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ASYMPTOTICS FOR ORTHOGONAL POLYNOMIALS BEYOND THE ANALYTIC BOUNDARY

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ABSTRACT. Let μ be a finite positive measure on the unit circle and $w_m(z) = \prod_{j=1}^m (z-\alpha_j)$ for $|\alpha_j| > 1$, $j = 1, \dots, m$. Using a relation between the orthonormal polynomials with respect to the measures $d\mu/(2\pi)$ and $|w_m(e^{i\theta})|^2\, d\mu/(2\pi)$, we derive the asymptotic behavior for orthonormal polynomials beyond the analytic boundary of the Szegő function. This is a generalization of some known results (see, e.g., [**4**] and [**8**]).

1. Introduction. Let μ be a finite positive measure on the unit circle. Let $\mu = \mu_a + \mu_s$ be its canonical decomposition into the absolutely continuous and the singular parts (with respect to Lebesgue measure on the unit circle). We denote by $\mu'(\theta)$ the Radon-Nikodym derivative of μ_a with respect to $d\theta$. Then $\mu' \in L^1[0, 2\pi)$, $\mu'(\theta) \ge 0$ almost everywhere, and we define its geometric mean $G(\mu')$ by

$$G(\mu') := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log \mu'(\theta)\, d\theta\right\}.$$

If $\log \mu' \in L^1[0, 2\pi)$ one can define the Szegő function for $|z| < 1$ by

$$D(z) = D(d\mu, z) := \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \log \mu'(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\theta\right\}.$$

It is known that (cf. [**11**, p. 276]) $D \in H^2(\{z : |z| < 1\})$ and for almost every $\theta \in [0, 2\pi)$

$$\lim_{r\nearrow 1} D(d\mu, re^{i\theta}) = D(d\mu, e^{i\theta})$$

exists and $|D(d\mu, e^{i\theta})|^2 = \mu'(\theta)$ for almost every $\theta \in [0, 2\pi)$. If $\log \mu' \notin L^1[0, 2\pi)$ we define $D(d\mu, z) \equiv 0$. Denote by $\mathcal{P}_n$ the set of polynomials of degree at most n. The $*$-transform $p^*(z)$ of a polynomial of degree n is defined as $p^*(z) = z^n \overline{p_n(1/\bar z)}$. Let $\phi_n(z) = \kappa_n z^n + \cdots \in \mathcal{P}_n$, $\kappa_n > 0$,

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be the nth orthonormal polynomial corresponding to $d\mu/(2\pi)$ on the unit circle, i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi}\phi_n(z)z^{-k}\,d\mu(\theta)=\kappa_n^{-1}\delta_{k,n},\qquad k=0,1,\dots,n,\quad z=e^{i\theta}.$$

It is well known that, from [**11**, Theorem 12.3.16],

$$\lim_{n\to\infty}\frac{\phi_n(z)}{z^n}=\overline{D^{-1}(1/\bar z)}, \tag{1}$$

holds locally uniformly for $|z|>1$ if $\log\mu'\in L^1[0,2\pi)$, here we say $D^{-1}(z)=1/D(z)$. Furthermore, in [**8**], Nevai and Totik proved that if $D^{-1}(z)$ has an analytic continuation to $|z|<\rho$, $\rho>1$, then (1) is also true for $|z|>1/\rho$. The purpose of this paper is to study the asymptotic behavior of $\phi_n(z)$ outside of the analytic region of $D^{-1}(z)$.

Instead of $d\mu/(2\pi)$, we consider the new measure $|w_m(e^{i\theta})|^2\,d\mu/(2\pi)$ on the unit circle, where $w_m(z):=\prod_{i=1}^m(z-\alpha_i)$. Suppose $D^{-1}(z)$ is analytic in $|z|<\rho$, $\rho>1$, and $\rho>\max_{1\le i\le m}\{|\alpha_i|\}$, then $D_w^{-1}:=D^{-1}(|w_m|^2\,d\mu,z)=D^{-1}(z)/w_m(z)$ will create some "bad" points at $\alpha_1,\dots,\alpha_m$. Let $\psi_n(z)=\eta_n z^n+\cdots\in\mathcal{P}_n$, $\eta_n>0$, be the nth orthonormal polynomial corresponding to $|w_m(e^{i\theta})|^2\,d\mu/(2\pi)$ on the unit circle, i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi}\psi_n(z)z^{-k}|w_m(z)|^2\,d\mu(\theta)=\eta_n^{-1}\delta_{k,n},$$
$$k=0,1,\dots,n,\quad z=e^{i\theta}.$$

