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THE DIFFERENTIAL INVARIANTS OF PARTICLE LAGRANGIANS UNDER EQUIVALENCE BY CONTACT TRANSFORMATIONS

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ABSTRACT. We use Élie Cartan's method of equivalence to derive the structure equations of the integral $\int L(x, y^1, \dots, y^m, dy^1/dx, \dots, dy^m/dx)dx$ under the group of contact transformations for the case $m > 1$. These equations define a complete set of local differential invariants of the integral under contact transformations. We obtain a differential quadratic form and an associated system of frames which are intrinsic to $\int L\, dx$ and interpret our results from the standpoint of Finsler spaces. In the last section we explore some of the consequences of the structure equations.

1. Introduction. This paper extends results obtained by Robert Gardner and Robert Bryant in an unpublished work [**3**] on the problem of finding local differential invariants of the integral

$$\int L(x, y^1, \dots, y^m, dy^1/dx, \dots, dy^m/dx)\, dx$$

under the group of contact transformations for the case $m > 1$ with the assumption that L is a regular Lagrangian. This problem was solved by Élie Cartan for the case $m = 1$ [**4**]; S.S. Chern [**6**] found the differential invariants of the integral

$$\int L(y^1, \dots, y^m, dy^1/dx, \dots, dy^m/dx)\, dx$$

where L is positively homogeneous of degree one in the variables $p^1, \dots, p^m$.

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The tool we use is Élie Cartan's method of equivalence as described in [**9**] and [**10**]. In Section 3 we describe the intrinsic calculations of Bryant and Gardner; these calculations are supplemented by the author's parametric calculations. In addition to the structure equations which define the invariants, we obtain a complete parallelism which is intrinsically associated to $\int L\,dx$.

The differential invariants produced by the structure equations mentioned above are the primary topic of this paper. However, the equivalence method produces some other interesting invariants. For example, the Hilbert invariant differential form is produced naturally from the computations (see Section 3). As another example we will see how the equivalence method produces natural frame fields associated to $\int L\,dx$. These frame fields diagonalize a differential quadratic form, which projects onto a Finsler metric having $\int L\,dx$ as its arc length integral (see Section 4).

The structure equations which define the invariants are written out in full in Section 5. We explore some of the implications of these equations, e.g., if all the invariants are constant, then $\int L\,dx$ is the arc length integral of a pseudo-Riemannian manifold of constant curvature.

The chief results of this paper are not new; in fact, they are all classical (see [**15**]). The tool that we use, Cartan's method of equivalence, was used by Chern [**6**], and everything in this paper (structure equations, Bianchi identities, etc.), is logically a special case of [**6**]. However, the results contained in this paper were obtained independently by Bryant, Gardner, and the author. The value of the results presented here is twofold: We show the intrinsic calculations that lead to the structure equations; these calculations are omitted from [**6**] where only the parametric calculations are given. Without the intrinsic calculations serving as a guide, the parametric calculations can leave a reader somewhat mystified. Secondly, the formulation of the equivalence problem that we give and the structure equations that we derive are specifically adapted to a nonhomogeneous Lagrangian function L.

We should mention that the definition of equivalence that we give is not the natural one from the standpoint of the calculus of variations, i.e., we do not consider divergence-equivalent Lagrangians (it is a standard result [**14**] that two Lagrangians have the same Euler-Lagrange equations if and only if they are divergence-equivalent). However, our definition of equivalence is the natural one from the standpoint of control theory. The reader who is interested in the problem of divergence-equivalent Lagrangians should see the papers of Bryant [**1**] and Kamran and Olver [**12**].

2. Preliminaries. We assume all functions, curves, manifolds, maps, etc., are C^∞. Everything in this paper is local, e.g., a diffeomorphism is a map with a nowhere vanishing Jacobian determinant.

Given positive integers m and n, $GL(n)$ denotes the general linear group of nonsingular real $n \times n$ matrices and $\mathcal{M}(m, n)$ denotes the vector space of all real $m \times n$ matrices.

Let $(x, y, p) = (x, y^1, \dots, y^m, p^1, \dots, p^m)$ denote the natural coordinates on $\mathbf{R}^{2m+1}$. Given a connected open subset U of $\mathbf{R}^{2m+1}$, let

$$\theta_U = \begin{pmatrix} (dy^1 - p^1\, dx)|_U \\ \vdots \\ (dy^m - p^m\, dx)|_U \end{pmatrix}.$$

The system of equations,

$$\theta_U = 0,$$

defines a subbundle of the tangent bundle TU called the *contact system of* U. Given a closed interval $[a, b]$ and a curve $\alpha : [a, b] \to U$, we call α a 1-*graph* if its field of tangent vectors belongs to the contact system of U; equivalently,

$$\alpha^* \theta_U = 0$$

where $*$ denotes the pullback operation by a mapping on a differential form.

Let V be a second connected open subset of $\mathbf{R}^{2m+1}$, and let $\phi : U \to V$ be a diffeomorphism. We call ϕ a *contact transformation* if the derivative of ϕ maps the contact system of U onto the contact system of V. Thus, if we let

$$\theta_V = \begin{pmatrix} (dy^1 - p^1\, dx)|_V \\ \vdots \\ (dy^m - p^m\, dx)|_V \end{pmatrix},$$

then ϕ is a contact transformation if and only if there exists a function $A_1 : U \to GL(2m+1)$ such that

$$\phi^*\theta_V = A_1\theta_U. \tag{2.1}$$

Let $L(x, y, p)$ be a function on U that we assume to be positively valued at each point of U. We regard the integral $\int L\,dx$ associated to L as a functional on 1-graphs:

$$\alpha \mapsto \int_\alpha L\,dx \stackrel{\text{def}}{=} \int_a^b \alpha^*(L\,dx)$$

where $\alpha : [a, b] \to U$ is a 1-graph.

Let K be an everywhere positive function defined on V. The integral $\int L\,dx$ will be said to be *equivalent* to the integral $\int K\,dx$ provided there exists a contact transformation $\phi : U \to V$ such that

$$\int_\alpha L\,dx = \int_{\phi\circ\alpha} K\,dx \tag{2.2}$$

for *all* 1-graphs $\alpha : [a, b] \to U$; the contact transformation ϕ will be called an *equivalence*. The requirement that equation (2.2) hold for all 1-graphs is equivalent to the existence of a function $b_1 : U \to \mathcal{M}(1, m)$ such that

$$\phi^*(K\,dx) = L\,dx + b_1\theta_U. \tag{2.3}$$

Let

$$\eta_U = \begin{pmatrix} dy^1|_U \\ \vdots \\ dy^m|_U \end{pmatrix},$$

and let η_V be defined in a similar fashion. Finally, let $\omega_U = L\,dx$ and $\omega_V = K\,dx$. Then a diffeomorphism $\phi : U \to V$ is an equivalence of the integrals $\int L\,dx$ and $\int K\,dx$ if and only if there exists, in addition to the functions A_1 and b_1 already defined, functions $B : U \to \mathcal{M}(m, m)$, $b_2 : U \to \mathcal{M}(m, 1)$, and $A_2 : U \to GL(m)$ such that

$$\phi^* \begin{pmatrix} \theta_V \\ \omega_V \\ \eta_V \end{pmatrix} = \begin{pmatrix} A_1 & 0 & 0 \\ b_1 & 1 & 0 \\ B & b_2 & A_2 \end{pmatrix} \begin{pmatrix} \theta_U \\ \omega_U \\ \eta_U \end{pmatrix}. \tag{2.4}$$

If we let G be the group of all matrices of the form

$$\begin{pmatrix} A_1 & 0 & 0 \\ b_1 & 1 & 0 \\ B & b_2 & A_2 \end{pmatrix}$$

where $A_1,\ A_2 \in GL(m)$, $B \in \mathcal{M}(m,m)$, $b_1 \in \mathcal{M}(1,m)$, and $b_2 \in \mathcal{M}(m,1)$, then equation (2.4) is an equivalence problem of Élie Cartan with G as the prescribed linear group (see [**10**, p. 1]).

