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ALGEBRAIC CHARACTERIZATION OF CONVOLUTION AND MULTIPLICATION OPERATORS ON HANKEL-TRANSFORMABLE FUNCTION AND DISTRIBUTION SPACES

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ABSTRACT. In this paper we characterize the continuous linear mapping from $\mathcal{B}_\mu$ into $\mathcal{H}'_\mu$ that commutes with Hankel convolutions (equivalently, with Hankel translations) as the convolution operators with symbol in $\mathcal{H}'_\mu$. As a consequence, we establish that $\mathcal{H}'_\mu$ and the space $\mathrm{Hom}\,(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#})$ of continuous linear mappings from $\mathcal{H}_\mu$ into $\mathcal{O}_{\mu,\#}$ commuting with Hankel translations are homeomorphic. Also, we prove that $\mathcal{O}'_{\mu,\#}$, respectively $\mathcal{O}$, is homeomorphic to $\mathrm{Hom}_{\#}(\mathcal{H}_\mu)$, respectively $\mathrm{Hom}_p(\mathcal{H}_\mu)$, where $\mathrm{Hom}_{\#}(\mathbf{H}_\mu)$, respectively $\mathrm{Hom}_p(\mathcal{H}_\mu)$, denotes the space of continuous linear mappings from $\mathcal{H}_\mu$ into itself that commute with Hankel convolutions, respectively with ordinary products.

1. Introduction and preliminaries. I.I. Hirschman [**12**], D.T. Haimo [**11**] and F.M. Cholewinski [**10**] investigated a convolution operation for the Hankel-type transformation

$$(h_\mu\psi)(t) = \int_0^\infty x^{2\mu+1}(xt)^{-\mu}J_\mu(xt)\psi(x)\,dx, \qquad t \in I.$$

Here, as usual, J_μ denotes the Bessel function of the first kind and order μ. Throughout this paper, the real parameter μ will be greater than or equal to $-1/2$.

In [**13**], we introduced a Hankel convolution closely connected with that cited above. The Hankel translation $\tau_x\psi$, $x \in I = (0,\infty)$, of a suitable function $\psi = \psi(x)$, $x \in I$, was defined by

$$(\tau_x\psi)(y) = \int_0^\infty \psi(z)D_\mu(x,y,z)\,dz, \qquad x, y \in I,$$

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where

$$D_\mu(x,y,z) = \int_0^\infty t^{-\mu-1/2}(xt)^{1/2}J_\mu(xt)(yt)^{1/2}J_\mu(yt)(zt)^{1/2}J_\mu(zt)\,dt,$$
$$x,y,z \in I.$$

Then the Hankel convolution $\psi\#\varphi$ of the functions $\psi = \psi(x)$ and $\varphi = \varphi(x)$ is given by

$$(\psi\#\varphi)(x) = \int_0^\infty \varphi(y)(\tau_x\psi)(y)\,dy, \qquad x \in I,$$

provided that the integral exists. The Hankel translation and the Hankel convolution relate to the Hankel integral transformation

$$(h_\mu\psi)(t) = \int_0^\infty (xt)^{1/2}J_\mu(xt)\psi(x)\,dx$$

by means of the formulae

$$(1) \qquad h_\mu(\tau_x\psi)(t) = t^{-\mu-1/2}(xt)^{1/2}J_\mu(xt)(h_\mu\psi)(t), \qquad x,t \in I,$$

$$(2) \qquad h_\mu(\psi\#\varphi)(t) = t^{-\mu-1/2}(h_\mu\psi)(t)(h_\mu\varphi)(t), \qquad t \in I,$$

valid for suitable ψ and φ.

In a series of papers [**5, 6, 7, 8, 9, 13**], the authors have investigated the #-convolution in certain spaces of generalized functions. We summarize below the main results in those papers that will be needed in the sequel.

A.H. Zemanian [**15**] introduced the function space $\mathcal{H}_\mu$ consisting of all those smooth, complex valued functions $\psi = \psi(x)$, $x \in I$, such that

$$\gamma^\mu_{m,k}(\psi) = \sup_{x\in I}\left| x^m \left(\frac{1}{x}D\right)^k (x^{-\mu-1/2}\psi(x))\right| < \infty$$

for every $m,k \in \mathbf{N}$. When endowed with the topology generated by the family of seminorms $\{\gamma^\mu_{m,k}\}_{m,k\in\mathbf{N}}$, $\mathcal{H}_\mu$ becomes a Fréchet space. Further properties of the space were found by the authors in [**3**].

The h_μ-transformation is an automorphism of $\mathcal{H}_\mu$, [**15**, Theorem 5.4-1]. As usual, $\mathcal{H}'_\mu$ will denote the dual space of $\mathcal{H}_\mu$. Throughout this paper $\mathcal{H}'_\mu$ will be equipped with its strong topology. The generalized Hankel transformation h'_μ is defined on $\mathcal{H}'_\mu$ as the adjoint of the h_μ-transformation.

In his study of the Hankel transformation, A.H. Zemanian [**14**] introduced the space $\mathcal{B}_\mu = \cup_{a>0}\beta_{\mu,a}$, where, for each $a > 0$, $\mathcal{B}_{\mu,a}$ consists of all those smooth, complex valued functions $\psi = \psi(x)$, $x \in I$, such that $\psi(x) = 0$, for $x > a$, and

$$\gamma_k^\mu(\psi) = \sup_{x\in I}\left|\left(\frac{1}{x}D\right)^k (x^{-\mu-1/2}\psi(x))\right| < \infty,$$

for every $k \in \mathbf{N}$. The space $\mathcal{B}_{\mu,a}$ is topologized by the family of seminorms $\{\gamma_k^\mu\}_{k\in\mathbf{N}}$, and $\mathcal{B}_\mu$ is endowed with the inductive topology associated to the collection $\{\mathcal{B}_{\mu,a}\}_{a>0}$.