From the theorems in [**8**], we only know that $\lim_{n\to\infty}\psi_n(z)/z^n=\overline{D_w^{-1}(1/\bar z)}$ in $|z|>1/\alpha$, $\alpha:=\min_{1\le i\le m}\{|\alpha_i|\}$. Our goal is to use a representation of $\psi_n(z)$ in terms of $\phi_n(z)$ (Theorem 1) in order to find the asymptotic behavior of $\psi_n(z)$ in $|z|>1/\rho$ (Theorem 7). The special case $d\mu=d\theta$ of this asymptotic result has been studied by Ismail and Ruedemann [**4**].

The asymptotic behavior of $\psi_n(z)$ in $|z|<1/\rho$ is discussed in Section 4. The representation of $\psi_n(z)$ is given in Section 2. Section 3 is devoted to some lemmas needed for the proof in Section 4.

2. Comparison of $\psi_n(z)$ and $\phi_n(z)$. Our aim is to investigate the relationship between the orthonormal polynomials $\phi_n(z)$ and $\psi_n(z)$.

The situation is very similar to adding m mass points distribution to the measure μ and comparing the corresponding polynomials. Surprisingly similar results are valid (cf. [**5**]). In this section we derive a representation of $\psi_n(z)$ in terms of $\phi_n(z)$. The determinant representation can be found in [**1, 3, 4**] and [**10**].

Theorem 1. *For $n > m$, we have*

$$w_m(z)\psi_{n-m}(z) = \frac{\eta_{n-m}}{\kappa_n}\phi_n(z) + \sum_{k=1}^{m} A_{n,k}K_{n-1}(z,\alpha_k), \tag{2}$$

and the ratio of the two leading coefficients satisfies

$$\frac{\eta_{n-m}}{\kappa_n} - \frac{\kappa_n}{\eta_{n-m}} = \sum_{k=1}^{m} A_{n,k}\overline{\phi_n(\alpha_k)}, \tag{3}$$

where

$$A_{n,k} := \frac{1}{2\pi}\int_0^{2\pi} w_m(z)\psi_{n-m}(z)\overline{l_k(z)}\, d\mu, \qquad z = e^{i\theta},$$

$l_k(z) := \prod_{i\neq k}(z-\alpha_i)/(\alpha_k - \alpha_i)$, *and*

$$K_n(z,\xi) := \sum_{i=0}^{n} \overline{\phi_i(\xi)}\phi_i(z)$$

is the kernel polynomial associated with the orthonormal polynomials $\phi_n(z)$.

Remark. One can obtain $A_{n,k}$ by letting $z = \alpha_k$, $k = 1, \ldots, m$ in (2), and solve for $A_{n,k}$ in terms of $\phi_n(\alpha_i)$ and $K_{n-1}(\alpha_i, \alpha_j)$, $i, j = 1, \ldots, m$.

Proof of Theorem 1. It is clear that a Fourier expansion

$$w_m(z)\psi_{n-m}(z) = \frac{\eta_{n-m}}{\kappa_n}\phi_n(z) + \sum_{j=0}^{n-1} a_j\phi_j(z) \tag{4}$$

always exists and that the Fourier coefficients are given by

$$a_j = \frac{1}{2\pi}\int_0^{2\pi} w_m(z)\psi_{n-m}(z)\overline{\phi_j(z)}\,d\mu, \qquad z = e^{i\theta},$$

for $j = 0, \dots, n-1$. We now express a_j in terms of $\phi_j(\alpha_k)$, $k = 1, \dots, m$. When $j = 0, \dots, m-1$, we have an expansion

$$\phi_j(z) = \sum_{k=1}^{m} \phi_j(\alpha_k) l_k(z),$$

and so

$$\begin{aligned} a_j &= \frac{1}{2\pi}\int_0^{2\pi} w_m(z)\psi_{n-m}(z)\overline{\phi_j(z)}\,d\mu \\ &= \frac{1}{2\pi}\int_0^{2\pi} w_m(z)\psi_{n-m}(z)\sum_{k=1}^{m}\overline{\phi_j(\alpha_k)l_k(z)}\,d\mu \\ &= \sum_{k=1}^{m} A_{n,k}\overline{\phi_j(\alpha_k)}, \qquad z = e^{i\theta}. \end{aligned} \tag{5}$$