Finally, we observe that if

$$\bar{x} = \bar{x}(x,y,p), \qquad \bar{y} = \bar{y}(x,y,p), \qquad \bar{p} = \bar{p}(x,y,p),$$

are the equations of an equivalence ϕ, then equations (2.1) and (2.3) imply that the functions $\bar{x}$ and $\bar{y}$ do not involve p and thus define a point transformation. Conversely, it is not difficult to see that any point transformation

$$\bar{x} = \bar{x}(x,y), \qquad \bar{y} = \bar{y}(x,y),$$

extends uniquely to a contact transformation, i.e., to a transformation satisfying equation (2.1); the contact transformations that are determined by point transformations are also called point transformations (although some authors prefer to call them extended point transformations). As not every contact transformation is a point transformation (equation (2.1) alone does not suffice to eliminate p from the equations for $\bar{x}$ and $\bar{y}$), we can more precisely define an equivalence between the integrals $\int L\,dx$ and $\int K\,dx$ as a point transformation ϕ satisfying equation (2.3). However, having said all this, equation (2.4) will be our operational definition of an equivalence.

3. Application of the method of equivalence. At this point we apply the method of equivalence algorithm with which we assume the reader is familiar; this algorithm is described in [**9**] and [**10**].

Evidently, the Lie algebra $\mathcal{G}$ of G consists of all matrices of the form

$$\begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & * & * \end{pmatrix}$$

where the $*$'s denote arbitrary matrices of the appropriate dimensions. We regard $U \times G \to U$ (natural projection) as a local principal bundle with G as the structure group. The action of G on $U \times G$ is the natural left action.

The canonical $\mathbf{R}^{2m+1}$-valued 1-form on $U \times G$ may be represented as a column vector:

$$\begin{pmatrix} \theta \\ \omega \\ \eta \end{pmatrix}$$

where

$$\theta = \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^m \end{pmatrix}, \qquad \eta = \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^m \end{pmatrix},$$

and $\theta^1, \dots, \theta^m, \omega, \eta^1, \dots, \eta^m$ are scalar 1-forms. By definition,

$$\begin{pmatrix} \theta \\ \omega \\ \eta \end{pmatrix} = \begin{pmatrix} A_1 & 0 & 0 \\ b_1 & 1 & 0 \\ B & b_2 & A_2 \end{pmatrix} \begin{pmatrix} \theta_U \\ \omega_U \\ \eta_U \end{pmatrix}. \tag{3.1}$$

We may write

$$d \begin{pmatrix} \theta \\ \omega \\ \eta \end{pmatrix} = \begin{pmatrix} \pi_1 & \alpha_1 & \alpha_2 \\ \beta_1 & \alpha_3 & \alpha_4 \\ \beta & \beta_2 & \pi_2 \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \omega \\ \eta \end{pmatrix} \tag{3.2}$$

where $\pi_1, \alpha_1, \dots, \pi_2$ are matrices (of the appropriate dimensions) of 1-forms. The form of the matrices belonging to $\mathcal{G}$ implies

$$\alpha_1, \alpha_2, \alpha_2, \alpha_4 \equiv 0 \quad \mathrm{mod}\{\theta, \omega, \eta\}. \tag{3.3}$$

Thus, $\alpha_1, \dots, \alpha_4$ are principal components. It is important to observe that we can infer the definition of $\mathcal{G}$ from equation (3.3).

Let us consider the first of equations (3.2):

$$d\theta = \pi_1 \wedge \theta + \alpha_1 \wedge \omega + \alpha_2 \wedge \eta. \tag{3.4}$$

Equation (3.3) tells us that the terms $\alpha_1 \wedge \omega$ and $\alpha_2 \wedge \eta$ are quadratic in θ, ω and η. The terms that have θ as a factor can be absorbed into the

$\pi_1 \wedge \theta$ term; this term will still have the form $\pi_1 \wedge \theta$ where π_1 has been modified by the addition of θ terms. Having made this absorption, and because $\omega \wedge \omega = 0$, we may assume

$$\alpha_1 \equiv 0 \bmod \{\eta\} \qquad \alpha_2 \equiv 0 \mod\{\omega, \eta\}. \tag{3.5}$$

The definition of θ and ω implies

$$\theta, \omega \equiv 0 \mod\{dx, dy\}.$$

Thus, the differential system $\theta = \omega = 0$ is completely integrable and so equation (3.4) cannot have any terms quadratic in η. Therefore we may assume

$$\alpha_2 \equiv 0 \mod\{\omega\}. \tag{3.6}$$

We conclude from equations (3.5) and (3.6) that equation (3.4) can be written as

$$d\theta = \pi_1 \wedge \theta + A\eta \wedge \omega \tag{3.7}$$

where $A = (A^i_j) : U \times G \to \mathcal{M}(m, m)$.

If we subject the second of equations (3.2) to a similar treatment we obtain the following structure equations:

$$d \begin{pmatrix} \theta \\ \omega \\ \eta \end{pmatrix} = \begin{pmatrix} \pi_1 & 0 & 0 \\ \beta_1 & 0 & 0 \\ \beta & \beta_2 & \pi_2 \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \omega \\ \eta \end{pmatrix} + \begin{pmatrix} A\eta \wedge \omega \\ b\eta \wedge \omega \\ 0 \end{pmatrix} \tag{3.8}$$

where $b : U \times G \to \mathcal{M}(1, m)$. These equations uniquely determine A and b as we will show explicitly for the case of A; b is handled in a similar manner.

Suppose

$$d\theta = \bar{\pi}_1 \wedge \theta + \bar{A}\eta \wedge \omega \tag{3.9}$$

where $\bar{A} = (\bar{A}^i_j)$ and

$$\bar{\pi}_1 \equiv \pi_1 \mod\{\theta, \omega, \eta\}. \tag{3.10}$$

At this point we write everything using indices where the range on all indices is from 1 to m. Equation (3.10) is equivalent to

$$(\bar{\pi}_1)^i_k = (\pi_1)^i_k + a^i_{jk}\theta^j + c^i_k\omega + e^i_{jk}\eta^j \tag{3.11}$$

where the a's, c's and e's are functions (for the rest of this paper it will be understood that in any indicial expression, an index which appears an *even* number of times in a term after all symbols of inclusion have been removed is summed; otherwise no summation is implied). We substitute equation (3.11) into equation (3.9):

$$\begin{aligned} d\theta^i &= ((\pi_1)^i_k + a^i_{jk}\theta^j + c^i_k\omega + e^i_{jk}\eta^j) \wedge \theta^k + \bar{A}^i_j\eta^j \wedge \omega \\ &= (\pi_1)^i_k \wedge \theta^k + \frac{1}{2}(a^i_{jk} - a^i_{kj})\theta^j \wedge \theta^k \\ &\quad + c^i_k\omega \wedge \theta^k + e^i_{jk}\eta^j \wedge \theta^k + \bar{A}^i_j\eta^j \wedge \omega. \end{aligned} \tag{3.12}$$

Comparing equation (3.12) with equation (3.7), which indicially is

$$d\theta^i = (\pi_1)^i_k \wedge \theta^k + A^i_j\eta^j \wedge \omega,$$

we conclude that $a^i_{jk} = a^i_{kj}$, $c^i_k = 0$, $e^i_{jk} = 0$, and $\bar{A} = A$. Note that π_1 is not uniquely determined: we may add to the terms $(\pi_1)^i_k$ terms of the form $a^i_{jk}\theta^j$, where the a^i_{jk} are symmetric in the indices j and k.