The space $\mathcal{O}$ of all those smooth, complex valued functions $\vartheta = \vartheta(x)$, $x \in I$, with the property that to every $k \in \mathbf{N}$ there corresponds $n = n(k) \in \mathbf{Z}$ such that $\sup_{x\in I}|(1+x^2)^n((1/x)D)^k\vartheta(x)| < \infty$, was characterized by the authors as the space of multipliers of $\mathcal{H}_\mu$ and of $\mathcal{H}'_\mu$ [**4**, Theorem 2.3]. Hence, $\mathcal{O}$ can be viewed as a subspace of the space $\mathcal{L}(\mathcal{H}_\mu)$ of all continuous linear mappings from $\mathcal{H}_\mu$ into itself. Denoting by $\mathcal{L}_b(\mathcal{H}_\mu)$, respectively $\mathcal{L}_s(\mathcal{H}_\mu)$, the space $\mathcal{L}(H_\mu)$ endowed with the topology of uniform convergence on bounded subsets of $\mathcal{H}_\mu$, respectively the topology of pointwise convergence on $\mathcal{H}_\mu$, we find that both $\mathcal{L}_b(\mathcal{H}_\mu)$ and $\mathcal{L}_s(\mathcal{H}_\mu)$ induce on $\mathcal{O}$ the same topology [**8**, Proposition 1], namely, the topology generated by the family of seminorms $\{\gamma^\mu_{m,k;\psi}\}_{m,k\in\mathbf{N},\psi\in\mathcal{H}_\mu}$, where $\gamma^\mu_{m,k;\psi}(\vartheta) = \gamma^\mu_{m,k}(\vartheta\psi)$ for every $m, k \in \mathbf{N}$ and $\psi \in \mathcal{H}_\mu$, with $\vartheta \in \mathcal{O}$. Then $\mathcal{O}$ is complete [**4**, Proposition 3.2].

In our study on the generalized #-convolution we introduced [**13**] the space $\mathcal{O}_{\mu,\#} = \cup_{m\in\mathbf{Z}}\mathcal{O}_{\mu,m,\#}$, where $\mathcal{O}_{\mu,m,\#}$, $m \in \mathbf{Z}$, is defined as follows. Given $m \in \mathbf{Z}$, the function space $O_{\mu,m,\#}$ is formed by all those smooth, complex valued functions $\phi = \phi(x)$, $x \in I$, such that the quantities

$$o_k^{\mu,m}(\phi) = \sup_{x\in I}|(1+x^2)^m x^{-\mu-1/2}S_\mu^k\phi(x)|$$

are finite, for every $k \in \mathbf{N}$. Here S_μ denotes the Bessel operator $x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2}$. We endow $O_{\mu,m,\#}$ with the topology generated by $\{o_k^{\mu,m}\}_{k\in\mathbf{N}}$. It is clear that $O_{\mu,m,\#}$ contains $\mathcal{H}_\mu$. The space $\mathcal{O}_{\mu,m,\#}$ is then defined as the closure of $\mathcal{H}_\mu$ in $O_{\mu,m,\#}$. On $\mathcal{O}_{\mu,\#}$ we consider the inductive limit topology associated to the family $\{\mathcal{O}_{\mu,m,\#}\}_{m\in\mathbf{Z}}$.

The dual $\mathcal{O}'_{\mu,\#}$ of $\mathcal{O}_{\mu,\#}$ was characterized as the subspace of $\mathcal{H}'_\mu$ formed by all those functionals T in $\mathcal{H}'_\mu$ such that, for every $\psi \in \mathcal{H}_\mu$, the function $(T\#\psi)(x) = \langle T, \tau_x\psi\rangle$, $x \in I$, lies in $\mathcal{H}_\mu$ [**6**, Proposition 2.5]. It turns out that $h'_\mu(x^{-\mu-1/2}\mathcal{O}) = \mathcal{O}'_{\mu,\#}$ [**13**, Proposition 4.2]. For every $T \in \mathcal{O}'_{\mu,\#}$ the mapping $\psi \mapsto T\#\psi$ belongs to $\mathcal{L}(\mathcal{H}_\mu)$. In this sense, $\mathcal{O}'_{\mu,\#}$ can be regarded as a subspace of $\mathcal{L}(\mathcal{H}_\mu)$. Both $\mathcal{L}_b(\mathcal{H}_\mu)$ and $\mathcal{L}_s(\mathcal{H}_\mu)$ induce in $\mathcal{O}'_{\mu,\#}$ its strong topology as a dual of $\mathcal{O}_{\mu,\#}$ [**8**, Propositions 4 and 9, Proposition 5, Corollary 1].

The Hankel convolution $T\#S \in \mathcal{H}'_\mu$ of $T \in \mathcal{H}'_\mu$ and $S \in \mathcal{O}'_{\mu,\#}$ is defined by the formula $\langle T\#S, \psi\rangle = \langle T, S\#\psi\rangle$, $\psi \in \mathcal{H}_\mu$. If $T \in \mathcal{H}'_\mu$ and $\psi \in \mathcal{H}_\mu$, then $T\#\psi \in \mathcal{O}_{\mu,\#}$ [**9**, Proposition 2]. This fact allows us to define $S\#T \in \mathcal{H}'_\mu$, when $S \in \mathcal{O}'_{\mu,\#}$ and $T \in \mathcal{H}'_\mu$, by $\langle S\#T, \psi\rangle = \langle S, T\#\psi\rangle$, $\psi \in \mathcal{H}_\mu$. In [**9**, Proposition 3] we proved that $T\#S = S\#T$, for each $T \in \mathcal{H}'_\mu$ and $S \in \mathcal{O}'_{\mu,\#}$.

In this paper we characterize the continuous linear mappings from $\mathcal{B}_\mu$ into $\mathcal{H}'_\mu$ that commute with Hankel translations (equivalently, with Hankel convolutions) as the convolution operators with symbol in $\mathcal{H}'_\mu$. Motivated by the work of S. Abdullah [**1, 2**], we prove that $\mathcal{H}'_\mu$ is homeomorphic to the space $\operatorname{Hom}(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#})$ of all continuous linear mappings from $\mathcal{H}_\mu$ into $\mathcal{O}_{\mu,\#}$ commuting with Hankel translations. Furthermore, we establish that the elements in $\mathcal{L}(\mathcal{H}_\mu)$ that commute with Hankel translations (equivalently, with Hankel convolutions) are precisely convolutions with elements of $\mathcal{O}'_{\mu,\#}$. We also prove that $\operatorname{Hom}_\#(\mathcal{H}_\mu)$, respectively $\operatorname{Hom}_p(\mathcal{H}_\mu)$, is homeomorphic to $\mathcal{O}'_{\mu,\#}$, respectively $\mathcal{O}$, where $\operatorname{Hom}_\#(\mathcal{H}_\mu)$, respectively $\operatorname{Hom}_p(\mathcal{H}_\mu)$ is the subspace of $\mathcal{L}(\mathcal{H}_\mu)$ consisting of all continuous linear mappings from $\mathcal{H}_\mu$ into itself that commute with Hankel convolutions, respectively with ordinary products. Finally, invertibility with respect to the Hankel convolution in $\mathcal{O}'_{\mu,\#}$, respectively to the ordinary product in $\mathcal{O}$, and invertibility in $\operatorname{Hom}_\#(\mathcal{H}_\mu)$, respectively in $\operatorname{Hom}_p(\mathcal{H}_\mu)$, are shown to be equivalent.

