ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 1, Winter 1996

ENTROPY VALUES OF CHAINS OF PARTITIONS OF INFINITE COUNTABLE SETS

SERVET MARTÍNEZ AND JAIME SAN MARTÍN

ABSTRACT. We study the structure of the chains of partitions Γ of countable sets. Our main result asserts that for any chain Γ and any probability measure μ the set of entropy values $h_\mu(\Gamma) = \{h_\mu(\alpha) : \alpha \in \Gamma\}$ is a totally disconnected set in $\mathbf{R}_+ \cup \{+\infty\}$ with null-Lebesgue measure. The complexity of the set $h_\mu(\Gamma)$ is exhibited in an example where $h_\mu(\Gamma)$ is a Cantor set with non null Hausdorff dimension.

0. Introduction. Let Ω be a countable set. In this work we deal with chains of partitions of Ω, i.e., totally ordered sets of partitions with respect to the relation "be finer than." Our main result is Theorem 2.1. There we show that for any probability measure μ on Ω and any chain of partitions Γ the set of entropy values $h_\mu(\Gamma)$ is of null Lebesgue measure.

For proving this result we made in Section 1 a previous study about the topological structure of the chains of partitions. In this context we define a topology on the whole set of partitions by means of the inf and sup operations. In Theorem 1.1 we prove that any chain is a totally disconnected set and a closed chain is also a compact metric space. These properties are transferred to the set of entropy values.

Our main theorem is obviously trivial if Ω is finite. But for Ω countably infinite this is not the case; in fact, there exist chains of partitions with the cardinality of the continuum. For showing the complexity of the sets dealt with we exhibit in Section 3 an example where $h_\mu(\Omega)$ is a Cantor set with strictly positive Hausdorff dimension.

1. Topology on chains of partitions. Let $\Omega = \{m\}$ be a nonempty countable set. To avoid trivial situations we assume it is not a singleton. We note by $\mathcal{A}(\Omega)$ (or simply by $\mathcal{A}$) the set of all partitions on Ω. For any partition $\alpha \in \mathcal{A}$ we write $\alpha = \{A\}$, each

Received by the editors on May 31, 1991.
Partially financed by FONDECYT grant 0553/88-90, 1237/90-92.

Copyright ©1996 Rocky Mountain Mathematics Consortium

$A \in \alpha$ being an atom of the partition. If $m \in \Omega$ we note by $A_\alpha(m)$ the atom of α which contains m. The set of partitions $\mathcal{A}$ is partially ordered by the relation "to be finer than," i.e., $\alpha \le \alpha'$ if and only if $A_\alpha(m) \supset A_{\alpha'}(m)$ for all $m \in \Omega$. By $\mathcal{N}(\Omega)$ and $\mathcal{W}(\Omega)$ (or simply $\mathcal{N}$ and $\mathcal{W}$) we note respectively the trivial and the discrete partition on Ω: $\mathcal{A}_\mathcal{N}(m) = \Omega$, $A_\mathcal{W}(m) = \{m\}$ for any $m \in \Omega$.

A subset $\Gamma \subset \mathcal{A}$ is called a chain if and only if it is totally ordered, i.e., any pair $\alpha, \alpha' \in \Gamma$ verifies the relation $[\alpha \le \alpha'$ or $\alpha' \le \alpha]$. If Γ is a chain the atoms of the partitions $\sup \Gamma$ and $\inf \Gamma$ are given by $A_{\sup \Gamma}(m) = \cap_{\alpha \in \Gamma} A_\alpha(m)$ and $A_{\inf \Gamma}(m) = \cup_{\alpha \in \Gamma} A_\alpha(m)$.

Let us define a topology on $\mathcal{A}$. A subset $\zeta \subset \mathcal{A}$ is closed if for any nonempty chain $\Gamma \subset \zeta$ the partitions $\sup \Gamma$ and $\inf \Gamma$ belong to ζ. It is easy to show that intersection and finite union of closed sets are also closed and that the sets $\phi, \mathcal{A}$ are closed. Then they generate a topology which we denote by $\mathcal{T}(\mathcal{A})$ (or simply by $\mathcal{T}$). This topology $\mathcal{T}$ will be the unique one that we shall consider on $\mathcal{A}$. If $\mathcal{B} \subset \mathcal{A}$ we denote by $\mathcal{T}(\mathcal{B})$ the topology induced by $\mathcal{T}$ on $\mathcal{B}$.