We now have a well-defined structure function

$$(A, b) : U \times G \to \mathcal{M}(m, m) \times \mathcal{M}(1, m).$$

Differentiating the structure equations (3.8) we obtain the equations

$$\left.\begin{aligned} dA &\equiv \pi_1 A - A\pi_2 \\ db &\equiv \beta_1 A - b\pi_2 \end{aligned}\right\} \quad \operatorname{mod}\{\theta, \omega, \eta\}. \tag{3.13}$$

Let I denote the $m \times m$ identity matrix and set $(A, b) = (\pm I, 0)$. Then equations (3.13) become

$$\left.\begin{aligned} \pi_1 - \pi_2 &\equiv 0 \\ \beta_1 &\equiv 0 \end{aligned}\right\} \quad \operatorname{mod}\{\theta, \omega, \eta\}. \tag{3.14}$$

To compute A parametrically, we consider the first of equations (3.1):

$$\theta = A_1 \theta_U. \tag{3.15}$$

Differentiating equation (3.15) gives

$$\begin{aligned} d\theta &= dA_1 \wedge \theta_U + A_1\, d\theta_U \\ &= dA_1 \wedge \theta_U - \frac{1}{L} A_1 \eta_U \wedge \omega_U. \end{aligned} \tag{3.16}$$

We invert equations (3.1) to express the righthand side of equation (3.16) in terms of θ, ω, and η; and then, we perform the same absorptions that led to equation (3.7). We may then read A from $\eta \wedge \omega$ term. We note that when doing the calculation, any terms involving θ may be ignored since equation (3.7) implies

$$d\theta \equiv A\eta \wedge \omega \quad \mathrm{mod}\{\theta\}.$$

The parametric equation for A is

$$A = -\frac{1}{L} A_1 A_2^{-1}. \tag{3.17}$$

It is worth noting that differentiation of equation (3.17) yields

$$dA = (dA_1 A_1^{-1})A - A(dA_2 A_2^{-1}) \quad \mathrm{mod}\{\theta, \omega, \eta\}$$

since L is constant $\mathrm{mod}\ \{\theta, \omega, \eta\}$ (compare to the first of equations (3.13)). It is clear now that the equation $A = \pm I$ has solutions since it is equivalent to $A_2 = \mp(1/L)A_1$ and we can let A_1 be any nonsingular $m \times m$ matrix. As a matter of convenience (to eliminate the $-$ sign), we set $A = -I$; then

$$A_2 = \frac{1}{L} A_1.$$

A computation similar to the one used to compute A parametrically yields

$$b = \frac{1}{L}(L_p - b_1)A_2^{-1}$$

where

$$L_p = (L_{p^1}, \ldots, L_{p^m}) \in \mathcal{M}(1, m)$$

is the matrix of partial derivatives of $L(x, y, p)$ with respect to $p^1, \dots, p^m$, respectively. The equation $b = 0$ has the unique solution

$$b_1 = L_p.$$

We return to the intrinsic calculations. Let $\mathcal{F}_1(U)$ denote the submanifold of $U \times G$ defined by the equation

$$(A, b) = (-I, 0).$$

To compute the new structure function we restrict the structure equations (3.8) to $\mathcal{F}_1(U)$ obtaining

$$d\begin{pmatrix}\theta\\ \omega\\ \eta\end{pmatrix} = \begin{pmatrix}\pi_1 & 0 & 0\\ \beta_1 & 0 & 0\\ \beta & \beta_2 & \pi_2\end{pmatrix} \wedge \begin{pmatrix}\theta\\ \omega\\ \eta\end{pmatrix} + \begin{pmatrix}-\eta\wedge\omega\\ 0\\ 0\end{pmatrix}. \tag{3.18}$$

We have new principal components $\pi_1 - \pi_2$ and β_1, which define a Lie subalgebra $\mathcal{G}_1$ of $\mathcal{G}$ with corresponding group $G_1 \subset G$.

If we differentiate the first of equations (3.18), we get

$$(\pi_2 - \pi_1) \wedge \eta \wedge \omega \equiv 0 \quad \mathrm{mod}\{\theta\},$$

and thus,

$$\pi_2 \wedge \eta \equiv \pi_1 \wedge \eta \quad \mathrm{mod}\{\theta, \omega\}.$$

Since quadratic terms which are congruent to zero modulo $\{\theta, \omega\}$ can be absorbed into $\beta \wedge \theta + \beta_2 \wedge \omega$, the third of equations (3.18) can be expressed as

$$d\eta = \beta \wedge \theta + \beta_2 \wedge \omega + \pi_1 \wedge \eta$$

where β and β_2 have been modified appropriately. Differentiating the second of the equations (3.18) gives

$$\beta_1 \wedge \eta \wedge \omega \equiv 0 \quad \mathrm{mod}\{\theta\}$$

and hence,

$$\beta_1 \wedge \eta \equiv 0 \quad \mathrm{mod}\{\theta, \omega\}. \tag{3.19}$$

As β_1 is a principal component (equations (3.14)), there exists $\rho \in \mathcal{M}(1,m)$ and C, $H \in \mathcal{M}(m,m)$ (ρ, C and H are matrices of functions) such that

$$\beta_1 = \omega\rho + {}^t\theta C + {}^t\eta H. \tag{3.20}$$

Substituting equation (3.20) into the second of equations (3.18) gives

$$d\omega = \omega\rho\theta + {}^t\theta C\theta + {}^t\eta H\theta \tag{3.21}$$

(the implied products of differential forms are $\wedge$ products), and it is evident that we may assume that C is a skew symmetric matrix. The matrices ρ, C and H are now uniquely determined by equation (3.21). Note that equation (3.19) implies that H must be a symmetric matrix. We now have new structure equations

$$d\begin{pmatrix}\theta\\ \omega\\ \eta\end{pmatrix} = \begin{pmatrix}\pi & 0 & 0\\ 0 & 0 & 0\\ \beta & \beta_2 & \pi\end{pmatrix} \wedge \begin{pmatrix}\theta\\ \omega\\ \eta\end{pmatrix} + \begin{pmatrix}-\eta\wedge\omega\\ \omega\rho\theta + {}^t\theta C\theta + {}^t\eta H\theta\\ 0\end{pmatrix}, \tag{3.22}$$

where $\pi = \pi_1$, and a new structure function (ρ, C, H).

Before investigating the fiber variation of the new structure function we observe that the method of equivalence has given us an interesting differential invariant, namely ω. To see that ω is invariant, let X be a vertical vector field, then

$$X \lrcorner\, \omega = 0 \quad \text{by equation (3.1)},$$

and

$$X \lrcorner\, d\omega = 0 \quad \text{by equations (3.21) and (3.1)};$$

thus,

$$\mathcal{L}_X\omega = d(X \lrcorner\, \omega) + X \lrcorner\, d\omega = 0$$

where $\mathcal{L}_X$ denotes Lie differentiation with respect to X and $\lrcorner$ denotes the interior product. Hence, ω drops down to U and is preserved by equivalences $\phi : U \to V$. If we compute ω parametrically in terms of the frame field ${}^t(\theta_U, \omega_U, \eta_U)$, we get

$$\omega = L\,dx + L_{p^i}(dy^i - p^i\,dx),$$

which the reader may recognize as Hilbert's invariant differential form.