Throughout this paper we will use Hankel approximate identities as considered in [**5**]. Also, we will refer to the space $\mathcal{E}_\mu$ and $\mathcal{W}^m_{\mu,a}$, $m \in \mathbf{N}$, $a > 0$, introduced in [**5**]. We recall that a smooth, complex valued function $\psi = \psi(x)$, $x \in I$, lies in $\mathcal{E}_\mu$ if, and only if, the limit $\lim_{x\to 0+}((1/x)D)^k(x^{-\mu-1/2}\psi(x))$ exists for every $k \in \mathbf{N}$. The space $\mathcal{E}_\mu$ becomes a Fréchet space when equipped with the topology generated by the system of seminorms $\{\eta^\mu_{k,n}\}_{k,n\in\mathbf{N}}$, where

$$\eta^\mu_{k,n}(\psi) = \sup_{0<x<n}\left|\left(\frac{1}{x}D\right)^k(x^{-\mu-1/2}\psi(x))\right|, \qquad \psi \in \mathcal{E}_\mu.$$

The space $\mathcal{W}^m_{\mu,a}$, $m \in \mathbf{N}$, $a > 0$, is constituted by all those complex valued functions $\psi \in C^{2m}(I)$ such that $\psi(x) = 0$ for $x \geq a$, and $\lim_{x\to 0+}((1/x)D)^k(x^{-\mu-1/2}\psi(x))$ exists for $k \in \mathbf{N}$, $0 \leq k \leq 2m$. This space is normed by

$$|\psi|_{\mu,\infty,m} = \max_{0\leq k\leq m}\sup_{x\in I}|x^{-\mu-1/2}S^k_\mu\psi(x)|, \qquad \psi \in \mathcal{W}^m_{\mu,a}.$$

2. Algebraic characterization of $\mathcal{H}'_\mu$. Our first aim in this section is to prove that the mappings from $\mathcal{B}_\mu$ into $\mathcal{H}'_\mu$ commuting with Hankel translations are precisely those which commute with Hankel convolutions. Moreover, we establish that such mappings are always (and only) convolution operators with symbol in $\mathcal{H}'_\mu$.

Two auxiliary results are previously needed.

Lemma 1. *Let $\{\phi_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathcal{H}_\mu$. If $\{\phi_n\}_{n\in\mathbf{N}}$ is bounded in $\mathcal{H}_\mu$ and there exists $\phi \in \mathcal{E}_\mu$ such that $\lim_{n\to\infty}\phi_n = \phi$ in the sense of convergence in $\mathcal{E}_\mu$, then $\phi \in \mathcal{H}_\mu$ and $\lim_{n\to\infty}\phi_n = \phi$ in the sense of convergence in $\mathcal{H}_\mu$.*

Proof. Since $\{\phi_n\}_{n\in\mathbf{N}}$ is a bounded sequence in $\mathcal{H}_\mu$, given $m, k \in \mathbf{N}$ there exists a positive constant $B_{m,k}$ such that $\gamma^\mu_{m,k}(\phi_n) \leq B_{m,k}$, for each $n \in \mathbf{N}$. Also, for every $x \in I$, $x^m((1/x)D)^k x^{-\mu-1/2}\phi_n(x) \to x^m((1/x)D)^k x^{-\mu-1/2}\phi(x)$, as $n \to \infty$, because $\phi_n \to \phi$ in $\mathcal{E}_\mu$, as $n \to \infty$. Then $\gamma^\mu_{m,k}(\phi) \leq B_{m,k}$, so that $\phi \in \mathcal{H}_\mu$.

Let $\varepsilon > 0$. It is easy to find $\rho > 0$ such that

$$(3) \qquad \left|x^m\left(\frac{1}{x}D\right)^k x^{-\mu-1/2}(\phi_n(x) - \phi(x))\right| < \varepsilon, \qquad n \in \mathbf{N}, x \geq \rho.$$

On the other hand, since $\phi_n \to \phi$ in $\mathcal{E}_\mu$ as $n \to \infty$, there exists $n_0 \in \mathbf{N}$ such that

$$\sup_{0<x<\rho} \left| x^m \left(\frac{1}{x}D\right)^k x^{-\mu-1/2}(\phi_n(x) - \phi(x)) \right| < \varepsilon, \qquad n \geq n_0. \tag{4}$$

By combining (3) and (4) we conclude that $\phi_n \to \phi$ in $\mathcal{H}_\mu$, as $n \to \infty$. □

Lemma 2. *Let $a, b > 0$. Then for every $\psi \in \mathcal{B}_{\mu,a}$ and $\varphi \in \mathcal{B}_{\mu,b}$ the convergence*

$$(\psi \# \varphi)(x) = \lim_{r\to\infty} \frac{a}{r} \sum_{j=1}^{r} \psi(y_{r,j})(\tau_{y_{r,j}}\varphi)(x) \tag{5}$$

takes place both in the topology of $\mathcal{H}_\mu$ and in that of $\mathcal{B}_\mu$, where $y_{r,j} = aj/r$, $r, j \in \mathbf{N}$, $1 \leq j \leq r$.