On the other hand, for $j = m, \dots, n-1$, we also have

$$\begin{aligned} a_j &= \frac{1}{2\pi}\int_0^{2\pi} w_m(z)\psi_{n-m}(z)\overline{\phi_j(z)}\,d\mu \\ &= \frac{1}{2\pi}\int_0^{2\pi} |w_m(z)|^2\psi_{n-m}(z)\left[\overline{\frac{\phi_j(z) - \sum_{k=1}^{m}\phi_j(\alpha_k)l_k(z)}{w_m(z)}}\right]d\mu \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} w_m(z)\psi_{n-m}(z)\sum_{k=1}^{m}\overline{\phi_j(\alpha_k)l_k(z)}\,d\mu \\ &= \frac{1}{2\pi}\int_0^{2\pi} w_m(z)\psi_{n-m}(z)\sum_{k=1}^{m}\overline{\phi_j(\alpha_k)l_k(z)}\,d\mu \\ &= \sum_{k=1}^{m} A_{n,k}\overline{\phi_j(\alpha_k)}, \qquad z = e^{i\theta}, \end{aligned} \tag{6}$$

the third equality being true because

$$\left\{\phi_j(z) - \sum_{k=1}^{m}\phi_j(\alpha_k)l_k(z)\right\}/w_m(z) \in \mathcal{P}_{n-m-1},$$

and then use the orthonormality of $\psi_{n-m}(z)$ with respect to $|w_m(z)|^2\,d\mu$. Thus, by (4), (5) and (6), we see that

$$\begin{aligned} w_m(z)\psi_{n-m}(z) &= \frac{\eta_{n-m}}{\kappa_n}\phi_n(z) \\ &\quad + \sum_{j=0}^{n-1}\biggl(\sum_{k=1}^{m} A_{n,k}\overline{\phi_j(\alpha_k)}\biggr)\phi_j(z) \\ &= \frac{\eta_{n-m}}{\kappa_n}\phi_n(z) + \sum_{k=1}^{m} A_{n,k}K_{n-1}(z,\alpha_k). \end{aligned}$$

In order to prove (3), from (4) we have

$$\begin{aligned} \frac{\eta_{n-m}}{\kappa_n} &= \frac{1}{2\pi}\int_0^{2\pi} w_m(z)\psi_{n-m}(z)\overline{\phi_n(z)}\,d\mu \\ &= \frac{1}{2\pi}\int_0^{2\pi} |w_m(z)|^2\psi_{n-m}(z)\left[\overline{\frac{\phi_n(z)-\sum_{k=1}^m \phi_n(\alpha_k)l_k(z)}{w_m(z)}}\right]d\mu \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} w_m(z)\psi_{n-m}(z)\sum_{k=1}^{m}\overline{\phi_n(\alpha_k)l_k(z)}\,d\mu \\ &= \frac{1}{2\pi}\int_0^{2\pi} |w_m(z)|^2\psi_{n-m}(z)\kappa_n z^{m-n}\,d\mu \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} w_m(z)\psi_{n-m}(z)\sum_{k=1}^{m}\overline{\phi_n(\alpha_k)l_k(z)}\,d\mu \\ &= \frac{\kappa_n}{\eta_{n-m}} + \sum_{k=1}^{m} A_{n,k}\overline{\phi_n(\alpha_k)}, \qquad z=e^{i\theta}, \end{aligned}$$

the third equality being valid because

$$\biggl\{\phi_n(z) - \sum_{k=1}^{m}\phi_n(\alpha_k)l_k(z)\biggr\}\Big/ w_m(z) = \kappa_n z^{n-m} + \cdots$$

and then use the orthonormality of $\psi_{n-m}(z)$ with respect to $|w_m(z)|^2\,d\mu$. □