Differentiating equation (3.21), we obtain the equations

$$\text{(3.23)}\qquad \left.\begin{aligned} d\rho &\equiv -\rho\pi - {}^t\beta_2 H \\ dC &\equiv -({}^t\pi C + C\pi) - \frac{1}{2}({}^t\beta H - H\beta) \\ dH &\equiv -({}^t\pi H + H\pi) \end{aligned}\right\} \quad \operatorname{mod}\{\theta, \omega, \eta\}.$$

The third of equations (3.23) tells us that H is an invariant piece of the structure function; in fact, the righthand side of this equation is the differential, modulo base variables, of ${}^tA^{-1}HA^{-1}$ where A denotes a variable element of $GL(m)$. Thus, by Sylvester's law of inertia, the orbit of H under this action by $GL(m)$ consists of all symmetric matrices having the same signature as H. We can make all of equations (3.23) independent by choosing H to be a nonsingular symmetric matrix and letting $C = 0$ and $\rho = 0$. We set $H = Q$ where Q is a diagonal matrix with ± 1's on the diagonal.

A parametric calculation in terms of the coframe ${}^t(\theta_U, \omega_U, \eta_U)$ gives

$$H = {}^tA_1^{-1}(LL_{pp})A_1^{-1}$$

where $L_{pp} = (L_{p^ip^j})$ is an $m \times m$ matrix. We see that our hypothesis, $\det H \neq 0$, is equivalent to the regular problem in the calculus of variations. If we assume that Q has the same signature as L_{pp}, then the equation $H = Q$ has solutions $A_1 \in GL(m)$. The parametric equation for ρ is

$$\rho = \frac{1}{L}\left(\frac{d}{dx}(L_p) - L_y\right)A_1^{-1} - {}^tb_2H$$

where

$$\begin{aligned} L_y &= (L_{y^1}, \ldots, L_{y^m}) \in \mathcal{M}(1, m), \\ \frac{d}{dx}(L_p) &= \left(\frac{d}{dx}(L_{p^1}), \ldots, \frac{d}{dx}(L_{p^m})\right) \in \mathcal{M}(1, m), \end{aligned}$$

and

$$\frac{d}{dx} = \frac{\partial}{\partial x} + p^i \frac{\partial}{\partial y^i}$$

is the "total derivative" operator. Hence, the normalizations $H = Q$ and $\rho = 0$ yield

$$^{t}b_2 = \frac{1}{L}\left(\frac{d}{dx}(L_p) - L_y\right)A_1^{-1}Q.$$

We will not write out the full equation for C, which is rather complicated. Suffice it to say that the normalization $(\rho, C, H) = (0, 0, Q)$ gives

$$^{t}(BA_1^{-1})Q - Q(BA_1^{-1}) = {}^{t}A_1^{-1}(L_{yp} - L_{py})A_1^{-1}$$

where

$$L_{yp} = (L_{y^i p^j}) \quad \text{and} \quad L_{py} = {}^{t}L_{yp}$$

are $m \times m$ matrices.

Let $\mathcal{F}_2(U)$ be the submanifold of $\mathcal{F}_1(U)$ defined by

$$(\rho, C, H) = (0, 0, Q).$$

Restrict the structure equations (3.22) to $\mathcal{F}_2(U)$. We then have new principal components

$$\beta_2, \qquad {}^{t}\beta Q - Q\beta, \qquad {}^{t}\pi Q + Q\pi;$$

new Lie algebra $\mathcal{G}_2 \subset \mathcal{G}_1$; and new group $G_2 \subset G_1$. Let us consider first the principal component ${}^{t}\pi Q + Q\pi$. Indicially, we may write

$$Q_{ik}\pi_j^k + Q_{jk}\pi_i^k = g_{kj}^i\theta^k + M_j^i\omega + S_{ijk}\eta^k. \tag{3.24}$$

Note that

$$g_{kj}^i = g_{ki}^j, \qquad M_j^i = M_i^j, \tag{3.25}$$

and

$$S_{ijk} = S_{jik}. \tag{3.26}$$

We can modify π to eliminate the θ terms in equation (3.24): Set

$$f_{jk}^i = \frac{1}{2}Q_{ii}(g_{kj}^i + g_{ji}^k - g_{ik}^j)$$

(note that i is *not* summed) and

$$\bar{\pi}^i_j = \pi^i_j - f^i_{kj}\theta^k,$$

then $f^i_{jk} = f^i_{kj}$ so that $\bar{\pi} \wedge \theta = \pi \wedge \theta$ and

$$Q_{ik}\bar{\pi}^k_j + Q_{jk}\bar{\pi}^k_i = M^i_j\omega + S_{ijk}\eta^k. \tag{3.27}$$

Now let $\pi = \bar{\pi}$. We note that π also appears in the third structure equation (3.22); however, we may absorb the θ modification of π into the $\beta \wedge \theta$ terms and this will leave the form of the third equation unchanged.

Now consider the principal component β_2. It is clear from the third structure equation (3.22) that we may assume that β_2 has no ω term, and the θ term may be absorbed into the $\beta \wedge \theta$ term. Thus we may assume that

$$\beta_2 = A\eta \tag{3.28}$$

where $A = (A^i_j)$ is an $m \times m$ matrix. Differentiating the second structure equation (3.22) we get

$$(Q_{kj}\beta^k_i \wedge \theta^i + Q_{kj}\beta^k_2 \wedge \omega + (Q_{ik}\pi^k_j + Q_{jk}\pi^k_i) \wedge \eta^i) \wedge \theta^j = 0,$$

which implies

$$Q_{kj}\beta^k_2 \wedge \omega + (Q_{ik}\pi^k_j + Q_{jk}\pi^k_i) \wedge \eta^i \equiv 0 \quad \mod\{\theta\}. \tag{3.29}$$

Substituting equations (3.27) and (3.28) into equation (3.29) gives

$$M \stackrel{\text{def}}{=} (M^i_j) = QA \quad \Rightarrow \quad \beta_2 = QM\eta,$$

and

$$S_{ijk} = S_{ikj}. \tag{3.30}$$

Comparing equations (3.30) and (3.26) we see that S_{ijk} is symmetric in all three indices.

We now compute dM modulo the base variables. Differentiating the third structure equation (3.22) and recalling that $\beta_2 = QM\eta$ we obtain

$$\pi\wedge\pi - d\pi + \beta\wedge\omega - \pi QM\omega + QM\pi\omega + Q\,dM\wedge\omega \equiv 0 \mod\{\theta,\eta\}. \tag{3.31}$$

However, differentiation of the first structure equation (3.22) gives

$$d\pi - \pi\wedge\pi + \beta\wedge\omega \equiv 0 \mod\{\theta,\eta\}. \tag{3.32}$$

Combining equations (3.31) and (3.32), we get

$$2\beta\wedge\omega + Q\,dM\wedge\omega + QM\pi\wedge\omega - \pi QM\omega \equiv 0 \mod\{\theta,\eta\},$$

and thus,

$$dM \equiv -M\pi + Q\pi QM - 2Q\beta \mod\{\theta,\omega,\eta\}. \tag{3.33}$$

We make the normalization

$$M = 0,$$

which reduces equation (3.33) to

$$\beta \equiv 0 \mod\{\theta,\omega,\eta\}.$$

The parametric calculation shows that the equation $M = 0$ is equivalent to

$$B = \frac{1}{2}Q^t A_1^{-1}((LL_{pp})_w + L_{py} - L_{yp}),$$

where $(LL_{pp})_\omega$ denotes the $m\times m$ matrix whose ith row and jth column entry is

$$\frac{1}{L}\frac{d}{dx}((LL_{pp})^{-1})_{kl}\left(\frac{d}{dx}L_{p^k} - L_{y^k}\right)(LL_{p^ip^j})_{p^l}.$$