Proof. In the first place, we note that (5) holds in the sense of convergence in $\mathcal{H}_\mu$ if and only if

$$h_\mu\left((\psi\#\varphi)(x) - \frac{a}{r}\sum_{j=1}^{r} \psi(y_{r,j})(\tau_{y_{r,j}}\varphi)(x) \right)(t) \to 0 \quad \text{in } \mathcal{H}_\mu,$$
$$\text{as } r \to \infty.$$

By virtue of (1) and (2), establishing the latter convergence is equivalent to proving that $(h_\mu\varphi)(t)\vartheta_r(t) \to 0$ in $\mathcal{H}_\mu$ as $r \to \infty$ where, for every $r \in \mathbf{N}$, $r \geq 1$, and $t \in I$, $\vartheta_r(t)$ is defined by

$$\vartheta_r(t) = t^{-\mu-1/2}\left((h_\mu\psi)(t) - \frac{a}{r}\sum_{j=1}^{r} \psi(y_{r,j})(ty_{r,j})^{1/2} J_\mu(ty_{r,j}) \right).$$

We shall proceed to show that $(h_\mu\varphi)(t)\vartheta_r(t) \to 0$ in $\mathcal{E}_\mu$, as $r \to \infty$. In fact, fix $k \in \mathbf{N}$. Leibniz's rule leads to

$$\begin{aligned} &\left(\frac{1}{t}D\right)^k (t^{-\mu-1/2}(h_\mu\varphi)(t)\vartheta_r(t)) \\ &\qquad = \sum_{i=0}^{k} \binom{k}{i} \left(\frac{1}{t}D\right)^{k-i} (t^{-\mu-1/2}(h_\mu\varphi)(t)) \left(\frac{1}{t}D\right)^{i} \vartheta_r(t). \end{aligned} \tag{6}$$

By using Equation 5.1 (6) in [**15**], for each fixed $i \in \mathbf{N}$, $0 \le i \le k$, one obtains

(7)
$$\begin{aligned}\Big(-\frac{1}{t}D\Big)^i \vartheta_r(t) = \int_0^a (xt)^{-\mu-i} J_{\mu+i}(xt) x^{2i+\mu+1/2}\psi(x)\,dx \\ - \frac{a}{r}\sum_{j=1}^{r} \psi(y_{r,j})(ty_{r,j})^{-\mu-i} J_{\mu+i}(ty_{r,j}) y_{r,j}^{2i+\mu+1/2}, \\ t \in I.\end{aligned}$$

Now let $n \in \mathbf{N}$. The (uniform) continuity of

$$h(x,t) = \begin{cases} (xt)^{-\mu-i} J_{\mu+i}(xt) x^{2i+\mu+1/2}\psi(x), & 0 \le t \le n \text{ and } 0 < x \le a \\ 0 & 0 \le t \le n \text{ and } x = 0 \end{cases}$$

in $\{(x,t) : 0 \le t \le n, 0 \le x \le a\}$ guarantees that $((1/t)D)^i \vartheta_r(t) \to 0$ uniformly in $0 \le t \le n$, as $r \to \infty$. Moreover, the function $((1/t)D)^{k-i}(t^{-\mu-1/2}(h_\mu\varphi)(t))$ is bounded on I, because $h_\mu\varphi \in \mathcal{H}_\mu$. Hence, $(h_\mu\varphi)(t)\vartheta_r(t) \to 0$ in $\mathcal{E}_\mu$, as $r \to \infty$.

The proof that $(h_\mu\varphi)(t)\vartheta_r(t) \to 0$ in $\mathcal{H}_\mu$, as $r \to \infty$, may now be accomplished by establishing the boundedness in $\mathcal{H}_\mu$ of the sequence $\{(h_\mu\varphi)(t)\vartheta_r(t)\}_{r\in\mathbf{N}}$ (Lemma 1). To this end, fix $m, k \in \mathbf{N}$. From (6) we infer that

$$\gamma^\mu_{m,k}((h_\mu\varphi)\vartheta_r) \le \sum_{i=0}^{k} \binom{k}{i} \gamma^\mu_{m,k-i}(h_\mu\varphi) \sup_{t\in I} \Big|\Big(\frac{1}{t}D\Big)^i \vartheta_r(t)\Big|.$$

The function $x^{-\mu}J_\mu(x)$ being bounded on I, (7) implies

$$\begin{aligned}\sup_{t\in I}\Big|\Big(\frac{1}{t}D\Big)^i \vartheta_r(t)\Big| \le A\Big(\int_0^a x^{2i+\mu+1/2}|\psi(x)|\,dx \\ + \frac{a}{r}\sum_{j=1}^{r} |\psi(y_{r,j})| y_{r,j}^{2i+\mu+1/2}\Big) \le B\end{aligned}$$

for certain $A, B > 0$. Thus (5) holds in the topology of $\mathcal{H}_\mu$.

Finally we are going to prove that (5) holds also in $\mathcal{B}_\mu$. According to [**5**, Proposition 3.4], $\psi\#\varphi$ belongs to $\mathcal{B}_{\mu,a+b}$ whenever $\psi \in \mathcal{B}_{\mu,a}$ and

$\varphi \in \mathcal{B}_{\mu,b}$. By [**5**, Corollary 3.3], $\tau_{y_{r,j}}\psi \in \mathcal{B}_{\mu,a+b}$ for every $r, j \in \mathbf{N}$, $1 \le j \le r$. Hence, $(\psi\#\varphi)(x) - (a/r)\sum_{j=1}^{r} \psi(y_{r,j})(\tau_{y_{r,j}}\varphi)(x) \in \mathcal{B}_{\mu,a+b}$ for every $r \in \mathbf{N}$, $r \ge 1$. Since (5) holds in $\mathcal{H}_\mu$, it is true also in $\mathcal{B}_{\mu,a+b}$. $\square$

We are now ready to establish the main result in this section.

Proposition 1. *Let $\mathcal{U} : \mathcal{B}_\mu \to \mathcal{H}'_\mu$ be a continuous linear mapping. Then the following are equivalent.*

(i) *$\mathcal{U}$ commutes with Hankel translations, that is, $\tau_x\mathcal{U}\psi = \mathcal{U}\tau_x\psi$ for $x \in I$ and $\psi \in \mathcal{B}_\mu$.*

(ii) *$\mathcal{U}$ commutes with Hankel convolutions, that is, $\mathcal{U}(\varphi\#\psi) = (\mathcal{U}\varphi)\#\psi$ for $\varphi, \psi \in \mathcal{B}_\mu$.*

(iii) *There exists a unique $T \in \mathcal{H}'_\mu$ such that $\mathcal{U}\psi = T\#\psi$, $\psi \in \mathcal{B}_\mu$.*

Proof. (i) $\Rightarrow$ (ii). Let $a, b > 0$ and take $\psi \in \mathcal{B}_{\mu,a}$, $\varphi \in \mathcal{B}_{\mu,b}$. According to Lemma 2, the identity

$$(\varphi\#\psi)(x) = \lim_{r\to\infty} \frac{a}{r} \sum_{j=1}^{r} \psi(y_{r,j})(\tau_{y_{r,j}}\varphi)(x)$$

holds both in the sense of convergence in $\mathcal{H}_\mu$ and in that of $\mathcal{B}_\mu$ where, as in Lemma 2, $y_{r,j} = aj/r$, $r, j \in \mathbf{N}$, $1 \le j \le r$.