Remark that a chain Γ is closed if and only if for any nonempty $\mathcal{H} \subset \Gamma : \sup \mathcal{H} \in \Gamma$ and $\inf \mathcal{H} \in \Gamma$. From Zorn's lemma any chain is contained in a maximal chain which, by maximality, is a closed chain. So we can define $\hat{\Gamma}$ to be the smallest closed chain containing Γ. Then the (topological) closure $\bar{\Gamma}$ of Γ is a subset of $\hat{\Gamma}$, so it is also a chain. We conclude that $\bar{\Gamma} = \hat{\Gamma}$ so $\bar{\Gamma}$ is the smallest closed chain containing Γ.

Let Γ be a closed chain and $\alpha \in \Gamma$. Define $\alpha^- = \sup\{\alpha' \in \Gamma : \alpha' < \alpha\}$ if $\alpha \ne \inf \Gamma$, $\alpha^+ = \inf\{\alpha' \in \Gamma : \alpha' > \alpha\}$ if $\alpha \ne \sup \Gamma$, $(\inf \Gamma)^- = \inf \Gamma$, $(\sup \Gamma)^+ = \sup \Gamma$. Since Γ is a closed chain the partitions α^- and α^+ belong to Γ. Remark that the operations α^-, α^+ depend not only on α but also on the chain Γ. The discontinuity sets are:

$$\Gamma^- = \{\alpha \in \Gamma : \alpha < \alpha^+\}, \qquad \Gamma^+ = \{\alpha \in \Gamma : \alpha^- < \alpha\}.$$

It is easy to see that if $\alpha \in \Gamma^-$ then $\alpha^+ \in \Gamma^+$ and $(\alpha^+)^- = \alpha$, analogously if $\alpha \in \Gamma^+$ then $\alpha^- \in \Gamma^-$ and $(\alpha^-)^+ = \alpha$.

The sets $\Gamma^-(m) = \{\alpha \in \Gamma^- : A_\alpha(m) \ne A_{\alpha^+}(m)\}$ are countable. Then the equalities $\Gamma^- = \cup_{m \in \Omega} \Gamma^-(m)$, $\Gamma^+ = \cup_{m \in \Gamma} \Gamma^+(m)$ imply that Γ^- and Γ^+ are also countable.

If Γ is a maximal chain then $\inf \Gamma = \mathcal{N}$, $\sup \Gamma = \mathcal{W}$ and the set Γ^-

satisfies

(1.1)

$$\forall\, \alpha \in \Gamma^{-} \exists! A^* \in \alpha \quad \text{such that} \quad \alpha^{+} = \{A \in \alpha : A \neq A^*\} \cup \{A_1^*, A_2^*\}$$

where $\{A_1^*, A_2^*\}$ is a nontrivial partition of the atom A^*.

Reciprocally, it is easy to show that any closed chain Γ whose Γ^- verifies (1.1) is maximal.

Lemma 1.1. *Let Γ be a chain. Then it is a totally disconnected set with respect to $\mathcal{T}(\Gamma)$.*

Proof. Let $\alpha' \leq \alpha''$. It is easy to show that the interval $[\alpha', \alpha''] = \{\alpha \in \mathcal{A} : \alpha' \leq \alpha \leq \alpha''\}$ is closed in $\mathcal{A}$. Then, for any subset $\mathcal{B} \subset \mathcal{A}$ the interval $[\alpha', \alpha'']_{\mathcal{B}} = [\alpha', \alpha''] \cap \mathcal{B}$ is closed in $\mathcal{T}(\mathcal{B})$. By elementary rules we deduce that $[\mathcal{N}, \alpha'')_\Gamma$, $(\alpha', \mathcal{W}]_\Gamma$, $(\alpha', \alpha'')_\Gamma$ are open sets in Γ. Now, take $\alpha' < \alpha''$ in Γ. Let $m \in \Omega$ be such that $A_{\alpha'}(m) \neq A_{\alpha''}(m)$ and $m' \in A_{\alpha'}(m) \backslash A_{\alpha''}(m)$. Define $\mathcal{H} = \{\alpha \in \Gamma : A_\alpha(m') = A_\alpha(m')\}$. The partitions $\alpha_1 = \sup \mathcal{H}$, $\alpha_2 = \inf(\Gamma \backslash \mathcal{H})$ belong to $\bar{\Gamma}$ and verify $\alpha_2 = \alpha_1^+$ (+ with respect to $\bar{\Gamma}$). It is easy to show that $\{[\mathcal{N}, \alpha_2)_\Gamma, (\alpha_1, \mathcal{W}|_\Gamma\}$ is an open partition of Γ. Hence the connected components of Γ are singletons. □