3. Estimation for $A_{n,k}$. In order to estimate $\psi_n(z)$, we would like to obtain an asymptotic expression for $A_{n,k}$. We say $\mu \in N$ if

$$\lim_{n\to\infty}\frac{\phi_n(0)}{\kappa_n} = 0.$$

It is well known that the condition $\mu' > 0$ almost everywhere on the unit circle implies $\mu \in N$ (cf. [**6**] or [**9**]). From now on, we will always assume that α's are all distinct. Let

$$B(z) := \prod_{j=1}^{m} \frac{(z-\alpha_j)}{(1-\bar{\alpha}_j z)}.$$

Since the set $\{(1-\bar{\alpha}_j z)^{-1}\}_{j=1}^{m}$ is linearly independent, we have the unique expansion

$$B(z) = \frac{1}{B(0)} + \sum_{j=1}^{m} \frac{b_j}{1-\bar{\alpha}_j z}. \tag{7}$$

We now discuss the asymptotic behavior for coefficients $A_{n,k}$.

Theorem 2. *If $\mu \in N$, then*

$$\lim_{n\to\infty} A_{n,k}\overline{\phi_n(\alpha_k)} = -\frac{\overline{B(0)}}{|B(0)|} b_k, \qquad k = 1, \dots, m,$$

where b_k, $k = 1, \dots, m$ are defined in (7).

Before we give the proof of the theorem, we need the following lemmas.

Lemma 3. *If $\mu \in N$, then*

$$\lim_{n\to\infty} \frac{K_{n-1}(z,\xi)}{\overline{\phi_n(\xi)}\phi_n(z)} = \frac{1}{\bar{\xi}z - 1},$$

holds locally uniformly for $|z| > 1$ and $|\xi| > 1$.

Proof. By the Christoffel-Darboux formula (cf. [**2**, p. 3]), we have

$$K_{n-1}(z,\xi) = \frac{\overline{\phi_n^*(\xi)}\phi_n^*(z) - \overline{\phi_n(\xi)}\phi_n(z)}{1-\bar{\xi}z}.$$

It is proven in [**7**] that $\mu \in N$ implies

$$\lim_{n\to\infty} \frac{\phi_n^*(z)}{\phi_n(z)} = 0,$$

locally uniformly for $|z| > 1$. The lemma then follows from the Christoffel-Darboux formula. □

Lemma 4. *For points $\alpha_1, \dots, \alpha_m$ outside the unit circle, the following matrix*

$$\mathbf{C}_m := \left(\frac{1}{1-\bar{\alpha}_j\alpha_i}\right)_{i,j=1}^{m}$$

is nonsingular. Furthermore,

$$\mathbf{C}_m^{-1}\mathbf{1} = -\overline{B(0)}(b_1, \dots, b_m)^T,$$

where $\mathbf{1} = \mathbf{1}_m = (1, \dots, 1)^T$, and the b_k's are defined in (7).

Proof. See [**5**]. □

Lemma 5. *For all $n \geq m$, we have $\eta_{n-m}/\kappa_n \leq 1$.*

Proof. From the extremal property of the monic polynomial $\kappa_n^{-1}\phi_n(z)$ (cf. [**11**, p. 289]),

$$\begin{aligned}\frac{1}{\kappa_n^2} &= \min_{p\in\mathcal{P}_{n-1}} \frac{1}{2\pi}\int_0^{2\pi} |z^n + p(z)|^2\, d\mu \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{\psi_{n-m}(z)}{\eta_{n-m}}\right|^2 |w_m(z)|^2\, d\mu \\ &= \frac{1}{\eta_{n-m}^2}. \qquad \square\end{aligned}$$

We are now ready to prove Theorem 2.