As $\mathcal{U}$ is a continuous linear mapping commuting with Hankel translations, for every $\phi \in \mathcal{B}_\mu$ we may write

$$\begin{aligned}
\langle \mathcal{U}(\varphi\#\psi), \phi\rangle &= \lim_{r\to\infty} \frac{a}{r} \sum_{j=1}^{r} \psi(y_{r,j})\langle \mathcal{U}(\tau_{y_{r,j}}\varphi), \phi\rangle \\
&= \lim_{r\to\infty} \frac{a}{r} \sum_{j=1}^{r} \psi(y_{r,j})\langle \mathcal{U}\varphi, \tau_{y_{r,j}}\phi\rangle \\
&= \langle \mathcal{U}\varphi, \lim_{r\to\infty} \frac{a}{r} \sum_{j=1}^{r} \psi(y_{r,j})\tau_{y_{r,j}}\phi\rangle \\
&= \langle \mathcal{U}\varphi, \psi\#\phi\rangle \\
&= \langle (\mathcal{U}\varphi)\#\psi, \phi\rangle.
\end{aligned}$$

The density of $\mathcal{B}_\mu$ in $\mathcal{H}_\mu$ establishes (ii).

(ii) $\Rightarrow$ (iii). Let $\{k_n\}_{n\in\mathbf{N}}$ be a Hankel approximate identity. Since $\mathcal{U}$ is continuous and commutes with convolutions, and since $k_n\#\psi \to \psi$ in $\mathcal{B}_\mu$, as $n\to\infty$, whenever $\psi\in\mathcal{B}_\mu$ [**5**, Proposition 3.6], one has

$$\mathcal{U}\psi = \lim_{n\to\infty}\mathcal{U}(k_n\#\psi) = \lim_{n\to\infty}(\mathcal{U}k_n)\#\psi, \qquad \psi\in\mathcal{B}_\mu \tag{8}$$

in the topology of $\mathcal{H}'_\mu$. Now, for each $n\in\mathbf{N}$, write $T_n = \mathcal{U}k_n$ and define

$$\begin{aligned}\mathcal{U}_n : \mathcal{B}_\mu &\longrightarrow \mathcal{H}'_\mu\\ \psi &\longmapsto T_n\#\psi.\end{aligned}$$

According to [**13**, Proposition 3.5], each $\mathcal{U}_n$ is a continuous linear mapping. Moreover, (8) guarantees that the sequence $\{\mathcal{U}_n\psi\}_{n\in\mathbf{N}}$ is bounded in $\mathcal{H}'_\mu$ for every $\psi\in\mathcal{B}_\mu$.

Fix $a>0$. By [**6**, Lemma 2.3] there exists $s\in\mathbf{N}$ such that each $\mathcal{U}_n$ can be continuously extended up to $\mathcal{W}^s_{\mu,a}$ (this extension will be denoted again by $\mathcal{U}_n$) to obtain an equicontinuous family of mappings from $\mathcal{W}^s_{\mu,a}$ into $\mathcal{H}'_\mu$. Choose $0<b<a$ and $m,r\in\mathbf{N}$ such that $m>\mu+s+2$ and $r>\mu+2m+1$. The identity $\delta_\mu=(1-S_\mu)^r\psi_0-\varphi_0$ holds for suitable $\psi_0\in\mathcal{W}^m_{\mu,b}$ and $\varphi_0\in\mathcal{B}_{\mu,b}$ [**6**, Lemma 2.1]], where $\delta_\mu\in\mathcal{O}'_{\mu,\#}$ is defined by

$$\langle\delta_\mu,\psi\rangle = c_\mu\lim_{x\to0+}x^{-\mu-1/2}\psi(x), \qquad \psi\in\mathcal{H}_\mu.$$

Then

$$\begin{aligned}T_n = T_n\#\delta_\mu &= (1-S_\mu)^r T_n\#\psi_0 - T_n\#\varphi_0\\ &= (1-S_\mu)^r\mathcal{U}_n\psi_0 - \mathcal{U}_n\varphi_0, \qquad n\in\mathbf{N},\end{aligned}$$

because S_μ commutes with the $\#$-convolution [**13**, Proposition 2.2]. Arguing as in the proof that (iii) $\Rightarrow$ (i) in [**6**, Proposition 2.4], we infer that the sequence $\{T_n\}_{n\in\mathbf{N}}$ is bounded in $\mathcal{H}'_\mu$. The space $\mathcal{H}'_\mu$ being Montel [**3**, Corollary 4.3], there exists a subsequence of $\{T_n\}_{n\in\mathbf{N}}$ (that we will denote again by $\{T_n\}_{n\in\mathbf{N}}$) which converges to a certain $T\in\mathcal{H}'_\mu$, as $n\to\infty$. According to [**5**, Propositions 3.6 and 4.1] we have that $T_n\#\psi\to T\#\psi$, as $n\to\infty$, whenever $\psi\in\mathcal{B}_\mu$, in the sense of convergence in $\mathcal{H}'_\mu$. From (8), we conclude that $\mathcal{U}\psi=T\#\psi$ for all $\psi\in\mathcal{B}_\mu$.

Uniqueness of T can be established as follows. If $T\#\psi = 0$ whenever $\psi \in \mathcal{B}_\mu$, then

$$\langle T, \psi\rangle = \langle T\#\delta_\mu, \psi\rangle = \langle \delta_\mu, T\#\psi\rangle = 0, \qquad \psi \in \mathcal{B}_\mu.$$

Since $\mathcal{B}_\mu$ is dense in $\mathcal{H}_\mu$, necessarily $T = 0$.

(iii) $\Rightarrow$ (i). This follows immediately from [**7**, Lemma 2.2]. □

An interesting consequence of Proposition 1 is the next.