Now consider $\zeta \subset \mathcal{A}$ a closed set. A mapping $\Psi : \zeta \to \mathbf{R}$ is monotone continuous if for any chain $\Gamma \subset \zeta : \sup \Psi(\Gamma) = \Psi(\sup \Gamma)$ and $\inf \Psi(\Gamma) = \Psi(\inf \Gamma)$. It is easy to prove that Ψ is monotone continuous if and only if it is a continuous increasing mapping.

The set of increasing continuous mappings $\Psi : \mathcal{A}(\Omega) \to \mathbf{R}$ is nonempty (consider the entropy functional for a strictly positive probability measure of finite total entropy). By canonical arguments we can prove that any nonempty chain Γ contains two sequences $(\alpha_n)_{n \in \mathbf{N}}$ and $(\alpha'_n)_{n \in \mathbf{N}}$, strictly increasing and decreasing respectively such that $\sup \Gamma = \sup_{n \in \mathbf{N}} \alpha_n$, $\inf \Gamma = \inf_{n \in \mathbf{N}} \alpha'_n$. So, an increasing function $\Psi : \Gamma \to \mathbf{R}$ defined on a closed chain Γ is continuous if and only if for any monotone sequence $(\alpha_n)_{n \in \mathbf{N}} \subset \Gamma$, $\Psi(\lim_{n \to \infty} \alpha_n) = \lim_{n \to \infty} \Psi(\alpha_n)$.

Lemma 1.2. *Let $\Gamma \subset \mathcal{A}$ be a chain and $\Psi : \bar{\Gamma} \to \mathbf{R}$ a strictly increasing continuous mapping. Define $d_\Psi(\alpha', \alpha'') = \Psi(\sup(\alpha', \alpha'')) -$*

$\Psi(inf(\alpha', \alpha''))$. Then d_Ψ is a metric whose topology on Γ is the same as $\mathcal{T}(\Gamma)$ and the metric space (Γ, d_Ψ) is totally bounded. Moreover, the restriction $\Psi : \Gamma \to \mathbf{R}$ is a metric order preserving homeomorphism between Γ and $\Psi(\Gamma)$ which send the metric d_Ψ on the Euclidean distance $|\cdot|$.

Proof. Ψ is strictly increasing so d_Ψ is a metric. Denote $B_\Psi(\alpha, \varepsilon) = \{\alpha' \in \Gamma : d_\Psi(\alpha, \alpha') < \varepsilon\}$. We have $[\mathcal{N}, \alpha)_\Gamma = B_\Psi(\inf \bar{\Gamma}, \Psi(\alpha) - \Psi(\inf \bar{\Gamma}))$, $(\alpha, \mathcal{W}]_\Gamma = B_\Psi(\sup \bar{\Gamma}, \Psi(\sup \bar{\Gamma}) - \Psi(\alpha))$. It is easy to show that the family $\{[\mathcal{N}, \alpha'')_\Gamma, (\alpha', \mathcal{W}]_\Gamma : \alpha'' > \mathcal{N}, \alpha' < \mathcal{W}\}$ is a sub-basis of open sets on Γ, then $\mathcal{T}(\Gamma)$ is contained in the topology generated by d_Ψ.