Let $\mathcal{F}_3(U)$ be the submanifold of $\mathcal{F}_2(U)$ defined by the equation

$$M = 0.$$

We restrict the structure equations (3.22) to $\mathcal{F}_3(U)$ and obtain the new structure equations

$$d\begin{pmatrix}\theta\\ \omega\\ \eta\end{pmatrix} = \begin{pmatrix}\pi & 0 & 0\\ 0 & 0 & 0\\ \beta & 0 & \pi\end{pmatrix}\wedge\begin{pmatrix}\theta\\ \omega\\ \eta\end{pmatrix} + \begin{pmatrix}-\eta\wedge\omega\\ {}^t\eta Q\theta\\ 0\end{pmatrix} \tag{3.34}$$

with new principal components

$$ {}^t\pi Q + Q\pi = (S_{ijk}\eta^k), \quad \beta; $$

new Lie algebra $\mathcal{G}_3 \subseteq \mathcal{G}_2$; and new group $G_3 \subset G_2$. Differentiating the second structure equation (3.34), we get

$$ \begin{aligned} 0 &= d^2\omega \\ &= {}^t(\beta\wedge\theta + \pi\wedge\eta)Q\theta - {}^t\eta Q\pi\theta \\ &= -{}^t\theta\,{}^t\beta Q\theta - {}^t\eta({}^t\pi Q + Q\pi)\theta && (3.35) \\ &= -{}^t\theta\,{}^t\beta Q\theta && (3.36) \\ &= {}^t\theta Q\beta\theta. && (3.37) \end{aligned} $$

In going from equation (3.35) to (3.36), we used the symmetry of S_{ijk} in the indices i and k; also, in a number of instances we used the following observation: Let Δ be a matrix of p-forms and Γ be a matrix of q-forms such that the product $\Delta\Gamma$ is defined. Then

$$ {}^t(\Delta\Gamma) = (-1)^{pq}\,{}^t\Gamma\,{}^t\Delta. $$

As β is a principal component, we may write

$$ (Q\beta)_{ij} = B_{ijk}\theta^k + S_{ij}\omega + T_{ijk}\eta^k. $$

The following identities are then implied by equation (3.37):

$$ B_{ijk} - B_{jik} + B_{jki} - B_{kji} + B_{kij} - B_{ikj} = 0, \qquad (3.38) $$

$$ S_{ij} = S_{ji}, $$

and

$$ T_{ijk} = T_{jik}. $$

Note that β only appears in the $\beta\wedge\theta$ term of the third structure equation (3.34). Thus the B_{ijk} are not well defined; however, we may make the B_{ijk} unique by demanding that β satisfy the third structure equation and the equation

$$ {}^t\beta Q + Q\beta \equiv 0 \quad \operatorname{mod}\{\omega,\eta\}. $$

The B_{ijk} are now well defined and satisfy $B_{ijk} = -B_{jik}$, and this identity reduces (3.38) to

$$B_{ijk} + B_{jki} + B_{kij} = 0.$$

The matrix π has the unique decomposition

$$\pi = \psi + \varphi$$

where

$$(3.39) \qquad {}^t\psi Q - Q\psi = 0, \qquad {}^t\varphi Q + Q\varphi = 0,$$

i.e., $Q\psi$ and $Q\varphi$ are, respectively, the symmetric and skew symmetric parts of the matrix $Q\pi$. Thus,

$$ {}^t\pi Q + Q\pi = 2\,{}^t\psi Q \quad \Longrightarrow \quad \psi^i_j = \frac{1}{2} Q_{ii} S_{ijk} \eta^k$$

(in the latter equation the i is not summed, but the k is) and hence, $\psi \wedge \eta = 0$ since S_{ijk} is symmetric in j and k. Therefore, we may write

$$\pi \wedge \eta = \varphi \wedge \eta.$$

We may now write the structure equations (3.34) in the form:

$$(3.40) \qquad d\begin{pmatrix} \theta \\ \omega \\ \eta \end{pmatrix} = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varphi \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \omega \\ \eta \end{pmatrix} + \begin{pmatrix} \psi \wedge \theta - \eta \wedge \omega \\ {}^t\eta Q \theta \\ \beta \wedge \theta \end{pmatrix},$$

where it is understood that φ satisfies equation (3.39). Notice that

$$\psi \equiv 0 \bmod \{\eta\} \Longrightarrow d\theta = \varphi \wedge \theta \quad \bmod\{\eta\};$$

this equation and equation (3.39) uniquely determine φ. The system of 1-forms θ, ω, η, and φ define a complete parallelism on $\mathcal{F}_3(U)$, which is a (local) generalization of the Levi-Civita parallelism (we will see later that the Levi-Civita case occurs when all the S_{ijk} vanish identically). This completes the intrinsic calculations of Gardner and Bryant. The parametric calculations were the author's.

4. The fundamental form and the Finsler metric. In this section we relate the results of our calculations to Finsler metrics. The reader who wants more information on Finsler metrics and geometry should consult [**15**]. The bibliography in [**15**] has an extensive list of the classical and foundational treatises on Finsler geometry.

Let $O(Q)$ be the group of all $S \in GL(m)$ such that ${}^tSQS = Q$, and let $o(Q)$ denote the Lie algebra of $O(Q)$. It is clear from equation (3.39) that $\varphi \in o(Q)$. Thus, G_3 is the group of all matrices

$$\begin{pmatrix} S & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S \end{pmatrix}$$

where $S \in O(Q)$ (for the rest of this section S will denote an element of $O(Q)$). Clearly, G_3 is isomorphic to $O(Q)$ and we will identify the two groups.

On $\mathcal{F}_3(U)$ we consider the differential quadratic form

$$({}^t\theta Q\theta) + (\omega^2) + ({}^t\eta Q\eta)$$

(where the parentheses denote the symmetric product of 1-forms), which we will call the *fundamental form* of $\int L\,dx$. As θ, ω and η are invariants of $\int L\,dx$ and Q is a constant matrix, the fundamental form is an invariant of $\int L\,dx$. We can use the parametric calculations of Section 3 to compute column vectors of 1-forms on U,

$$\alpha = {}^t(\alpha^1, \dots, \alpha^m) \quad \text{and} \quad \beta = {}^t(\beta^1, \dots, \beta^m),$$

such that (α, ω, β) is an "orthonormal frame" i.e.,

$$({}^t\alpha Q\alpha) + (\omega^2) + ({}^t\beta Q\beta)$$

equals the fundamental form (this will show that the fundamental form drops to U). We do this as follows: Recall from the parametric calculations that

$$H = {}^tA_1^{-1}(LL_{pp})A_1^{-1},$$

and we made the normalization $H = Q$. Thus, A_1 is restricted by the equation

$$(4.1) \qquad {}^tA_1QA_1 = LL_{pp}.$$

Let $A_1 = P = P(x, y, p) : U \to GL(m)$ be a solution to equation (4.1). Then the general solution to (4.1) is easily seen to be

$$A_1(x, y, p, S) = SP(x, y, p) : U \times O(Q) \to GL(m).$$

Recall $\theta = A_1\theta_U$. Thus,

$$\theta = S\alpha \quad \text{where} \quad \alpha \stackrel{\text{def}}{=} P\theta_U.$$

Evidently, $({}^t\theta Q\theta) = ({}^t\alpha Q\alpha)$; it follows that $({}^t\theta Q\theta)$ drops to U. Recalling from Section 3 the equations giving A_2, b_2 and B in terms of L, its partial derivatives, and A_1, we easily see that

$$A_2 = S(P/L), \tag{4.2}$$

$${}^tb_2 = \left(\frac{1}{L}\left(\frac{d}{dx}(L_p) - L_y\right)P^{-1}Q\right){}^tS, \tag{4.3}$$

and

$$B = S\left(\frac{1}{2}Q^tP^{-1}((LL_{pp})_\omega + L_{py} - L_{yp})\right). \tag{4.4}$$