Proof of Theorem 2. We first show the existence of $\lim_{n\to\infty} \eta_{n-m}/\kappa_n$ and calculate the value of this limit. By Lemma 5, every subsequence of $\{\eta_{n-m}/\kappa_n\}_{n=m}^{\infty}$ contains a convergent subsequence. Let $r \geq 0$ be a limit point of this sequence, and $\Lambda \subseteq \{m, m+1, \dots\}$ satisfy

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} \eta_{n-m}/\kappa_n = r. \tag{8}$$

From Theorem 1, we know that

$$\begin{aligned} &\frac{w_m(z)\psi_{n-m}(z)}{\phi_n(z)} \\ &\quad = \frac{\eta_{n-m}}{\kappa_n} + \sum_{j=1}^{m} \left\{ A_{n,j}\overline{\phi_n(\alpha_j)}\frac{(1-\bar{\alpha}_j z)K_{n-1}(z,\alpha_j)}{\overline{\phi_n(\alpha_j)}\phi_n(z)} \right\} \frac{1}{1-\bar{\alpha}_j z}. \end{aligned} \tag{9}$$

Setting $z = \alpha_1, \dots, \alpha_m$, by Lemma 3, we can obtain

$$\frac{\eta_{n-m}}{\kappa_n}\mathbf{1} = \mathbf{C}_m[\mathbf{X}(1+o(1))], \qquad n \to \infty,$$

where $\mathbf{C}_m$ is defined as in Lemma 4, $\mathbf{X} := (x_1, x_2, \dots, x_m)^T$, and $x_i := A_{n,i}\overline{\phi_n(\alpha_i)}$, $i = 1, \dots, m$. From Lemma 4, we have

$$\mathbf{X}(1+o(1)) = \frac{\eta_{n-m}}{\kappa_n}\mathbf{C}_m^{-1}\mathbf{1} = -\frac{\eta_{n-m}}{\kappa_n}\overline{B(0)}(b_1, \dots, b_m)^T,$$

and so, by use of (8),

$$\begin{aligned} \lim_{\substack{n\to\infty \\ n\in\Lambda}} x_j = \lim_{\substack{n\to\infty \\ n\in\Lambda}} A_{n,j}\overline{\phi_n(\alpha_j)} = -r\overline{B(0)}b_j, \\ j = 1, \dots, m. \end{aligned} \tag{10}$$

Now let $n \to \infty$ and $n \in \Lambda$ in (3) yield

$$r - \frac{1}{r} = -r\overline{B(0)}\sum_{j=1}^{m} b_j = -r\overline{B(0)}[B(0) - 1/\overline{B(0)}].$$

Hence, $r = |B(0)|^{-1}$. Since r is an arbitrary limit point of $\{\eta_{n-m}/\kappa_n\}$, we see that the $\lim_{n\to\infty} \eta_{n-m}/\kappa_n$ must exist and be equal to $|B(0)|^{-1}$. Finally from (10) we see that

$$\lim_{n\to\infty} A_{n,k}\overline{\phi_n(\alpha_k)} = -\frac{\overline{B(0)}}{|B(0)|}b_k, \qquad k = 1, \dots, m.$$

This completes the proof of the theorem. □

The following result is a special case of theorems in [**7**].

Corollary 6. *If $\mu \in N$, then*

$$\lim_{n\to\infty} \frac{\psi_{n-m}(z)}{\phi_n(z)} = \frac{\overline{B(0)}}{|B(0)|} \frac{1}{\prod_{j=1}^m (1-\bar{\alpha}_j z)}$$

holds locally uniformly in $|z| > 1$.

Proof. The proof follows from (9) in the proof of Theorem 2, Theorem 2, Lemma 3 and the expansion of $B(z)$ in (7). □

4. Asymptotic behavior outside the analytic region. We now have all results needed for proving the asymptotic behavior of $\psi_n(z)$ in $|z| > 1/\rho$.

Theorem 7. *Suppose $D^{-1}(z)$ has an analytic continuation to the disk $|z| < \rho$, and $\rho > \max_{1\le i\le m}\{|\alpha_i|\}$. Then*

$$\psi_{n-m}(z)w_m(z) = \frac{\overline{B(0)}}{|B(0)|}\left[\frac{B(z)z^n}{\overline{\overline{D(1/z)}}} - \sum_{j=1}^m \frac{b_j}{(1-\bar{\alpha}_j z)} \frac{D^2(1/\bar{\alpha}_j)}{D(z)} \frac{1}{\bar{\alpha}_j^n}\right] + o(\alpha_z^n)$$

holds locally uniformly for $1/\rho < |z| < \rho$, where $\alpha_z := \max\{|z|, 1/\alpha\}$ and $\alpha := \min_{1\le i\le m}\{|\alpha_i|\}$.