Corollary 1. *Let $\mathcal{U}$ be a continuous linear mapping from $\mathcal{H}_\mu$ into $\mathcal{O}_{\mu,\#}$. The following are equivalent.*

(i) *$\mathcal{U}$ commutes with Hankel translations.*

(ii) *$\mathcal{U}$ commutes with Hankel convolutions.*

(iii) *There exists a unique $T \in \mathcal{H}'_\mu$ such that $\mathcal{U}\psi = T\#\psi$ whenever $\psi \in \mathcal{H}_\mu$.*

Proof. It is easily seen that $\mathcal{H}'_\mu$ contains $\mathcal{O}_{\mu,\#}$. Moreover, if $\mathcal{T}_s$ denotes the topology induced on $\mathcal{O}_{\mu,\#}$ by the strong topology of $\mathcal{H}'_\mu$, then $\mathcal{O}_{\mu,m,\#}$ is continuously embedded in $(\mathcal{O}_{\mu,\#}, \mathcal{T}_s)$, for every $m \in \mathbf{Z}$. In fact, let $m \in \mathbf{Z}$. For $f \in \mathcal{O}_{\mu,m,\#}$ and $\psi \in \mathcal{H}_\mu$, we have

$$|\langle f, \psi\rangle| \leq \int_0^\infty x^{2\mu+1}|(1+x^2)^m x^{-\mu-1/2} f(x)| \cdot |(1+x^2)^r x^{-\mu-1/2}\psi(x)|(1+x^2)^{-r-m}\,dx,$$

where r is a nonnegative integer such that $m + r - 2\mu > 2$. Hence, the inductive topology of $\mathcal{O}_{\mu,\#}$ is stronger than $\mathcal{T}_s$. Corollary 1 may now be derived from Proposition 1 by taking into account the density of $\mathcal{B}_\mu$ in $\mathcal{H}_\mu$. □

The next Proposition 2 yields an algebraic characterization of $\mathcal{H}'_\mu$. The space $\operatorname{Hom}(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#})$ consists of all those continuous linear mappings from $\mathcal{H}_\mu$ into $\mathcal{O}_{\mu,\#}$ commuting with Hankel translations. A 0-neighborhood base for the topology of $\operatorname{Hom}(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#})$ is given by the sets

$$V(B, G) = \{F \in \operatorname{Hom}(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#}) : F(\psi) \in G, \psi \in B\},$$

where B runs over the family of all bounded subsets of $\mathcal{H}_\mu$ and G denotes any neighborhood of the origin in $\mathcal{O}_{\mu,\#}$.

Proposition 2. *The spaces $\mathcal{H}'_\mu$ and $\mathrm{Hom}\,(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#})$ are homeomorphic.*

Proof. It is clear that, given $T \in \mathcal{H}'_\mu$, the mapping $\mathbf{U}(T)$, defined by $\mathbf{U}(T)\psi = T\#\psi$, $\psi \in \mathcal{H}_\mu$, belongs to $\mathrm{Hom}\,(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#})$. Moreover, by virtue of Corollary 1, to every $F \in \mathrm{Hom}\,(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#})$ there corresponds a unique $T \in \mathcal{H}'_\mu$ such that $F(\psi) = T\#\psi$, $\psi \in \mathcal{H}_\mu$. Hence $\mathbf{U}$ establishes an algebraic isomorphism from $\mathcal{H}'_\mu$ onto $\mathrm{Hom}\,(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#})$. We claim this isomorphism is also topological.

To begin with, $\mathbf{U}$ is continuous. Let V denote the neighborhood of the origin in $\mathrm{Hom}\,(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#})$ defined by

$$V = V(B, W^\circ) = \{F \in \mathrm{Hom}\,(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#}) : F(\psi) \in W^\circ, \psi \in B\},$$

where B is a bounded set in $\mathcal{H}_\mu$ and W is a bounded set in $\mathcal{O}'_{\mu,\#}$. We want to show that $\mathbf{U}^{-1}(V)$ is a neighborhood of the origin in $\mathcal{H}'_\mu$. To this end, consider $U = \{S\#\psi : S \in W, \psi \in B\}$. Then U is bounded in $\mathcal{H}_\mu$. For, given a 0-neighborhood D in $\mathcal{H}_\mu$, the set

$$C = C(B, D) = \{S \in \mathcal{O}'_{\mu,\#} : S\#\psi \in D, \psi \in B\}$$

defines a 0-neighborhood in $\mathcal{O}'_{\mu,\#}$ so that $\lambda W \subseteq C$ for some $\lambda > 0$. Consequently $\lambda(S\#\psi) = (\lambda S)\#\psi \in D$, for each $S \in W$ and $\psi \in B$, and we conclude that $\lambda U \subseteq D$. Now $T \in \mathbf{U}^{-1}(V)$ if and only if

$$|\langle T\#\psi, S\rangle| = |\langle S\#T, \psi\rangle| = |\langle T\#S, \psi\rangle| = |\langle T, S\#\psi\rangle| < 1$$

for every $S \in W$ and $\psi \in B$. Therefore, $\mathbf{U}^{-1}(V) = U^\circ$, thus proving that $\mathbf{U}^{-1}(V)$ is a neighborhood of the origin in $\mathcal{H}'_\mu$.

In order to establish the continuity of $\mathbf{U}^{-1}$, let B be a bounded set in $\mathcal{H}_\mu$. We aim to show that $\mathbf{U}(B^\circ)$ is a neighborhood of the origin in $\mathrm{Hom}\,(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#})$. This will be done by proving that

$$\mathbf{U}(B^\circ) = V(B, \{\delta_\mu\}^\circ) = \{F \in \mathrm{Hom}\,(\mathcal{H}_\mu, \mathcal{O}_{\mu,\#}) : F(\psi) \in \{\delta_\mu\}^\circ, \psi \in B\}.$$

In fact, $F \in V(B, \{\delta_\mu\}^\circ)$ if and only if $F(\psi) \in \{\delta_\mu\}^\circ$ for every $\psi \in B$ or, equivalently,

$$|\langle \delta_\mu, T\#\psi\rangle| = |\langle T\#\delta_\mu, \psi\rangle| = |\langle T, \psi\rangle| < 1$$

for every $\psi \in B$, where $T = \mathbf{U}^{-1}(F)$. Thus $\mathbf{U}(B^\circ) = V(B, \{\delta_\mu\}^\circ)$, as asserted. □

Proposition 2 suggests that we define in $\mathcal{H}'_\mu$ the topology $\mathcal{T}_H$ which has as 0-neighborhood base the family of all sets of the form

$$V(B, W) = \{T \in \mathcal{H}'_\mu : T\#\psi \in W, \psi \in B\},$$

where B is any bounded set in $\mathcal{H}_\mu$ and W is any neighborhood of the origin in $\mathcal{O}_{\mu,\#}$. As an immediate consequence of Proposition 2, we have the following.

