \\ &= O(1)\sum_{v=1}^{m} v^{\delta k+(2-\alpha)k-1}|\Delta T_{v-1}|^k \\ &= O(1)\sum_{v=1}^{m}(P_v/p_v)^{\delta k+(2-\alpha)k-1}|\Delta T_{v-1}| = O(1)\end{aligned}$$

as $m \to \infty$, by virtue of the hypotheses of Theorem 2 and Lemma 3. Finally, we have that

$$\sum_{n=2}^{m+1} n^{\delta k-1}|t_{n,4}^{\alpha}|^k \leq \sum_{n=2}^{m-1} n^{\delta k-1} \frac{1}{(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} \frac{P_{v-1}}{p_v} A_{n-v-1}^{\alpha-1}|\Delta T_{v-1}|\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1} \frac{1}{n^{1-\delta k}(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1}(P_v/p_v)A_{n-v}^{\alpha-1}|\Delta T_{v-1}|\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1} \frac{1}{n^{1-\delta k}A_n^{\alpha}}\left\{\sum_{v=1}^{n-1}(P_v/p_v)^k A_{n-v}^{\alpha-1}|\Delta T_{v-1}|^k\right\} \times \left\{\frac{1}{A_n^{\alpha}}\sum_{v=1}^{n-1} A_{n-v}^{\alpha-1}\right\}^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1} \frac{1}{n^{\alpha+1-\delta k}}\left\{\sum_{v=1}^{n-1}(P_v/p_v)^k(n-v)^{\alpha-1}|\Delta T_{v-1}|^k\right\}$$
$$= O(1)\sum_{v=1}^{m}(P_v/p_v)^k|\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{(n-v)^{\alpha-1}}{n^{\alpha+1-\delta k}}$$
$$= O(1)\sum_{v=1}^{m}(P_v/p_v)^k|\Delta T_{v-1}|^k v^{\delta k-1}.$$

Hence, as in $t_{n,3}^{\alpha}$, we have that

$$\sum_{n=2}^{m+1} n^{\delta k-1}|t_{n,4}^{\alpha}|^k = O(1)\sum_{v=1}^{m}(P_v/p_v)^k|\Delta T_{v-1}|^k v^{\delta k-1}v^{(1-\alpha)k}$$
$$= O(1)\sum_{v=1}^{m}(P_v/p_v)^{\delta k+(2-\alpha)k-1}|\Delta T_{v-1}|^k$$
$$= O(1) \quad \text{as } m \longrightarrow \infty$$

by virtue of the hypothese of Theorem 2 and Lemma 3. Therefore, (5.4) holds. The case $k = 1$ can be dealt with in a similar manner. This completes the proof of Theorem 2. □

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