

**EQUIMEASURABLE REARRANGEMENTS OF
FUNCTIONS AND FOURTH ORDER
BOUNDARY VALUE PROBLEMS**

PHILIP SAVOYE

ABSTRACT. The buckling properties of a vibrating beam subject to axial compressive and elastic destructive forces are investigated in this paper. In particular, lower bounds for the eigenvalues of the corresponding boundary value problem are obtained and expressed in terms of equimeasurable rearrangements of the associated differential equation's coefficients.

1. Introduction. In this paper, we investigate the buckling properties of a vibrating beam. Our interest in this fourth order boundary value problem stems from earlier work done by Barnes [2, 3] in obtaining spectral inequalities for second and fourth order problems.

The beam investigated in this paper has stiffness $p(x)$ and is subject to an axial compressive load λ which causes it to buckle. The beam is supported on an elastic foundation which provides, at each point x , an elastic destructive force $F(x)y$, $F(x) < 0$, which opposes restoration toward the line of no deflection and is directly proportional to the displacement y . From elementary beam theory, the natural modes of buckling of our problem are the eigenfunctions of the differential equation

$$(1) \quad (p(x)y'')''(x) + \lambda y''(x) + F(x)y = 0, \quad x \in (0, l)$$

subject to elastically constrained boundary conditions. We will assume hinged-hinged boundary conditions

$$(2) \quad y(0) = y(l) = y''(0) = y''(l) = 0.$$

It can be shown [8, 1] using a Prüffer transformation that the n th eigenfunction y_n has $n - 1$ zeros η_i interlaced as follows

$$(3) \quad 0 = \eta_1 < \eta_2 < \cdots < \eta_{n-1} = l.$$

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For our eigenvalue problem, it follows from a generalized Courant-Hilbert variational principle [4, 8] that our n th eigenvalue is determined by

$$(4) \quad \lambda_n = \min_{u \in G} \frac{\int_0^l [F(x)u^2(x) + p(x)(u''(x))^2] dx}{\int_0^l (u'(x))^2 dx},$$

where G is defined to be the class of piecewise twice differentiable functions satisfying

$$u(\eta_i) = 0, \quad i = 1, 2, \dots, n-1; \quad u(0) = u(l) = u''(0) = u''(l) = 0.$$

The lowest eigenvalue λ_1 represents the smallest axial compressive force necessary to cause the beam to buckle. We will assume throughout this paper that the elastic destructive force (per unit length) $F(x)$ is sufficiently small in absolute value to ensure that the eigenvalues λ are positive. As this is clearly the case when $F(x) = 0$, it follows from the continuous dependence of the eigenvalues λ on $F(x)$ (shown in an earlier work [5,6]) that our assumption is valid.

2. Concavity of the eigenfunctions. The concavity of the eigenfunctions y_n of the eigenvalue problem determined by (1) and (2) is analyzed in the following lemma.

Lemma 1. *Assume (as motivated above) that $\lambda_1(p, F) > 0$ and that the n th eigenfunction of (1) subject to boundary conditions (2) is positive in (η_{i-1}, η_i) . Then y_n is concave in (η_{i-1}, η_i) and $y_n''(\eta_i) = 0$.*

Proof. Proceeding towards contradiction, we suppose that our n th eigenfunction y_n is not concave on (η_{i-1}, η_i) . Then there exists a subinterval (z_1, z_2) of (η_{i-1}, η_i) such that $y_n''(x) \geq 0$ on (z_1, z_2) .

We construct a new function $y_n^*(x)$ as follows

$$y_n^*(x) = \begin{cases} y_n(x), & \text{if } x \notin (z_1, z_2); \\ 2 \left[y_n(z_1) + (y_n(z_2) - y_n(z_1)) \frac{x-z_1}{z_2-z_1} \right] - y_n(x), & \text{if } x \in (z_1, z_2), \end{cases}$$

i.e., $y_n^*(x)$ is the symmetric reflection of y_n about the line

$$y(x) = y_n(z_1) + (y_n(z_2) - y_n(z_1)) \frac{x-z_1}{z_2-z_1} - y_n(x)$$

on (z_1, z_2) but coincides with $y_n(x)$ elsewhere.