Corollary 2. *The strong topology of $\mathcal{H}'_\mu$ coincides with $\mathcal{T}_H$.*

3. Algebraic characterization of $\mathcal{H}_\mu$. In [**7**, Theorem 2.3] the authors established that any continuous linear mapping from $\mathcal{H}_\mu$ into itself commuting with Hankel translations can be represented as a convolution with an element of $\mathcal{O}'_{\mu,\#}$. This result can be improved as follows.

Proposition 3. *Let $\mathcal{U}$ be a continuous linear mapping from $\mathcal{H}_\mu$ into itself. Then the following are equivalent.*

(i) *$\mathcal{U}$ commutes with Hankel translations.*

(ii) *$\mathcal{U}$ commutes with Hankel convolutions.*

(iii) *There exists a unique $S \in \mathcal{O}'_{\mu,\#}$ such that $\mathcal{U}\psi = S\#\psi$, for every $\psi \in \mathcal{H}_\mu$.*

Proof. The equivalence between (i) and (iii) is Theorem 2.3 in [**7**]. That (iii) implies (ii) follows immediately from [**7**, Lemma 2.2]. To finish the proof we shall establish that (ii) implies (i). Assume that $\mathcal{U}$ satisfies (ii). Let $\{k_n\}_{n\in\mathbf{N}}$ be a Hankel approximate identity, and let $\psi \in \mathcal{H}_\mu$. An argument similar to the one developed in the proof of

[**5**, Proposition 3.5] reveals that $k_n\#\psi \to \psi$ in $\mathcal{H}_\mu$, as $n \to \infty$. Since τ_x, $x \in I$, is a continuous linear mapping from $\mathcal{H}_\mu$ into itself which commutes with Hankel convolutions, for every $x \in I$ we may write

$$\begin{aligned}\tau_x\mathcal{U}\psi &= \tau_x\mathcal{U}(\lim_{n\to\infty} k_n\#\psi) = \lim_{n\to\infty} \tau_x\mathcal{U}(k_n\#\psi) \\ &= \lim_{n\to\infty} \tau_x((\mathcal{U}k_n)\#\psi)) = \lim_{n\to\infty} (\mathcal{U}k_n)\#\tau_x\psi \\ &= \lim_{n\to\infty} \mathcal{U}(k_n\#\tau_x\psi) = \mathcal{U}(\lim_{n\to\infty} (k_n\#\tau_x\psi)) \\ &= \mathcal{U}(\tau_x\psi).\end{aligned}$$

The proof is thus complete. □

We denote by $\mathrm{Hom}_{\#}(\mathcal{H}_\mu)$ the subspace of $\mathcal{L}(\mathcal{H}_\mu)$ constituted by all continuous linear mappings from $\mathcal{H}_\mu$ into itself that commute with Hankel convolutions. The space $\mathrm{Hom}_{\#}(\mathcal{H}_\mu)$ is endowed with the topology it inherits from $\mathcal{L}_b(\mathcal{H}_\mu)$.

Proposition 3 yields the following algebraic characterization of $\mathcal{O}'_{\mu,\#}$.

Corollary 3. *The spaces* $\mathcal{O}'_{\mu,\#}$ *and* $\mathrm{Hom}_{\#}(\mathcal{H}_\mu)$ *are homeomorphic.*

Proof. By virtue of Proposition 3, the mapping

$$\begin{aligned}\mathbf{L} : \mathcal{O}'_{\mu,\#} &\longrightarrow \mathrm{Hom}_{\#}(\mathcal{H}_\mu) \\ S &\longmapsto \mathbf{L}(S) : \mathcal{H}_\mu \longrightarrow \mathcal{H}_\mu \\ &\qquad\qquad\qquad \psi \longmapsto S\#\psi\end{aligned}$$

is an algebraic isomorphism. This isomorphism is also topological because the topology of $\mathcal{O}'_{\mu,\#}$ is precisely that inherited from $\mathcal{L}_b(\mathcal{H}_\mu)$ via **L**. □

The isomorphisms in $\mathrm{Hom}_{\#}(\mathcal{H}_\mu)$ may be identified with the invertible elements in $\mathcal{O}'_{\mu,\#}$ with respect to the convolution product, as Proposition 4 shows.

Proposition 4. *Let* $S \in \mathcal{O}'_{\mu,\#}$, *and denote by* F_s *the member of* $\mathrm{Hom}_{\#}(\mathcal{H}_\mu)$ *defined by* $F_s(\psi) = S\#\psi$, $\psi \in \mathcal{H}_\mu$. *Then* $S\#\mathcal{O}'_{\mu,\#} = \mathcal{O}'_{\mu,\#}$ *if and only if* F_s *is an isomorphism.*

Proof. First assume that $S\#\mathcal{O}'_{\mu,\#} = \mathcal{O}'_{\mu,\#}$. Then some $u \in \mathcal{O}'_{\mu,\#}$ satisfies $S\#u = \delta_\mu$, so that $S\#(u\#\psi) = (S\#u)\#\psi = \delta_\mu\#\psi = \psi$, for every $\psi \in \mathcal{H}_\mu$. As $u\#\psi \in \mathcal{H}_\mu$ whenever $\psi \in \mathcal{H}_\mu$ [**13**, Proposition 4.3], we conclude that F_s is onto. On the other hand, if $\psi \in \mathcal{H}_\mu$ and $S\#\psi = 0$ then $\psi = \delta_\mu\#\psi = (S\#u)\#\psi = u\#(S\#\psi) = 0$. Therefore, F_s is one-to-one.