Reciprocally, take $\tilde{\alpha} \in \Gamma \backslash \{\mathcal{N}, \mathcal{W}\}$ and consider $B = B_\Psi(\tilde{\alpha}, \varepsilon)$. If there exists $\alpha_1 < \tilde{\alpha}$ (respectively, $\alpha_2 > \tilde{\alpha}$) in B set $\alpha' = \alpha_1$ (respectively, $\alpha'' = \alpha_2$); if not, $\alpha' = \sup[\mathcal{N}, \tilde{\alpha})_\Gamma$ (respectively $\alpha'' = \inf(\tilde{\alpha}, \mathcal{W})_\Gamma$). By construction, $\tilde{\alpha} \in (\alpha', \alpha'')_\Gamma \subset B$. Analogously, we show $[\mathcal{N}, \alpha'')_\Gamma \subset B$ and $(\alpha', \mathcal{W}]_\Gamma \subset B$. Then the topology generated by d_Ψ on Γ is $\mathcal{T}(\Gamma)$.

For proving Γ is d_Ψ-totally bounded, take $0 < \varepsilon < 1/2$ and α_n a partition in Γ such that $\Psi(\alpha_n) \in (n\varepsilon/2 - \varepsilon/3, n\varepsilon/2 + \varepsilon/3)$. Let N be the integer part of $(2(\Psi(\sup(\bar{\Gamma})) - \Psi(\inf(\bar{\Gamma}))))/\varepsilon$. The set $\Gamma_\varepsilon = \{\alpha_n \in \Gamma : n = -(N+1), \dots, N+1\}$ verifies $d_\Psi(\alpha, \Gamma_\varepsilon) \leq \varepsilon$ for any $\alpha \in \Gamma$.

The last statement of the lemma follows directly. □

Theorem 1.1. *Let Γ be a closed chain and $\Psi : \Gamma \to \mathbf{R}$ strictly increasing and continuous. Then the metric d_Ψ induced by Ψ generates the topology $\mathcal{T}(\Gamma)$. The metric space (Γ, d_Ψ) is a compact totally disconnected set and Ψ is an order and a metric preserving homeomorphism between (Γ, d_Ψ) and $(\Psi(\Gamma), |\cdot|)$.*

Proof. By Lemmas 1.1 and 1.2, the only thing left to prove is that (Γ, d_Ψ) is complete. Let $(\alpha_n)_{n\geq 1} \subset \Gamma$ be a d_Ψ-Cauchy sequence. Take $\tilde{\alpha} = \inf_{k\geq 1}(\sup_{n\geq k} \alpha_n)$. Since Γ is closed, we get $\tilde{\alpha} \in \Gamma$. Also it is easy to construct a subsequence $(\alpha_{n_j}) \subset (\alpha_n)$ such that $\alpha_{n_j} \to_{j\to\infty} \tilde{\alpha}$ in d_Ψ. Hence Γ is d_Ψ-complete. Since Γ is d_Ψ-totally bounded it is d_Ψ-compact. □

A Cantor set is a compact totally disconnected perfect metric space. Let Γ be a closed chain and $\Psi : \Gamma \to \mathbf{R}$ be continuous strictly increasing. The set of isolated points is $(\Gamma^- \cup \{\sup \Gamma\}) \cap (\Gamma^+ \cup \{\inf \Gamma\})$, then the space (Γ, d_Ψ) is a Cantor set if and only if $(\Gamma^- \cup \{\sup \Gamma\}) \cap (\Gamma^+ \cup \{\inf \Gamma\}) = \phi$.

2. Main result. Let X be a compact totally disconnected real set. Set $a = \inf X$, $b = \sup X$. For $x \in X$ define $x^- = \{x' \in X : x' < x\}$ if $x \neq a$, $x^+ = \inf \{x' \in X : x' > x\}$ if $x \neq b$, $a^- = a$, $b^+ = b$. Since X is a closed set the points x^-, x^+ belong to X. Denote:

$$X^- = \{x \in X : x < x^+\}, \qquad X^+ = \{x \in X : x^- < x\}.$$

If $x \in X^-$ the open set (x, x^+) is called a gap of X. The set X is closed so $[a, b] \backslash X = \cup_{x \in X^-} (x, x^+)$. On the other hand, X is a totally disconnected set so for $x_1 < x_2 \in X$ there exists $x \in X^-$ such that $(x, x^+) \subset [x_1, x_2]$. Then $\cup_{x \in X^+} (x, x^+)$ is dense everywhere in $[a, b]$. The set of isolated points is $x \in (X^- \cup \{b\}) \cap (X^+ \cup \{a\})$.