Parametrically, we have (cf. equation (3.1))

$$\eta = B\theta_U + b_2\omega_U + A_2\eta_U;$$

it is clear from equations (4.2), (4.3) and (4.4) how we should define β so that $\eta = S\beta$. Then $({}^t\eta Q\eta) = ({}^t\beta Q\beta)$; hence, $({}^t\eta Q\eta)$ drops to U. Thus,

$$({}^t\alpha Q\alpha) + (\omega^2) + ({}^t\beta Q\beta) = ({}^t\theta Q\theta) + (\omega^2) + ({}^t\eta Q\eta).$$

It is now clear that a diffeomorphism $\phi : U \to V$ is an equivalence of the integrals $\int L\,dx$ and $\int K\,dx$ if and only if the Jacobian matrix of ϕ with respect to a choice of orthonormal frame on each of U and V has the form:

$$\begin{pmatrix} S & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S \end{pmatrix},$$

where $S = S(x, y, p) : U \to O(Q)$. We have the following

Theorem 4.1. *A diffeomorphism $\phi : U \to V$ is an equivalence of the integral $\int L\,dx$ and $\int K\,dx$ if and only if it preserves the fundamental form.*

Recall that $\int L\,dx$ is being regarded as a functional on 1-graphs $\alpha : [a, b] \to U$. Generally, one wants to consider only those 1-graphs α such that $\alpha^*(dx) \neq 0$. In this case we may assume that α has the form

$$\alpha(x) = (x, y(x), y'(x));$$

we then call α the 1-graph of the function $y(x)$. Let U_1 be the image of U under the natural projection

$$(x, y, p) \mapsto (x, y) : \mathbf{R}^{2m+1} \to \mathbf{R}^{m+1}.$$

Each $(x, y, p) \in U$ determines a vector,

$$X = \frac{\partial}{\partial x}\bigg|_{(x,y)} + p^i \frac{\partial}{\partial y^i}\bigg|_{(x,y)},$$

tangent to U_1 at (x, y). Define the *length* of X, $||X||$, by

$$||X|| = L(x, y, p).$$

If

$$Y = \dot{x}\frac{\partial}{\partial x}\bigg|_{(x,y)} + \dot{y}^i \frac{\partial}{\partial y^i}\bigg|_{(x,y)}$$

where $\dot{x} \neq 0$, then homogeneity requires that we define $||Y||$ by

$$||Y|| = |\dot{x}|L(x, y, \dot{y}/\dot{x}) = F(x, y, \dot{x}, \dot{y});$$

if $\dot{x} = 0$, then $||Y||$ is undefined (the case $\dot{x} \neq 0$ covers all vectors which are tangent to curves $y = y(x)$). The function F is positively homogeneous of degree one and thus defines a Finsler metric on U_1 having $\int L\,dx$ as its associated arc length integral; the corresponding differential quadratic form is

$$ds^2 = a(dx^2) + 2b_j(dx dy^j) + c_{ij}(dy^i dy^j) \tag{4.5}$$

where

$$a = \frac{1}{2}\frac{\partial^2 F^2}{\partial \dot{x}^2} = (L - p^i L_{p^i})^2 + p^i p^j L L_{p^i p^j},$$
$$b_j = \frac{1}{2}\frac{\partial^2 F^2}{\partial \dot{x}\partial \dot{y}^j} = L L_{p^j} - p^i L_{p^i} L_{p^j} - p^i L L_{p^i p^j},$$

and

$$c_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial \dot{y}^i \partial \dot{y}^j} = L_{p^i} L_{p^j} + L L_{p^i p^j}.$$

Proposition 4.2. $ds^2 = ({}^t\theta Q \theta) + (\omega^2)$.

Proof. Recall from the parametric calculations that $\theta = S\alpha$ where $\alpha = P(x,y,p)\theta_U$ and $A_1 = P(x,y,p)$ is a solution to equation (4.1). Thus,

$$({}^t\theta Q\theta) = ({}^t\alpha Q \alpha) = ({}^t\theta_U {}^t P Q P \theta_U) = ({}^t\theta_U L L_{pp} \theta_U).$$

Recalling that $\theta_U^i = dy^i - p^i\, dx$ and $\omega = L\, dx + L_{p^i}(dy^i - p^i\, dx)$ and using the equations above for a, b_j and c_{ij}, a direct calculation shows that $ds^2 = ({}^t\theta_U L L_{pp} \theta_U) + (\omega^2)$. □

5. Structure equations and Bianchi identities. A full set of structure equations on $\mathcal{F}_3(U)$ consists of equations (3.40) and the equation for $d\varphi$. To emphasize the invariants, S_{ijk}, T_{ijk}, etc., we will write the structure equations in indicial form. Recall that φ is a matrix valued 1-form; therefore, let $\varphi = (\varphi^i_j)$. Then equation (3.39) is equivalent to

$$Q_{ii}\varphi^i_j + Q_{jj}\varphi^j_i = 0 \quad \text{where} \quad 1 \le i, j \le m;$$

i.e., the tensor $Q_{ii}\varphi^i_j$ is skew symmetric in i and j (note that i is not summed in $Q_{ii}\varphi^i_j$; see the parenthetical remark following equation (3.11)). The structure equations are

$$d\theta^i = \varphi^i_j \wedge \theta^j + \frac{1}{2} Q_{ii} S_{ijk} \eta^k \wedge \theta^j - \eta^i \wedge \omega, \tag{5.1}$$

$$d\omega = Q_{jj}\eta^j \wedge \theta^j, \tag{5.2}$$

$$(5.3)\qquad \begin{aligned} d\eta^i &= \varphi^i_j \wedge \eta^j + \frac{1}{2} Q_{ii}(B_{ijk} - B_{ikj})\theta^k \wedge \theta^j \\ &\quad + Q_{ii} S_{ij} \omega \wedge \theta^j + Q_{ii} T_{ijk} \eta^k \wedge \theta^j, \end{aligned}$$

and

$$(5.4)\qquad \begin{aligned} d\varphi^i_j &= \varphi^i_k \wedge \varphi^k_j \\ &\quad + \frac{1}{2} Q_{rr} a^{is}_{jrk} \varphi^r_s \wedge \theta^k + \frac{1}{2} Q_{rr} b^{is}_{jr} \varphi^r_s \wedge \omega \\ &\quad + \frac{1}{2} Q_{rr} c^{is}_{jrk} \varphi^r_s \wedge \eta^k \\ &\quad + \frac{1}{2} A^i_{jlk} \theta^l \wedge \theta^k + B^i_{jl} \theta^l \wedge \omega + C^i_{jlk} \theta^l \wedge \eta^k \\ &\quad + D^i_{jl} \eta^l \wedge \omega + \frac{1}{2} E^i_{jlk} \eta^l \wedge \eta^k, \end{aligned}$$

where all indices range from 1 to m. The coefficient functions appearing in (5.4) are defined by this equation, and to ensure their uniqueness, we assume that they have the same symmetry in i and j as do the φ^i_j, e.g.,

$$Q_{ii} a^{is}_{jrk} + Q_{jj} a^{js}_{irk} = 0;$$

also, the functions a^{is}_{jrk}, b^{is}_{jr}, and c^{is}_{jrk} are skew symmetric in r and s, and the functions A^i_{jlk} and E^i_{jlk} are skew symmetric in l and k.

Given a real valued function, f, on $\mathcal{F}_3(U)$ we define the covariant derivatives, f_{θ^i}, f_ω, f_{η^i}, and f^s_r of f by

$$(5.5)\qquad df = f_{\theta^i}\theta^i + f_\omega \omega + f_{\eta^i}\eta^i + \frac{1}{2} Q_{rr} f^s_r \varphi^r_s$$

(all indices are summed) where it is assumed that the f^s_r are skew symmetric in r and s.