Proof of Theorem 7. From Theorem 1, we obtain that

$$w_m(z)\psi_{n-m}(z) = \frac{\eta_{n-m}}{\kappa_n}\phi_n(z) + \sum_{k=1}^m A_{n,k}K_{n-1}(z,\alpha_k). \tag{11}$$

Next we want to estimate the righthand side of (11). First from the theorems in [**8**], we have

$$\lim_{n\to\infty} \frac{\phi_n(z)}{z^n} = \overline{D^{-1}(1/\bar{z})} \tag{12}$$

locally uniformly for $|z| > 1/\rho$. Thus, for $1/\rho < |z|$, we see that

$$\phi_n(z) = z^n \overline{D^{-1}(1/\bar{z})} + o(|z|^n). \tag{13}$$

On the other hand, from (1), we have

$$\phi_n^*(z) = D^{-1}(z) + o(1) \tag{14}$$

for $|z| < \rho$, and so

$$\phi_n^*(\alpha_k) = D^{-1}(\alpha_k) + o(1) \tag{15}$$

for $k = 1, \dots, m$. Also from Theorem 2 and (13), we have

$$A_{n,k} = -\frac{\overline{B(0)}}{|B(0)|}\frac{b_k D(1/\bar{\alpha}_k)}{\bar{\alpha}_k^n} + o\left(\frac{1}{|\alpha_k|^n}\right) \tag{16}$$

for $k = 1, 2, \dots, m$. According to the proof of Theorem 2, we see that

$$\frac{\eta_{n-m}}{\kappa_n} = \frac{1}{|B(0)|} + o(1). \tag{17}$$

From Christoffel-Darboux formula, (11), (13), (14), (15), (16) and (17) we have, for $|z| > 1/\rho$,

$$\begin{aligned}
w_m(z)\psi_{n-m}(z) &= \frac{\eta_{n-m}}{\kappa_n}\phi_n(z) \\
&\quad + \sum_{k=1}^m A_{n,k}\frac{\overline{\phi_n^*(\alpha_k)}\phi_n^*(z) - \overline{\phi_n(\alpha_k)}\phi_n(z)}{1 - \bar{\alpha}_k z} \\
&= \left[\frac{\eta_{n-m}}{\kappa_n} - \sum_{k=1}^m \frac{A_{n,k}\overline{\phi_n(\alpha_k)}}{1 - \bar{\alpha}_k z}\right]\phi_n(z) \\
&\quad + \left[\sum_{k=1}^m \frac{A_{n,k}\overline{\phi_n^*(\alpha_k)}}{1 - \bar{\alpha}_k z}\right]\phi_n^*(z) \\
&= \frac{\overline{B(0)}}{|B(0)|}\left[\frac{1}{\overline{B(0)}} + \sum_{k=1}^m \frac{b_k}{1 - \bar{\alpha}_k z}\right]\frac{z^n}{\overline{D(1/\bar{z})}} \\
&\quad - \frac{\overline{B(0)}}{|B(0)|}\sum_{k=1}^m \frac{b_j D(1/\bar{\alpha}_k)}{(1 - \bar{\alpha}_k z)\bar{\alpha}_k^n \overline{D(\alpha_k)} D(z)} \\
&\quad + o(|z|^n) + o(\alpha^{-n}) \\
&= \frac{\overline{B(0)}}{|B(0)|}\left[\frac{B(z) z^n}{\overline{D(1/\bar{z})}}\right. \\
&\qquad \left. - \sum_{j=1}^m \frac{b_j D(1/\bar{\alpha}_j)}{(1 - \bar{\alpha}_j z)\bar{\alpha}_j^n \overline{D(\alpha_j)} D(z)}\right] + o(\alpha_z^n),
\end{aligned}$$

the last equality being valid because of the expansion of $B(z)$ in (7). Together with $\overline{D(\alpha_j)} = D^{-1}(1/\bar{\alpha}_j)$, $j = 1, \dots, m$, we complete the proof of the theorem. □

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