We can characterize the compact totally disconnected real sets X of null Lebesgue measure, $\lambda(X) = 0$ in terms of the length of their gaps. In fact we have the equivalence

$$\lambda(X) = 0 \quad \Longleftrightarrow \quad (x'' - x') = \sum_{x \in X^- \cap [x', x'')} (x^+ - x) \quad \text{for any } x' \leq x'' \text{ in } X. \tag{2.1}$$

Let Γ be a closed chain and $\Psi : \Gamma \to \mathbf{R}$ a strictly increasing continuous mapping. From Theorem 1.1 we deduce $\Psi(\Gamma^-) = (\Psi(\Gamma))^-$, $\Psi(\Gamma^+) = (\Psi(\Gamma))^+$. So (2.1) implies

$$\lambda(\Psi(\Gamma)) = 0 \quad \Longleftrightarrow \quad \forall\, \alpha' < \alpha'' \text{ in } \Gamma : \quad \Psi(\alpha'') - \Psi(\alpha') = \sum_{\alpha \in \Gamma^- \cap [\alpha', \alpha'')} (\Psi(\alpha^+) - \Psi(\alpha)) \tag{2.2}$$

Denote by $\mathbf{1}_A$ the characteristic function of the set A.

Lemma 2.1. *Let Γ be a closed chain. For $\alpha' < \alpha''$ in Γ and any $m \in \Omega$ we have*

$$\mathbf{1}_{A_{\alpha'}(m)} - \mathbf{1}_{A_{\alpha''}(m)} = \sum_{\alpha \in \Gamma^- \cap [\alpha', \alpha'')} (\mathbf{1}_{A_\alpha(m)} - \mathbf{1}_{A_{\alpha^+}(m)}) \tag{2.3}$$

Proof. If $m' \in A_{\alpha''}(m)$ or $m' \notin A_{\alpha'}(m)$ both functions of (2.3) evaluated at m' are equal to 0. Then suppose $m' \in A_{\alpha'}(m)\backslash A_{\alpha''}(m)$. Let $\mathcal{H} = \{\alpha \in \Gamma : A_\alpha(m') = A_\alpha(m)\}$. The partitions $\alpha_1 = \sup \mathcal{H}$, $\alpha_2 = \inf(\Gamma\backslash\mathcal{H})$ belong to Γ. For any $\alpha_1' \in \mathcal{H}$, $\alpha_2' \in \Gamma\backslash\mathcal{H}$ we have $\alpha_1' < \alpha_2'$ then $\alpha_1 \le \alpha_2$, but $A_{\alpha_1}(m) = A_{\alpha_1}(m')$, $A_{\alpha_2}(m) \neq A_{\alpha_2}(m')$ so $\alpha_1 < \alpha_2$. By construction $\alpha_2 = \alpha_1^+$ so $\alpha_1 \in \Gamma^- \cap [\alpha', \alpha'')$.

Let $\alpha \in \Gamma^-$, $\alpha' \le \alpha < \alpha''$. If $\alpha < \alpha_1$ or $\alpha > \alpha_1$ we have $\mathbf{1}_{A_\alpha(m)}(m') = \mathbf{1}_{A_{\alpha^+}(m)}(m')$. For $\alpha = \alpha_1$ we get: $\mathbf{1}_{A_\alpha(m)}(m') - \mathbf{1}_{A_{\alpha^+}(m)}(m') = 1$. Then (2.3) is verified. □

Let μ be a probability measure on Ω and Γ a closed chain. For $\alpha' \le \alpha''$ in Γ and $m \in \Omega$ define the interval $I(m, \alpha', \alpha'') = (\mu(A_{\alpha''}(m)), \mu(A_{\alpha'}(m)))$. Then $I(m, \alpha', \alpha'')$ increases when α' decreases or α'' increases. If $A_{\alpha'}(m) = A_{\alpha''}(m)$ the set $I(m, \alpha', \alpha'')$ is empty. On the other hand, when $\alpha' \le \alpha'' \le \alpha_1' \le \alpha_1''$ the intervals $I(m, \alpha', \alpha'')$ and $I(m, \alpha_1', \alpha_1'')$ are disjoint for any $m \in \Omega$. Hence, the intervals $\{I(m, \alpha, \alpha^+) : \alpha \in \Gamma^- \cap [\alpha', \alpha'')\}$ are disjoint and all of them are contained in $I(m, \alpha', \alpha'')$.

