ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 1, Winter 1996

MOMENT MAPS FOR TORUS ACTIONS

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ABSTRACT. A surface of revolution in $\mathbf{R}^3$ has a natural symplectic S^1 action. A moment map for this action is constructed. Also, a converse to a theorem of Marsden and Weinstein on the existence of moment maps for torus actions is derived for compact surfaces.

1. Introduction. This paper examines when a moment map exists for a symplectic torus action on a compact manifold. This is an important question because these actions are useful tools in mathematical physics and in algebraic geometry (where the compact case is of more interest; see [**3**]). Most of the results apply to compact surfaces (2-manifolds); in fact, a lot of insight is gained by considering the rather simple case of the obvious S^1 action on a surface of revolution in $\mathbf{R}^3$. The most important result is a converse (in the case of a compact surface M) to a long-known result of Marsden and Weinstein [**6**] that a moment map for a torus action on M exists if $H^1(M, \mathbf{Q}) = 0$.

All manifolds are assumed differentiable, connected, compact and orientable. Singular cohomology is denoted H^i, while de Rham cohomology is denoted H^i_{dR}. G is an abelian Lie group, usually a torus (i.e., a product of S^1's), with Lie algebra $\underline{g}$.

Here are the basic definitions (to fix notation), mostly following [**1**]. A closed nondegenerate 2-form ω on the manifold M is called a *symplectic structure* on M. A trivial consequence of this definition is that every orientable 2-manifold has a symplectic structure given by the area form dA. (In fact, if M is a compact surface, then $H^2_{dR}(M, \mathbf{Q})$ has rank 1, so ω is a multiple of dA modulo exact forms.)

When G acts on a symplectic manifold (M, ω) we will consider each $g \in G$ as a diffeomorphism $g : M \to M$ and simply write gm instead of $g(m)$. The set $G.m = \{gm : g \in G\}$ is called the *orbit* through m. A G-action on M is *symplectic* if for all $g \in G$ we have $g^*(\omega) = \omega$. For each

Received by the editors on July 18, 1994.
AMS (MOS) 1991 *Mathematics Subject Classification.* 58F05, 70H33, 55M20.
Research partially supported by the AFOSR.

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$a \in \underline{g}$ and $m \in M$, define a curve γ through m by $\gamma(t) := \exp(ta)m$. Then $a_m := \gamma'(0)$ in $T_\varepsilon(M)$ is an *infinitesimal generator* for the action.

Let $\underline{g}^*$ denote the vector space dual of $\underline{g}$. A differentiable map $\mu : M \to \underline{g}^*$ is a *moment map* if it satisfies

$$\mu(gm) = \mu(m) \tag{1}$$

for all $g \in G$ (keep in mind that G is abelian) and, for all $a \in \underline{g}$, $m \in M$, and $v \in T_m(M)$,

$$d\mu(m)(v).a = \omega(m)(v, a_m). \tag{2}$$

Here are some simple but useful results.

Proposition 1. Ker $(d\mu(m))$ *is the (symplectic) orthogonal complement to the tangent space of* $G.m$. (*Both* Ker $(d\mu(m))$ *and the tangent space of* $G.m$ *are considered as vector subspaces of* $T_m(M)$.)

Proof. Suppose $v \in \operatorname{Ker}(d\mu(m))$. Then, for all $a \in \underline{g}$, $d\mu(m).a = 0$, so $\omega(m)(v, a_m) = 0$. Of course, each a_m is in $T_m(G.m)$, so v is in the symplectic complement of $T_m(G.m)$.

Conversely, every vector tangent to the orbit $G.m$ is of the form a_m for some $a \in \underline{g}$. If $\omega(m)(v, a_m) = 0$ for all a, then $d\mu(m)(v).a = 0$ as well, so $v \in \operatorname{Ker}(d\mu(m))$. □

Proposition 2. $d\mu(m)$ *is identically* 0 *if and only if* m *is a fixed point for the* G *action.*

Proof. The point m is fixed if and only if $a_m = 0$ for all a in $\underline{g}$. Hence, at a fixed point m, $d\mu(m)(v).a = \omega(m)(v, a_m) = \omega(m)(v, 0) = 0$. Conversely, if $d\mu(m)$ is identically zero, then $\omega(m)(v, a_m) = 0$ for all $v \in T_m(M)$ and $a \in \underline{g}$. If m is not a fixed point, then there is some a for which $a_m \neq 0$; but then $\omega(m)(v, a_m) = 0$ for all $v \in T_m(M)$, which is impossible due to the nondegeneracy of ω. □

2. Surfaces of revolution. Now consider a surface of revolution $M \subset \mathbf{R}^3$, with symplectic structure given by the area form dA. Use standard cylindrical coordinates (r, θ, z) on $\mathbf{R}^3$. The rotational axis of symmetry is taken to be the z-axis, and the orbits are all in planes perpendicular to this axis. (To see that this is a special case of a surface with S^1-action, consider the meridianal flow on the torus, all of whose orbits are circles. There is no axis of symmetry perpendicular to the orbits.) There is a curve, given in cylindrical coordinates by $f(r, z) = 0$ and $\theta = 0$, such that M is the locus formed by rotating the curve around the z-axis. If M is a differentiable manifold, then f must be differentiable and $f = 0$ must have a horizontal tangent when $r = 0$.

It is easy to check from the definition that elements of $\underline{g} \cong \mathbf{R}$ correspond to infinitesimal generators at m which are in the plane perpendicular to the z-axis at m, where $m = (r, \theta, z)$ in cylindrical coordinates.