Note that $y_n^{*''}(x) = -y_n''(x)$, and that since

$$y_n^{*'}(x) = 2 \frac{(y_n(z_2) - y_n(z_1))}{(z_2 - z_1)} - y_n'(x),$$

it follows that

$$\begin{aligned} (y_n^{*'}(x))^2 &= 4 \left[\frac{(y_n(z_2) - y_n(z_1))}{(z_2 - z_1)} \right]^2 \\ &\quad - 4 \left[\frac{(y_n(z_2) - y_n(z_1))}{(z_2 - z_1)} \right] y_n'(x) + (y_n'(x))^2. \end{aligned}$$

Consequently,

$$\int_{z_1}^{z_2} (y_n^{*'}(x))^2 dx = \int_{z_1}^{z_2} (y_n'(x))^2 dx.$$

Likewise, $y_n^*(x) \geq y_n(x)$ on (z_1, z_2) and so, since $F(x) < 0$,

$$\frac{\int_{z_1}^{z_2} F(x) y_n^2(x) dx}{\int_{z_1}^{z_2} (y_n'(x))^2 dx} \geq \frac{\int_{z_1}^{z_2} F(x) y_n^{*2}(x) dx}{\int_{z_1}^{z_2} (y_n^{*'}(x))^2 dx}$$

Since $(y_n^{*''}(x))^2 = (y_n''(x))^2$ for $x \in (z_1, z_2)$, we see that

$$\begin{aligned} \frac{\int_{z_1}^{z_2} [p(x)(y_n''(x))^2 + F(x)y_n^2(x)] dx}{\int_{z_1}^{z_2} (y_n'(x))^2 dx} \\ \geq \frac{\int_{z_1}^{z_2} [p(x)(y_n^{*''}(x))^2 + F(x)y_n^{*2}(x)] dx}{\int_{z_1}^{z_2} (y_n^{*'}(x))^2 dx} \end{aligned}$$

so

$$\frac{\int_0^l [p(x)(y_n''(x))^2 + F(x)y_n^2(x)] dx}{\int_0^l (y_n'(x))^2 dx} \geq \frac{\int_0^l [p(x)(y_n^{*''}(x))^2 + F(x)y_n^{*2}(x)] dx}{\int_0^l (y_n^{*'}(x))^2 dx}.$$

The existence of the function $y_n^*(x)$ contradicts the minimizing property (4). Thus, $y_n^* \geq 0$ implies that y_n is concave. In a like manner, we can

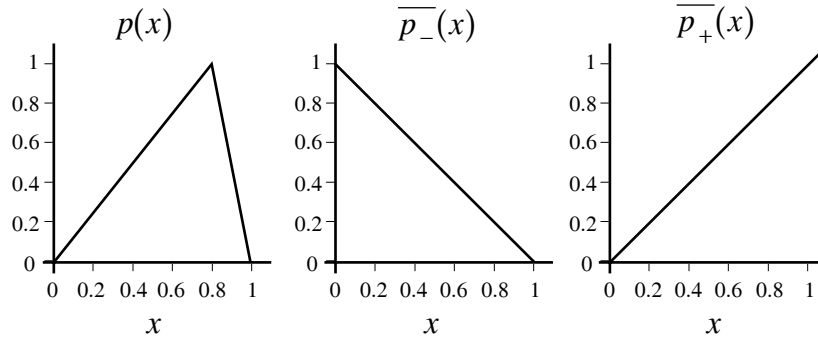


FIGURE 1.

show that $y_n \leq 0$ implies that y_n is convex. Therefore, y_n'' vanishes if y_n does, and consequently vanishes at the points η_i , $i = 1, 2, \dots, n-1$. This proves the lemma. \square

3. Rearrangements of functions. In the remainder of this paper, a lower bound for the n th eigenvalue $\lambda_n(p, F)$ is obtained using rearrangements of eigenfunctions first given by B. Schwarz [7]. A summary of these rearrangements is given next.

Definition 1. Two functions $f_1(x)$ and $f_2(x)$ are *equimeasurable* if, for all $t \geq 0$,

$$\text{measure of } \{x : f_1(x) \geq t\} = \text{measure of } \{x : f_2(x) \geq t\}.$$

Definition 2. The equimeasurable rearrangements of a function $f(x)$ on an interval $(0, l)$ in *increasing* and *decreasing* orders are denoted by $\overline{f}_\pm(x)$, respectively. The two rearrangements satisfy

$$\overline{f}_+(x) = \overline{f}_-(l - x).$$

Illustrations of these rearrangements for the function

$$p(x) = \begin{cases} 1.33x, & \text{if } 0 \leq x \leq 3/4; \\ 4 - 4x, & \text{if } 3/4 \leq x \leq 1 \end{cases}$$

are provided in Figure 1.

