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THE DIOPHANTINE EQUATION $x^2 + 119 = 15 \cdot 2^n$ HAS EXACTLY SIX SOLUTIONS

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Introduction. F. Beukers proved in [**1**] that the diophantine equation

$$x^2 + D = 2^n,$$

where D is an odd integer and $x, n \geq 1$, has at most four solutions for the case $D \neq 7$. For the case $D = 7$ we have the well-known Ramanujan-Nagell equation which has five solutions, namely,

$n =$	3	4	5	7	15
$x =$	1	3	5	11	181

Consider the diophantine equation

$$x^2 + D = A \cdot 2^n,$$

where D is an odd integer, $A \geq 3$ a positive odd integer, $\gcd(A, D) = 1$ and $x, n \geq 1$. Are there equations with more than five solutions? We want to prove the following theorem:

Theorem. *The diophantine equation* $x^2 + 119 = 15 \cdot 2^n$ *has exactly the six solutions*

$n =$	3	4	5	6	8	15
$x =$	1	11	19	29	61	701

for $x, n \geq 1$.

Proof. The proof is based on ideas of P. Bundschuh [**2**], and all we need are some calculations that can be done by a home computer.

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Let us first consider the case that n is even. Then our equation can be written as

$$15y^2 - x^2 = 119, \qquad y = 2^{n/2}. \tag{1}$$

(1) has the solutions

$x = 4$	11	16	29	61	104	139	236	484	821	$\cdots$
$y = 3$	4	5	8	16	27	36	61	125	212	$\cdots$

and the y's form the two two-sided infinite sequences

$$\begin{array}{c} \dots, 125, 16, 3, 8, 61, \dots \\ \dots, 36, 5, 4, 27, 212, \dots . \end{array} \tag{2}$$

(2) can be written as

$$a_{k+2} = 8a_{k+1} - a_k, \qquad a_0 = 3, \qquad a_1 = 8 \tag{3}$$

$$b_{k+2} = 8b_{k+1} - b_k, \qquad b_0 = 4, \qquad b_1 = 27 \tag{4}$$

with $k \in \mathbf{Z}$. We have to prove that 4, 8 and 16 are the only powers of 2 in these sequences.

Let a_k, with $|k| > 1$ be a power of 2. Since a_k is strictly increasing with $|k|$, then a_k must be at least 2^5; in particular, $a_k \equiv 0 \pmod{32}$. The residues of the a_k's modulo 32 form a periodic sequence of order 8:

$k \equiv 0$	1	2	3	4	5	6	7	mod 8
$a_k \equiv 3$	8	29	0	3	24	29	16	mod 32

Considering the a_{3+8l}'s, $l \in \mathbf{Z}$, modulo 23 we get a periodic sequence of residues of order 3:

$l \equiv$	0	1	2	mod 3
$a_{3+8l} \equiv$	20	7	19	mod 23

Let $R(m) := \{r_m(2^n) : m, h \in \mathbf{N}\}$, where $r_m(k) \equiv k \pmod{m}$ with $0 \leq r_m(k) < m$ and $k \in \mathbf{Z}$. Then $R(23) = \{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}$

and from $R(23) \cap \{7, 19, 20\} = \varnothing$, it follows that $a_{-1} = 16$ and $a_1 = 8$ are the only powers of 2 in (3) which leads to $n = 6, 8$.

From (4) we get $b_k \equiv 3, 4, 5 \pmod 8$ for all $k \in \mathbf{Z}$, so $b_0 = 4$ is the only power of 2 in (4) which leads to $n = 4$.

Next we consider the case that n is odd. Our equation can be written as

$$(5) \qquad 30y^2 - x^2 = 119, \qquad y = 2^{(n-1)/2},$$

and the solutions of (5) are

$x = 1$	19	31	109	131	449	701	2399	2881	9859	$\cdots$
$y = 2$	4	6	20	24	82	128	438	526	1800	$\cdots$

and the y's form the two two-sided infinite sequences

$$(6) \qquad c_{k+2} = 22c_{k+1} - c_k, \qquad c_0 = 2, \qquad c_1 = 20$$

$$(7) \qquad d_{k+2} = 22d_{k+1} - d_k, \qquad d_0 = 4, \qquad d_1 = 82$$

with $k \in \mathbf{Z}$.

It follows from (6) that $c_{1+2l} \equiv 0 \pmod 4$ and $c_{1+2l} \equiv 3, 5, 6, \pmod 7$ for all $l \in \mathbf{Z}$. $R(7) = \{1, 2, 4\}$ and from $R(7) \cap \{3, 5, 6\} = \varnothing$ we obtain that $c_0 = 2$ is the only power of 2 in (6) which leads to $n = 3$.

It follows from (7) that $d_{62+128l} \equiv 0 \pmod{256}$ and $d_{62+128l} \equiv 120, 137 \pmod{257}$ for all $l \in \mathbf{Z}$. $R(257) = \{1, 2, 4, 8, 16, 32, 64, 128, 129, 193, 225, 241, 249, 253, 255, 256\}$ and from $R(257) \cap \{120, 137\} = \varnothing$ we obtain that $d_{-2} = 128$ and $d_0 = 4$ are the only powers of 2 in (7) which leads to $n = 5, 15$. □

Remark . In the same way we can prove that the diophantine equation

$$x^2 + 391 = 35 \cdot 2^n$$

has the five solutions $(x, n) = (13, 4), (27, 5), (43, 6), (267, 11), (757, 14)$. For n even we obtain the sequences

$$a_{k+2} = 12a_{k+1} - a_k, \qquad a_0 = 5, \qquad a_1 = 52$$

$$b_{k+2} = 12b_{k+1} - b_k, \qquad b_0 = 4, \qquad b_1 = 37$$

and for n odd,

$$\begin{aligned} c_{k+2} &= 502c_{k+1} - c_k, \qquad c_0 = 4, \qquad c_1 = 194 \\ d_{k+2} &= 502d_{k+1} - d_k, \qquad d_0 = 22, \qquad d_1 = 11012 \end{aligned}$$

with $k \in \mathbf{Z}$. Only the a_{3+8l}'s are divisible by 16 and $a_{3+8l} \equiv 63 \pmod{71}$ for all $l \in \mathbf{Z}$. But $R(71) \cap \{63\} = \varnothing$. Only the $b_{62+128l}$'s are divisible by 256 and $b_{62+128l} \equiv 86,\ 171 \pmod{257}$ for all $l \in \mathbf{Z}$. But $R(257) \cap \{86, 171\} = \varnothing$. Only the c_{2+4l}'s are divisible by 8 and $c_{2+4l} \equiv 3, 7, 10, 14 \pmod{17}$ for all $l \in \mathbf{Z}$. But $R(17) \cap \{3, 7, 10, 14\} = \varnothing$. Only the d_{15+32l}'s are divisible by 64 and $d_{15+32l} \equiv 6 \pmod{17}$ for all $l \in \mathbf{Z}$. But $R(17) \cap \{6\} = \varnothing$.

REFERENCES

1. F. Beukers, *On the generalized Ramanujan-Nagell equation* I, Acta Arith. **38** (1980/1981), 389–410.

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