Notice that a moment map for an S^1 action can have no critical points other than fixed points. This is because the dimension of $\mathbf{R}$, the Lie algebra, is 1, so the image of μ can have rank 0 or 1. Hence, at a critical point for μ, $d\mu$ is identically zero.

There is an obvious (but incorrect!) candidate for a moment map $\mu : M \to \underline{g}^*$, namely, the map $\mu(r, \theta, z) = z$. Why is this the case? Recall that the kernel of $d\mu$ must be in the symplectic orthogonal complement of the orbit. Since $\omega = dA$ is nonzero on any pair of independent vectors in the tangent space to M, it follows that the tangent space to the orbit is its own symplectic orthogonal complement. Thus, $d\mu$ must take horizontal vectors (i.e., those perpendicular to the z-axis) to zero; thus, μ must be a function of z. If we knew that all orbits were connected, then we could see that μ must be a function of z alone since it must be constant on orbits; however, we have not made this assumption.

This is incorrect because we are only specifying the kernel of $d\mu$; μ itself also has to have certain values, according to (2). By definition, if v and w are orthogonal unit tangent vectors at $m \in M$, then $dA(v, w) = \pm 1$ (the sign is determined by the orientation). For a_m to be a unit vector, $|a| = 1/2\pi r$. Suppose $v \in T_m(M)$ is orthogonal (in the Euclidean sense) to the orbit $G.m$, and let ϕ denote the angle between v and the vertical. If $\mu = z$, $d\mu(v)$ has length $\sin(\phi)$ (as is easy to verify). Thus,

$$dz(v).a = \frac{\sin(\phi)}{2\pi r}$$

while

$$dA(v, a_m) = \pm 1.$$

Proposition 3. *Suppose the surface of revolution M is compact and simple connected. Then*

$$\mu(r_0, \theta_0, z_0) = \int \frac{2\pi r ds}{\sin(\phi)}$$

is a moment map for the S^1 action on M, where the integral is taken over the path from the minimum for z_0 to (r_0, θ_0, z_0) in the plane $\theta = \theta_0$, i.e., in a generator of the surface, and ds is arc length.

Proof. There is a minimum for z since the surface is compact; the integral is well defined since it is simply connected, and μ has the properties of a moment map by construction. □

(Related results appear in [**5**] and [**2**].)

This proposition also has an interesting geometric interpretation in the case when the generator has the form $r = f(z)$. It is a simple calculus exercise to verify that

$$\frac{r}{\sin(\phi)} = f(z)\sqrt{1 + (f'(z))^2}.$$

Suppose that the minimum value of z is a and that the maximum value is b; then the surface area of M is

$$\int_a^b 2\pi f(z)\sqrt{1 + f'(z))^2}\, dz.$$

Thus,

Corollary 4. *If $r = f(z)$, then the function $\mu(r_0, \theta_0, z_0)$ given by the area of the part of the surface with $z < z_0$ is a moment map.*

3. Torus actions on other manifolds. A moment map for a G-action restricts to a moment map for the action of a Lie subgroup $H \subset G$. Thus, nonexistence results for S^1-actions extend to nonexistence results for torus actions.

Let M be a compact manifold with a symplectic S^1 action, and suppose that there is a moment map μ for the action. Since M is compact and connected, so is $\mu(M) \subset \mathbf{R}$, so $\mu(M)$ is a closed interval. In particular, μ must have a maximum value and a minimum value. But at such points $d\mu$ is identically zero; these points are therefore fixed points for the action, by Proposition 2.

Proposition 5. *A fixed-point-free action of S^1 on a compact, connected, manifold cannot have a moment map.*

An S^1 action on a manifold M determines a vector field X on M by $X_m = 1_m$, where we consider 1 to be the generator of $\underline{g} \cong \mathbf{R}$. An isolated fixed point of the action corresponds to an isolated zero of the vector field. Any isolated zero of a vector filed has an *index*, and these are subject to the following topological constraint, proven in [**4**].

Theorem 6 (The Poincaré Hopf index theorem). *The sum over all zeros of the indices of a vector field on M with isolated zeros is the Euler characteristic of M.*

Dennis Stowe pointed out the following.

Lemma 7. *Suppose S^1 acts smoothly on a compact manifold M. Then all of the fixed points are isolated and of index* 1*.*

Proof. This is proven in [**6**], but here is a sketch. Choose local coordinates around a fixed point such that the fixed point is at the origin. Then modify the coordinates so the orbits are round circles and the action is rotation; this is possible unless the action is trivial. If the action is trivial,then a whole neighborhood of the fixed point consists of fixed points. It is trivial to check that the index of a rotation is 1. $\square$

Marsden and Weinstein showed that if $H^1(M, \mathbf{Q}) = 0$ then a torus action on M has a moment map. Here is a partial converse for surfaces.

Corollary 8. *Suppose S^1 acts symplectically on a compact surface M and that $H^1(M, \mathbf{Q}) \neq 0$ (i.e., M is a surface with genus $g \geq 1$). Then there can be no moment map for the action.*

Proof. If $H^1(M, \mathbf{Q}) \neq 0$, then g is at least 1 and χ is negative or zero. The vector field corresponding to the S^1 action must have fixed points, and by Lemma 7 these are isolated and of positive index. Furthermore, if there is a μ, then μ must have a maximum and a minimum, and every zero of the vector field must be a critical point for μ. Thus, χ must be positive, a contradiction. $\square$

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