If we differentiate (5.1) we obtain the following Bianchi identities:

$$(5.6)\qquad a^{is}_{jrk} - a^{is}_{krj} = 0,$$

$$(5.7)\qquad b^{is}_{jr} = 0,$$

(5.8)
$$\begin{aligned}2Q_{rr}Q_{ss}Q_{ii}c^{is}_{jrk} &= (Q_{ri}S_{sjk} + Q_{rj}S_{ski} + Q_{rk}S_{sij}) \\ &\quad - (Q_{si}S_{rjk} + Q_{sj}S_{rki} + Q_{sk}S_{rij}) \\ &\quad + \frac{1}{2}Q_{rr}Q_{ss}(S_{ijk})^r_s,\end{aligned}$$

(5.9)
$$Q_{ii}B^i_{jl} - Q_{ii}B^i_{lj} = \frac{1}{2}Q_{kk}(S_{ijk}S_{kl} - S_{ilk}S_{kj}) + B_{ilj} - B_{ijl},$$

(5.10)
$$\begin{aligned}Q_{ii}C^i_{ljk} - Q_{ii}C^i_{jlk} &= \frac{1}{2}((S_{ijk})_{\theta^l} - (S_{ilk})_{\theta^j}) \\ &\quad + \frac{1}{2}Q_{rr}(S_{ilr}T_{rjk} - S_{ijr}T_{rlk}),\end{aligned}$$

(5.11)
$$Q_{ii}D^i_{jk} = \frac{1}{2}(S_{ijk})_\omega - T_{ijk},$$

and

(5.12)
$$\begin{aligned}Q_{ii}E^i_{jlk} &= \frac{1}{2}((S_{ijl})_{\eta^k} - (S_{ijk})_{\eta^l}) \\ &\quad + \frac{1}{4}Q_{rr}(S_{irl}S_{rkj} - S_{irk}S_{rlj}) \\ &\quad + Q_{ik}Q_{jl} - Q_{il}Q_{jk}.\end{aligned}$$

Lemma 5.1. *Let n be a positive integer. Suppose (b_{ijk}) and (c_{ijk}), where $1 \le i, j, k \le n$, are tensors such that (b_{ijk}) is skew symmetric in j and k, and (c_{ijk}) is symmetric in i and j. Then there exists a unique tensor, (a_{ijk}), such that*

$$\begin{aligned}a_{ijk} + a_{jik} &= c_{ijk}, \\ a_{ijk} - a_{ikj} &= b_{ijk},\end{aligned}$$

for all $1 \le i, j, k \le n$.

Proof. It is an easy exercise in index juggling to show that the solution is
$$a_{ijk} = \frac{1}{2}(b_{ijk} - b_{kij} + b_{jki}) + \frac{1}{2}(c_{ijk} + c_{kij} - c_{jki}).$$
□

Corollary 5.2. *If (a_{ijk}) is a tensor which is symmetric in one pair of indices and skew symmetric in another pair, then $(a_{ijk}) = 0$.*

Equations (5.6) and (5.8) tell us that $Q_{ii}a^{is}_{jrk}$ and $Q_{ii}c^{is}_{jrk}$ are symmetric in j and k. Thus,

$$a^{is}_{jrk} = c^{is}_{jrk} = 0$$

by Corollary 5.2. Equation (5.8) becomes

$$\begin{aligned} \frac{1}{2}Q_{rr}Q_{ss}(S_{ijk})^r_s = {} & (Q_{si}S_{rjk} + Q_{sj}S_{rki} + Q_{sk}S_{rij}) \\ & - (Q_{ri}S_{sjk} + Q_{rj}S_{ski} + Q_{rk}S_{sij}). \end{aligned} \tag{5.13}$$

If we apply Lemma 5.1 to equations (5.9) and (5.10), then we easily see that B^i_{jl} and C^i_{jlk} can be solved for in terms of S_{ijk}, T_{ijk}, S_{ij}, B_{ijk}, and their covariant derivatives.

In Section 3 we differentiated equation (5.2) and obtained the corresponding Bianchi identities; we repeat them here for convenience:

$$S_{ij} = S_{ji}, \tag{5.14}$$

$$T_{ijk} = T_{jik}, \tag{5.15}$$

and

$$B_{ijk} + B_{jki} + B_{kij} = 0. \tag{5.16}$$

Also, by definition,

$$B_{ijk} = -B_{jik}. \tag{5.17}$$

Recall that S_{ijk} is symmetric in all three indices. Thus, equations (5.15) and (5.11) imply that $Q_{ii}D^i_{jk}$ is symmetric in i and j, but we know that this tensor is skew symmetric in i and j as well. Therefore, by Corollary 5.2,

$$D^i_{jk} = 0,$$

and

$$T_{ijk} = \frac{1}{2}(S_{ijk})_\omega. \tag{5.18}$$

Structure equations (5.4) become

$$\begin{aligned} d\varphi^i_j = \varphi^i_k \wedge \varphi^k_j + \frac{1}{2}A^i_{jlk}\theta^l \wedge \theta^k + B^i_{jl}\theta^l \wedge \omega \\ + C^i_{jlk}\theta^l \wedge \eta^k + \frac{1}{2}E^i_{jlk}\eta^l \wedge \eta^k. \end{aligned} \tag{5.4$'$}$$

Differentiating equation (5.3) gives

$$\begin{aligned} Q_{ii}A^i_{jlk} = {} & (B_{ilk} - B_{ikl})_{\eta^j} + Q_{jk}S_{il} - Q_{jl}S_{ik} \\ & + (T_{ijk})_{\theta^l} - (T_{ijl})_{\theta^k} + Q_{rr}(T_{ilr}T_{rkj} - T_{ikr}T_{rjl}) \\ & + \frac{1}{2}Q_{rr}(S_{rjl}(B_{irk} - B_{ikr}) - S_{rjk}(B_{irl} - B_{ilr})), \end{aligned} \tag{5.19}$$

$$Q_{ii}B^i_{kj} = B_{ikj} - B_{ijk} + (S_{ij})_{\eta^k} + \frac{1}{2}Q_{ll}S_{il}S_{lkj} - (T_{ijk})_\omega, \tag{5.20}$$

(5.21)

$$Q_{ii}C^i_{klj} - Q_{ii}C^i_{jlk} = (T_{ilj})_{\eta^k} - (T_{ilk})_{\eta^j} + \frac{1}{2}Q_{rr}(S_{rkl}T_{irj} - S_{rjl}T_{irk}),$$

and

$$Q_{rr}Q_{ss}(S_{ij})^r_s = (Q_{ir}S_{js} + Q_{jr}S_{is}) - (Q_{is}S_{jr} + Q_{js}S_{ir}). \tag{5.22}$$

Theorem 5.1. *The functions S_{ijk}, S_{ij} and B_{ijk} are a complete set of invariants.*

Proof. It is clear from equations (5.18), (5.19), (5.20) and (5.12) that T_{ijk}, A^i_{jlk}, B^i_{jk}, and E^i_{jlk} can be expressed in terms of S_{ijk}, S_{ij}, B_{ijk} and their covariant derivatives with respect to θ, ω, and η. Either equation (5.21) or (5.10) and Lemma 5.1 imply that the C^i_{jlk} can also be expressed in terms of S_{ijk}, S_{ij}, B_{ijk}, and their covariant derivatives with respect to θ, ω and η. □

Recall that U_1 is the image of U under the natural projection: $(x, y, p) \to (x, y)$.