Lemma 2.2. *Let μ be a probability measure with support equal to Ω and Γ be a closed chain. Then for any $\alpha' \le \alpha''$ in Γ and $m \in \Omega$ we have*

$$\lambda\left(I(m, \alpha', \alpha'') - \bigcup_{\alpha \in \Gamma^- \cap [\alpha', \alpha'')} I(m, \alpha, \alpha^+)\right) = 0. \tag{2.4}$$

So $I(m, \alpha', \alpha'') = \cup_{\alpha \in \Gamma \cap [\alpha', \alpha'')} I(m, \alpha, \alpha^+)$ λ almost everywhere.

Proof. Sum the equality of the functions (2.3) of Lemma 2.1 with respect to μ. Since the countable class of intervals $\{I(m, \alpha, \alpha^+) : \alpha \in \Gamma^- \cap [\alpha', \alpha'')\}$ is disjoint, we get $\mu(A_{\alpha'}(m)) - \mu(A_{\alpha''}(m)) = \sum_{\alpha \in \Gamma^- \cap [\alpha', \alpha'')} \mu(A_\alpha(m)) - \mu(A_{\alpha^+}(m))$. So (2.4) holds. □

Theorem 2.1. *Let Ω be a countable set. Then for any chain of partitions $\Gamma \subset \mathcal{A}(\Omega)$ and any probability measure μ on Ω the set $h_\mu(\Gamma)$ is a totally disconnected set with null Lebesgue measure. If Γ is also closed then $h_\mu(\Gamma)$ is compact in $\mathbf{R}_+ \cup \{+\infty\}$.*

Proof. Let us prove that $h_\mu(\Gamma)$ is of null Lebesgue measure. Obviously it suffices to show the result for Γ a closed chain and a probability measure with support equal to Ω. For any $N > 0$ consider $\Gamma^{(N)} = \{\alpha \in \Gamma : h_{\mu(\alpha)}(\Gamma) \le N\}$. Then h_μ restricted to the chain $\Gamma^{(N)}$ is a strictly increasing continuous real function. According to (2.2) the condition $\lambda(h_\mu(\Gamma^{(N)})) = 0$ is equivalent to

$$\begin{aligned} \forall\, \alpha' < \alpha'' \quad \text{in} \quad \Gamma^{(N)} : h_\mu(\alpha'') - h_\mu(\alpha') \\ = \sum_{\alpha \in (\Gamma^{(N)})^- \cap [\alpha', \alpha'')} (h_\mu(\alpha^+) - h_\mu(\alpha)). \end{aligned} \tag{2.5}$$

Let us prove (2.5). We have

$$\begin{aligned} h_\mu(\alpha'') - h_\mu(\alpha') &= \sum_{m \in \Omega} \mu(m)(\log \mu(A_{\alpha'}(m)) - \log \mu(A_{\alpha''}(m))) \\ &= \sum_{m \in \Omega} \mu(m) \int_{0^+}^1 \frac{1}{u} \mathbf{1}_{I(m,\alpha',\alpha'')}(u)\, d\lambda(u) \\ &= \sum_{m \in \Omega} \mu(m) \sum_{\alpha \in (\Gamma^{(N)}) - \cap [\alpha', \alpha'')} \int_{0^+}^1 \frac{1}{u} \mathbf{1}_{I(m,\alpha,\alpha^+)}(u)\, d\lambda(u). \end{aligned}$$

Since $h_\mu(\alpha^+) - h_\mu(\alpha) = \sum_{m \in \Omega} \mu(m) \int_{0^+}^1 (1/u) \mathbf{1}_{I(m,\alpha,\alpha^+)}(u)\, d\lambda(u)$ the equality (2.5) is shown.