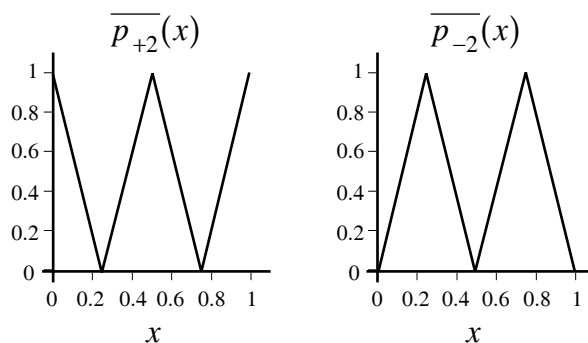


FIGURE 2.

From these definitions and observations, we deduce the following relation for nonnegative functions f and g :

$$\begin{aligned}
 (5) \quad \int_0^l \bar{f} + \bar{g}_- dx &= \int_0^l \bar{f}_- \bar{g}_+ dx \leq \int_0^l fg dx \\
 &\leq \int_0^l \bar{f}_+ \bar{g}_+ dx = \int_0^l \bar{f}_- \bar{g}_- dx.
 \end{aligned}$$

This follows intuitively from the fact that the products $\bar{f}_+ \bar{g}_+$ and $\bar{f}_- \bar{g}_-$ match large values of f with large values of g , while the reverse is true for the products $\bar{f}_+ \bar{g}_-$ and $\bar{f}_- \bar{g}_+$. Likewise, if E is a set of measure x , then

$$(6) \quad \int_0^x \bar{f}_+(t) dt \leq \int_E f(t) dt \leq \int_0^x \bar{f}_-(t) dt.$$

Schwarz [7] has presented many other rearrangements of functions f , including \bar{f}_{+n} and \bar{f}_{-n} . These rearrangements are equimeasurable with f , are periodic in $[0, l]$ with period l/n , and satisfy the symmetry condition

$$(7) \quad \bar{f}_{\pm n} \left(\frac{l}{2n} + x \right) = \bar{f}_{\pm n} \left(\frac{l}{2n} - x \right); \quad x \in \left[0, \frac{l}{2n} \right],$$

with \bar{f}_{+n} decreasing and \bar{f}_{-n} increasing in $[0, l/(2n)]$.

Illustrations of such rearrangements are provided in Figure 2 for the function

$$p(x) = \begin{cases} 1.33x, & \text{if } 0 \leq x \leq 3/4; \\ 4 - 4x, & \text{if } 3/4 \leq x \leq 1. \end{cases}$$

Other rearrangements used in this paper include $\tilde{f}_{\pm n}$ and $\hat{f}_{\pm n}$.

\tilde{f}_{+n} is defined by the following conditions:

- (i) $|Df(x)|$ and $|D\tilde{f}_{+n}(x)|$ are equimeasurable.
- (ii) \tilde{f}_{+n} is periodic in $[0, l]$ with period l/n .
- (iii) $\tilde{f}_{+n}(l/(2n) - x) = \tilde{f}_{+n}(l/(2n) + x)$; $x \in [0, l/(2n)]$.
- (iv) $\tilde{f}_{+n}(x)$ is convex in $[0, l/2n]$.
- (v) $\tilde{f}_{+n}(l/(2n)) = 0$.

It is easily verified that

$$(8) \quad \tilde{f}_{+n} = \int_x^{1/(2n)} \overline{|Df(t)|}_{-n} dt \text{ on } [0, l/2n].$$

\tilde{f}_{-n} is defined in an identical manner except that conditions (iv) and (v) are replaced by the conditions

- (vi) $\tilde{f}_{-n}(x)$ is concave in $[0, l/(2n)]$.
- (vii) $\tilde{f}_{-n}(0) = 0$.