Theorem 5.2. *If $(S_{ijk})_\omega = 0$, then (η, φ) is a connection in $\mathcal{F}_3(U) \to U_1$.*

Proof. Recall that

$$\theta, \omega \equiv 0 \mod\{dx, dy\}.$$

Thus, equations (5.1), (5.2), (5.3) and (5.4)$'$ are the structure equations of a connection with the η^i and φ^i_j as the connection 1-forms and with x and $y = (y^1, \dots, y^m)$ as base variables if $T_{ijk} = 0$ and $C^i_{jlk} = 0$. Equation (5.18) implies $T_{ijk} = 0$. It follows from equation (5.21) that $Q_{ii}C^i_{jlk}$ is symmetric in j and k, but $Q_{ii}C^i_{jlk}$ is also skew symmetric in i and j. Therefore, $C^i_{jlk} = 0$ by Corollary 5.2. □

Recall that $ds^2 = ({}^t\theta Q\theta) + (\omega^2)$ is a Finsler metric on U_1.

Theorem 5.3. *If $S_{ijk} = 0$, then the Finsler metric $ds^2 = ({}^t\theta Q\theta) + (\omega^2)$ is pseudo-Riemannian.*

Proof. Recall the coefficients a, b_j and c_{ij} of ds^2 (cf. equation (4.5)). The metric ds^2 is pseudo-Riemannian provided

$$\frac{\partial a}{\partial p^k} = \frac{\partial b_j}{\partial p^k} = \frac{\partial c_{ij}}{\partial p^k} = 0$$

$(k = 1, \dots, m)$, or equivalently,

$$\mathcal{L}_{\partial/\partial p^k}(ds^2) = 0. \tag{5.23}$$

The vector fields $\partial/\partial p^1, \dots, \partial/\partial p^m$ solve the system of equations

$$\omega = \theta^i = \varphi^j_k = 0. \tag{5.24}$$

Thus, if we define vector fields $X_1, \dots, X_m$ by equations (5.24) together with the equations

$$\eta^i = \delta^i_j,$$

then equations (5.23) are equivalent to

$$\mathcal{L}_{X_k}(ds^2) = 0.$$

A direct calculation using equations (5.1) and (5.2) shows that

$$\mathcal{L}_{X_k}(ds^2) = S_{ijk}(\theta^i\theta^j).$$

□

Theorem 5.4. *Suppose that $m \geq 2$. Then*

$$dS_{ijk} \equiv 0 \quad \mathrm{mod}\{\theta, \omega, \eta\}$$

implies $S_{ijk} = 0$.

Proof. By hypothesis $(S_{ijk})^r_s = 0$. Identity (5.13) can be written:

$$Q_{ri}S_{sjk} + Q_{rj}S_{ski} + Q_{rk}S_{sij} = Q_{si}S_{rjk} + Q_{sj}S_{rki} + Q_{sk}S_{rij}. \tag{5.25}$$

Let $i = j = k = r \neq s$. Then (5.25) reduces to $3Q_{ii}S_{sii} = 0$. Thus, $S_{ijj} = 0$ if $i \neq j$. Let $i = j = r \neq k = s$. The symmetry of S_{ijk} in all three indices implies that (5.25) reduces to $Q_{kk}S_{iii} = 0$; hence, $S_{iii} = 0$. If $m = 2$, then $S_{ijk} = 0$ for all i, j and k. If $m > 2$, then let $i = j = r$ and i, k and s be distinct. Then (5.25) reduces to $2Q_{ii}S_{sik} = 0$. Thus, $S_{ijk} = 0$ for all i, j and k. □

Theorem 5.5. *Suppose $m \geq 2$ and S_{ijk}, B_{ijk} and S_{ij} are constants. Then (U_1, ds^2) is a pseudo-Riemannian manifold of constant curvature, K, where $K = Q_{11}S_{11}$.*

Proof. It follows from the preceding theorem that $S_{ijk} = 0$; thus ds^2 is pseudo-Riemannian. From the proof of Theorem 5.2, we know $T_{ijk} = 0$ and $C^i_{jlk} = 0$. Identity (5.20) reduces to

$$Q_{ii}B^i_{kj} = B_{ikj} - B_{ijk}.$$

Thus, the tensor $Q_{ii}B^i_{jk}$ is skew symmetric in j and k and has the same cyclic symmetry in i, j and k that B_{ijk} has (see equation (5.16)); we also know that $Q_{ii}B^i_{jk}$ is skew symmetric in i and j. It is easy to see that any three index tensor will all these symmetries has to vanish. So $B^i_{jk} = 0$ and B_{ijk} is symmetric in j and k. Recall (equation (5.17)) that B_{ijk} is skew symmetric in i and j. Therefore by Corollary 5.2, $B_{ijk} = 0$.

Identity (5.22) becomes

$$Q_{ir}S_{js} + Q_{jr}S_{is} = Q_{is}S_{jr} + Q_{js}S_{ir}. \tag{5.26}$$

Let $i = j = r \neq s$. Then (5.26) reduces to $2Q_{ii}S_{is} = 0$; thus, $S_{ij} = 0$ if $i \neq j$. Let $r = i \neq j = s$. Then (5.26) becomes: $Q_{ii}S_{jj} = Q_{jj}S_{ii}$, and therefore, $Q_{ii}S_{ii} = Q_{jj}S_{jj}$ for all i and j. Let $K = Q_{11}S_{11}$. Then

$$S_{ij} = Q_{ij}K.$$

Identity (5.19) becomes

$$Q_{ii}A^i_{jlk} = Q_{jk}S_{il} - Q_{jl}S_{ik} = (Q_{jk}Q_{il} - Q_{jl}Q_{ik})K.$$

Identity (5.12) becomes $Q_{ii}E^i_{jlk} = Q_{ik}Q_{jl} - Q_{il}Q_{jk}$. Thus,

$$A^i_{jlk} = -E^i_{jlk}K.$$

The structure equations now take the form:

$$\begin{aligned}
d\theta^i &= \varphi^i_j \wedge \theta^j - \eta^i \wedge \omega, \\
d\omega &= Q_{jj}\eta^j \wedge \theta^j, \\
d\eta^i &= \varphi^i_j \wedge \eta^j + K\omega \wedge \theta^i, \\
d\varphi^i_j &= \varphi^i_k \wedge \varphi^k_j + Q_{jj}K\theta^i \wedge \theta^j - Q_{jj}\eta^i \wedge \eta^j.
\end{aligned}$$

We may write these equations in the following matrix form:

$$d\begin{pmatrix} \theta \\ \omega \end{pmatrix} = \begin{pmatrix} \varphi & -\eta \\ {}^t\eta Q & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \omega \end{pmatrix},$$

$$\begin{aligned}
d\begin{pmatrix} \varphi & -\eta \\ {}^t\eta Q & 0 \end{pmatrix} &= \begin{pmatrix} \varphi & -\eta \\ {}^t\eta Q & 0 \end{pmatrix} \wedge \begin{pmatrix} \varphi & -\eta \\ {}^t\eta Q & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} K\theta^t\theta Q & -K\omega\theta \\ -K^t\theta Q\omega & 0 \end{pmatrix}.
\end{aligned}$$

In matrix form, the metric is

$$ds^2 = \begin{pmatrix} {}^t\theta & \omega \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \theta \\ \omega \end{pmatrix}.$$

The transpose of the matrix,

$$\begin{pmatrix} \varphi & -\eta \\ {}^t\eta Q & 0 \end{pmatrix},$$

is

$$\begin{pmatrix} {}^t\varphi & Q\eta \\ -{}^t\eta & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} {}^t\varphi & Q\eta \\ -{}^t\eta & 0 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi & -\eta \\ {}^t\eta Q & 0 \end{pmatrix} = 0.$$

□

As a final observation, we note that Riemann's normal form for the metric of a constant curvature manifold [**19**, p. 69] determines a normal form for each integral $\int L\,dx$ whose invariants satisfy the hypothesis of Theorem 5.5.

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