Now suppose that F_s is bijective. It is easily checked that the inverse F_s^{-1} of F_s also belongs to $\mathrm{Hom}_{\#}(\mathcal{H}_\mu)$. Then the linear functional

$$\begin{aligned} u : \mathcal{H}_\mu &\longrightarrow \mathbf{C} \\ \psi &\longmapsto \langle \delta_\mu, F_s^{-1}(\psi)\rangle \end{aligned}$$

lies in $\mathcal{H}'_\mu$. Moreover,

$$\langle u\#S, \psi\rangle = \langle u, S\#\psi\rangle = \langle \delta_\mu, F_s^{-1}(S\#\psi)\rangle = \langle \delta_\mu, \psi\rangle, \qquad \psi \in \mathcal{H}_\mu.$$

This means that $u\#S = \delta_\mu$, whence $u\#(S\#\psi) = (u\#S)\#\psi = \psi$ for every $\psi \in \mathcal{H}_\mu$. The suprajectivity of F_s, along with [**6**, Proposition 2.5], yields that $u \in \mathcal{O}'_{\mu,\#}$. Now, if $T \in \mathcal{O}'_{\mu,\#}$ then $u\#T \in \mathcal{O}'_{\mu,\#}$ and $S\#(u\#T) = (S\#u)\#T = \delta_\mu\#T = T$. This completes the proof. □

At this point we introduce the space $\mathrm{Hom}_{p}(\mathcal{H}_\mu)$ consisting of all those continuous linear mappings F from $\mathcal{H}_\mu$ into itself, such that

$$F(x^{-\mu-1/2}\varphi(x)\psi(x)) = x^{-\mu-1/2}(F\varphi)(x)\psi(x), \qquad \varphi, \psi \in \mathcal{H}_\mu.$$

We endow $\mathrm{Hom}_{p}(\mathcal{H}_\mu)$ with the relative topology inherited from $\mathcal{L}_b(\mathcal{H}_\mu)$. Our next objective is to prove that $\mathrm{Hom}_{p}(\mathcal{H}_\mu)$ is homeomorphic to the space $\mathcal{O}$ of multipliers of $\mathcal{H}_\mu$. Previously, the following must be established.

Lemma 3. *For every $\vartheta \in \mathcal{O}$ there exists a sequence $\{\psi_n\}_{n\in\mathbf{N}}$ in $\mathcal{H}_\mu$ such that $\lim_{n\to\infty} x^{-\mu-1/2}\psi_n = \vartheta$ in $\mathcal{O}$.*

Proof. Let $\{k_n\}_{n\in\mathbf{N}}$ be a Hankel approximate identity. An argument similar to that developed in the proof of [**5**, Proposition 3.5] reveals that $k_n\#\psi \to \psi$ in $\mathcal{H}_\mu$, as $n \to \infty$, for $\psi \in \mathcal{H}_\mu$. Then h_μ being an automorphism of $\mathcal{H}_\mu$, and taking into account (2), we infer that

$x^{-\mu-1/2}(h_\mu k_n)(x)\phi(x) \to \phi(x)$ in $\mathcal{H}_\mu$ as $n \to \infty$, for $\phi \in \mathcal{H}_\mu$, or, in other words, that $x^{-\mu-1/2}(h_\mu k_n)(x) \to 1$ in $\mathcal{O}$, as $n \to \infty$. Hence, for each $\vartheta \in \mathcal{O}$ we have that $x^{-\mu-1/2}(h_\mu k_n)(x)\vartheta(x) \to \vartheta(x)$ in $\mathcal{O}$ as $n \to \infty$, with $(h_\mu k_n)\vartheta \in \mathcal{H}_\mu$, for every $n \in \mathbf{N}$. The proof is thus complete. □

Proposition 5. *The spaces $\mathcal{O}$ and $\operatorname{Hom}_p(\mathcal{H}_\mu)$ are homeomorphic.*

Proof. Define

$$\begin{aligned} \mathbf{H} : \mathcal{O} &\longrightarrow \operatorname{Hom}_p(\mathcal{H}_\mu) \\ \vartheta &\longmapsto \mathbf{H}(\vartheta) : \mathcal{H}_\mu \longrightarrow \mathcal{H}_\mu \\ &\qquad\qquad \psi \longmapsto \vartheta\psi. \end{aligned}$$

Clearly, $\mathbf{H}$ is one-to-one. Moreover, $\mathbf{H}$ is onto. In fact, let $F \in \operatorname{Hom}_p(\mathcal{H}_\mu)$. By virtue of Lemma 3, there exists a sequence $\{\psi_n\}_{n\in\mathbf{N}}$ in $\mathcal{H}_\mu$ such that $x^{-\mu-1/2}\psi_n(x) \to 1$ in $\mathcal{O}$, as $n \to \infty$. Then, for every $\varphi \in \mathcal{H}_\mu$, $x^{-\mu-1/2}\psi_n(x)\varphi(x) \to \varphi(x)$ in $\mathcal{H}_\mu$, as $n \to \infty$, whence $x^{-\mu-1/2}F(\psi_n)(x)\varphi(x) = F(x^{-\mu-1/2}\psi_n(x)\varphi(x)) \to F(\varphi)(x)$ in $\mathcal{H}_\mu$, as $n \to \infty$. This means that the sequence $\{x^{-\mu-1/2}F(\psi_n)\}_{n\in\mathbf{N}}$ is Cauchy in $\mathcal{O}$. Since $\mathcal{O}$ is complete [**4**, Proposition 3.2], there exists $\vartheta \in \mathcal{O}$ such that $x^{-\mu-1/2}F(\psi_n)(x) \to \vartheta(x)$ in $\mathcal{O}$, as $n \to \infty$. Hence $F(\varphi) = \vartheta\varphi$, $\varphi \in \mathcal{H}_\mu$. This proves that $\mathbf{H}$ defines an algebraic isomorphism. That this isomorphism is also topological can be seen easily. □

Concerning invertibility in the spaces $\operatorname{Hom}_p(\mathcal{H}_\mu)$ and $\mathcal{O}$, the following analogue of Proposition 4 holds.

Proposition 6. *Let $\vartheta \in \mathcal{O}$, and let F_ϑ denote the element of $\operatorname{Hom}_p(\mathcal{H}_\mu)$ defined by $F_\vartheta(\psi) = \vartheta\psi$, $\psi \in \mathcal{H}_\mu$. Then $\vartheta\mathcal{O} = \mathcal{O}$ if and only if F_ϑ is an isomorphism.*

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