A thorough discussion of these rearrangements (with illustrations) is to be found in [2] and [3]. The following generalization of relation (5) can be obtained from these rearrangements:

$$(9) \quad \int_0^l \bar{f}_{+n}(x) \bar{g}_{-n}(x) dx \leq \int_0^l f(x)g(x) dx \\ \leq \int_0^l \tilde{f}_{-n}(x) \bar{g}_{-n}(x) dx.$$

Inequality (9) follows from (5) and (7).

The following theorem, due to Barnes [3], relates the two kinds of rearrangements \bar{f}_{-n} and \tilde{f}_{-n} .

Theorem 1. *Suppose $f(x)$ is piecewise differentiable and has $n - 1$ zeros $\eta_i \in [0, l]$; $0 = \eta_1 < \eta_2 < \dots < \eta_{n-1} = l$. Further, suppose that in each interval $[\eta_i, \eta_{i+1}]$, f increases to its maximum value at α_i and decreases in $[\alpha_i, \eta_{i+1}]$ and that $f(\alpha_i) = 1$ for all i . Then*

$$\tilde{f}_{-n}(x) \geq \bar{f}_{-n}(x).$$

Proof. See [3]. □

By identical reasoning, we obtain an analogous result in terms of $\tilde{f}_{+n}\tilde{f}_{+n}$.

Theorem 2. *Suppose f satisfies the hypotheses of Theorem 1. Then*

$$\tilde{f}_{+n}(x) \leq \bar{f}_{+n}(x).$$

In analyzing the fourth order eigenvalue problem (1) and (2), it is desirable to define the second derivative rearrangements $\hat{f}_{\pm n}$ of f .

Suppose Df is continuous and D^2f is piecewise continuous on $[0, l]$. The second derivative rearrangement of f into symmetrically decreasing order of degree n is a function $\hat{f}_{-n}(x)$ defined by the following conditions:

- (i) $|D^2f(x)|$ and $|D^2\hat{f}_{+n}(x)|$ are equimeasurable,
- (ii) $\hat{f}_{+n}(x)$ is periodic in $[0, l]$ with period l/n ,
- (iii) $\hat{f}_{+n}(l/(2n) - x) = \hat{f}_{+n}(l/(2n) + x), x \in [0, l/(2n)]$,
- (iv) \hat{f}_{+n} is convex in $[0, l/(2n)]$,
- (v) $\hat{f}_{+n}(l/(2n)) = 0, D\hat{f}_{+n}(0) = 0$.

It is easily verified that

$$(10) \quad \hat{f}_{+n}(x) = \int_x^{1/(2n)} \int_0^t \overline{|D^2f(s)|}_{+n} ds dt, \quad \text{for } x \in [0, 1/(2n)].$$

The rearrangement $\hat{f}_{-n}(x)$ is defined similarly.

Barnes [3] has presented the following theorem which relates \hat{f}_{-n} and its derivative to \bar{f}_{-n} and its derivative:

Theorem 3. *Suppose Df is continuous and D^2f is piecewise continuous and f has $n - 1$ zeros $\eta_i \in [0, l], 0 = \eta_1 < \eta_2 < \dots < \eta_{n-1} = l$. If f is also concave in each interval $[\eta_i, \eta_{i+1}]$ and if the maximum value of f occurs at $x = \alpha_i \in [\eta_i, \eta_{i+1}]$ where $f(\alpha_i) = 1$, then $|D\hat{f}_{-n}(x)| \geq |D\bar{f}_{-n}(x)|$ and $\hat{f}_{-n}(x) \geq \bar{f}_{-n}(x)$.*

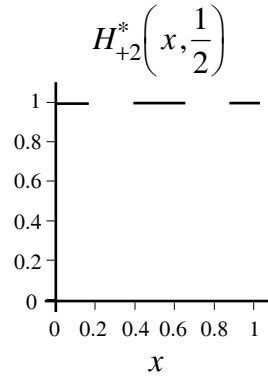


FIGURE 3.

Proof. See [3]. \square

We next introduce the periodic square wave $H_{+n}^*(x, \xi)$, which is the periodic generalization of a step function, with period l/n . For $x \in [0, l/n]$, $H_{+n}^*(x, \xi)$ is defined as follows:

$$H_{+n}^*(x, \xi) = \begin{cases} 1/\xi, & \text{if } x \notin [(l\varepsilon)/(2n), (2l - l\varepsilon)/(2n)]; \\ 0, & \text{if } x \in [(l\xi)/(2n), (2l - l\xi)/(2n)], \end{cases}$$

and $H_{+n}^*(x) = H_{+n}^*(x + l/n)$ for $x \in [0, l]$.

An illustration of $H_{+n}^*(x, \xi)$ is provided in Figure 3 for $n = 2$ and $\xi = 1/2$.

We see that H_{+n}^* is a sequence of very narrow rectangular pulses whose support lies where $\hat{y}_{+n}^2(x)$ attains its largest values.

We introduce a new rearrangement $-\overline{F}_{+n}^*(x)$ of the nonnegative function $-F(x)$ as follows:

$$-\overline{F}_{+n}^*(x) = -\overline{F}_{+n}(x)H_{+n}^*(x) \quad \text{for } x \in [0, l].$$

An illustration of such a rearrangement for the function

$$-F(x) = \begin{cases} 20x, & \text{if } 0 \leq x \leq 1/2; \\ 20 - 20x, & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

is provided in Figure 4 with $n = 2$ and $\xi = 1/2$.

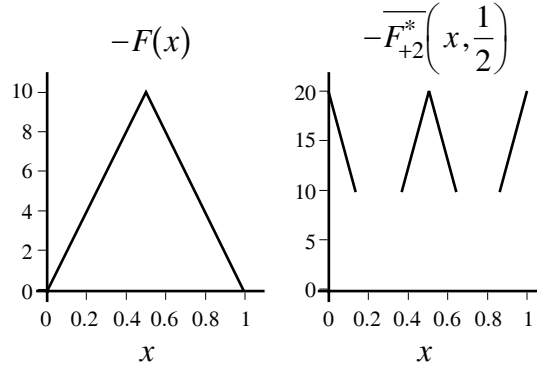


FIGURE 4.

We require that the parameter ξ be taken to be small enough to satisfy the inequality

$$\int_0^{l/n} (-\bar{F}_{+n}(x)) \bar{y}_{+n}^2(x) dx \leq 2/\xi \left(\int_0^{(l\xi)/(2n)} (-\bar{F}_{+n}(x)) \hat{y}_{+n}^2(x) dx + \int_{(2l-l\xi)/(2n)}^{1/(2n)} (-\bar{F}_{+n}(x)) \hat{f}_{+n}^2(x) dx \right).$$

4. Lower bounds. These results are used to establish a lower bound for the n th eigenvalue $\lambda_n(p, F)$ in the following theorem.

Theorem 4. *Let y_n be the n th eigenfunction of (1), (2), and let $\lambda_n(p, F)$ be the corresponding eigenvalue. Then*

$$\lambda_n(p, F) \geq \lambda_n(\bar{p}_{-n}, \bar{F}_{+n}^*),$$

where $F_{+n}^*(x, \xi)$ is the rearrangement of $F(x)$ defined above.

Proof. Let y_n be the n th eigenfunction of buckling problem (1), (2) corresponding to $\lambda_n(F, p)$. We define a function f so that $f(x) = c_i y_n(x)$ for $x \in (\eta_{i-1}, \eta_i)$, $i = 2, \dots, n - 2$. We select the constants c_i such that

$$\text{Max}_{[0, \eta_2]} f(x) = \dots = \text{Max}_{[\eta_{i-1}, \eta_i]} f(x) = \dots = \text{Max}_{[\eta_{n-2}, l]} f(x) = 1.$$

It follows from the homogeneity of (1) that f is a solution of

$$f(x)D^2[p(x)D^2f(x)] + \lambda f(x)D[f(x)] + F(x)f^2(x) = 0$$

on $(0, \eta_2), \dots, (\eta_{i-1}, \eta_i), \dots$, and (η_{n-2}, l) . Integrating repeatedly by parts over each interval, using boundary conditions (2) and the fact that $y''(\eta_i) = 0$, we find that

$$\lambda_n(p, F) = \frac{\int_0^l [F(x)f^2(x) + p(x)(D^2f(x))^2] dx}{\int_0^l (Df(x))^2 dx}.$$

We will next establish the identity

$$|D(\widetilde{f(x)})|_{-n}^2 = |D(\hat{f}_{+n}(x))|^2.$$

Differentiating $\hat{f}_{+n}(x)$ (defined in equation (10)), we find that

$$|D(\hat{f}_{+n}(x))|^2 = \left| - \int_0^x \overline{|D^2f(s)|_{+n}} ds \right|^2.$$

Applying Barnes' [3] definition

$$\tilde{f}_{-n}(x) = \int_0^x \overline{|Df(s)|_{+n}} ds$$

to $Df(x)$, we find that

$$|D(\widetilde{f(x)})|_{-n}^2 = \left| \int_0^x \overline{|D(Df(s))|_{+n}} ds \right|^2,$$

from which the identity follows.

Using this identity and Theorem 1, we see that

$$\begin{aligned} \int_0^l |D(f(x))|^2 dx &= \int_0^l \overline{|D(f(x))|_{-n}^2} dx = \int_0^l \overline{|D(f(x))|_{-n}^2} dx \\ &\leq \int_0^l |D(\widetilde{f(x)})|_{-n}^2 dx = \int_0^l |D(\hat{f}_{+n}(x))|^2 dx. \end{aligned}$$

Likewise, using (5) and the identity $\overline{|D^2 f|}_{+n}^2 = (D^2 \hat{f}_{+n})^2$, we obtain the inequality

$$\begin{aligned} \int_0^l p(x)(D^2(f(x)))^2 dx &\geq \int_0^l \bar{p}_{-n}(x) \overline{|D^2(f(x))|}_{+n}^2 dx \\ &= \int_0^l \bar{p}_{-n}(x)(D^2(\hat{f}_{+n}(x)))^2 dx. \end{aligned}$$

It also follows from (5) that

$$\begin{aligned} \int_0^l (-F(x))f^2(x) dx &\leq \int_0^l \overline{(-F(x))}_{+n} \overline{f^2(x)}_{+n} dx \\ &\leq \int_0^l (-\bar{F}_{+n}^*(x, \xi)) \hat{f}_{+n}^2(x) dx \\ &= -\frac{2n}{\xi} \left(\int_0^{(l\varepsilon)/(2n)} \bar{F}_{+n}(x) \hat{f}_{+n}^2(x) dx \right. \\ &\quad \left. + \int_{(2l-l\varepsilon)/(2n)}^{l/2n} \bar{F}_{+n}(x) \hat{f}_{+n}^2(x) dx \right) \end{aligned}$$

where

$$-\bar{F}_{+n}^*(x, \xi) = -H_{+n}^*(x, \xi) \bar{F}_{+n}(x).$$

Combining these three inequalities, we find that

$$(11) \quad \lambda(p, F) \geq \frac{\int_0^l [\bar{F}_{+n}^*(x, \xi)(\hat{f}_{+n}(x))^2 + \bar{p}_{-n}(x)(D^2 \hat{f}_{+n}(x))^2] dx}{\int_0^l (D(\hat{f}_{+n}(x)))^2 dx}.$$

It follows from the minimization property

$$\lambda_m(\bar{p}_{-n}, \bar{F}_{+n}^*) = \min_{u \in G} \frac{\int_0^l [\bar{F}_{+n}^*(x, \xi)u^2(x) + \bar{p}_{-n}(x)(u''(x))^2] dx}{\int_0^l (u'(x))^2 dx}$$

(where G is as defined in (4)) that the quantity on the right of (11) is not less than $\lambda_n(\bar{p}_{-n}, \bar{F}_{+n}^*)$. This proves our theorem. \square

The Rayleigh-Ritz procedure was used with fourth degree trial functions to estimate the first eigenvalue $\lambda_1(p, F)$ in an example in which

$$p(x) = \begin{cases} 1.33x, & \text{if } 0 \leq x \leq 3/4; \\ 4 - 4x, & \text{if } 3/4 \leq x \leq 1 \end{cases}$$

and

$$-F(x) = \begin{cases} 20x, & \text{if } 0 \leq x \leq 1/2; \\ 20 - 20x, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

It was determined that the first eigenvalue λ_1 was approximately 4.8471. A MAPLE procedure was written to determine the largest value of ξ for which the inequality

$$\lambda_1(\bar{p}_{-2}, \bar{F}_{+2}^*(\xi)) \leq \lambda_1(p, F)$$

held. It was found that the above inequality held for all $\xi \leq 0.601$.

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DEPARTMENT OF MATHEMATICS, MANSFIELD UNIVERSITY, MANSFIELD, PA 16933