Then if $\sup h_\mu(\Gamma) < \infty$ the measure property of $h_\mu(\Gamma)$ is proved. If $\sup h_\mu(\Gamma) = +\infty$, the equalities $h_\mu(\Gamma) = \lim_{N \to \infty} h_\mu(\Gamma^{(N)}) \cup \{+\infty\}$, $\lambda\{+\infty\} = 0$, imply $\lambda(h_\mu(\Gamma)) = 0$.

Let us show the topological properties asserted in the theorem. If $h_\mu(\sup \Gamma) < +\infty$ they are direct consequences of Theorem 1.1 because $h_\mu : \Gamma \to \mathbf{R}$ is continuous strictly increasing. Assume that $h_\mu(\sup \Gamma) = +\infty$. Let $\mathcal{H} = \{\alpha \in \Gamma : h_\mu(\alpha) < \infty\}$. Assume $\mathcal{H} \ne \phi$; otherwise the result is trivial. Set $\tilde{\alpha} = \sup \mathcal{H}$, so $\overline{\mathcal{H}} = \mathcal{H} \cup \{\tilde{\alpha}\}$. If $h_\mu(\tilde{\alpha}) < \infty$, then $h_\mu(\Gamma) = h_\mu(\overline{\mathcal{H}}) \cup \{+\infty\}$ and the result holds because $h_\mu(\overline{\mathcal{H}})$ is a compact totally disconnected set. So, assume that $h_\mu(\tilde{\alpha}) = +\infty$. In this case $h_\mu(\Gamma) = h_\mu(\overline{\mathcal{H}})$. The restriction $h_\mu : \overline{\mathcal{H}} \to \mathbf{R}_+ \cup \{+\infty\}$ is strictly increasing because the only partition in $\overline{\mathcal{H}}$ having entropy value $+\infty$ is $\tilde{\alpha}$. We can show that h_μ is a homeomorphism between $\overline{\mathcal{H}}$ and $h_\mu(\overline{\mathcal{H}})$, so the latter is a compact and totally disconnected set in $\mathbf{R}_+ \cup \{+\infty\}$. $\square$

Let μ be a strictly positive probability measure Ω and ϕ, Θ be strictly positive functions defined on $(0,1]$. Also assume that Θ is an absolutely continuous function with $d\Theta/du < 0$ in $(0,1]$. For $\alpha \in \mathcal{A}$ define

$$\Psi(\alpha) = \sum_{m\in\Omega} \phi(\mu(m))\Theta(\mu(A_\alpha(m))) \in [0,+\infty]. \tag{2.6}$$

Remark. This functional extends the entropy. In fact it suffices to take ϕ the identity and $\Theta(u) = -\log u$ in $(0,1]$.

The analog of Theorem 2.1 is

Theorem 2.2. *For any chain Γ the set $\Psi(\Gamma)$ is a totally disconnected set with null Lebesgue measure. If Γ is also closed, then $\Psi(\Gamma)$ is compact in $\mathbf{R} \cup \{+\infty\}$.*

Now let us show that the above class of mappings Ψ include some interesting functionals, in particular the following ones:

$$\Psi_q(\alpha) = \sum_{m\in\Omega} ||\mathbf{E}_\mu^\alpha \mathbf{1}_{\{m\}}||_q^q \quad \text{for } q > 1, \tag{2.7}$$

where $\mathbf{E}_\mu^\alpha$ is the mean expected value operator with respect to the σ-field generated by the partition α and $||\ ||_q$ is the q-norm. In fact, $\mathbf{E}_\mu^\alpha \mathbf{1}_{\{m\}} = (\mu(m)/\mu(A_\alpha(m)))1_{A_\alpha}(m)$. So $\Psi_q(a) = \sum_{m\in\Omega} \mu(m)^q \times (\mu(A_\alpha(m)))^{-(q-1)}$ for $q > 1$. Then it suffices to take in (2.6) $\phi_q(u) = u^q$, $\Theta(u) = u^{-(q-1)}$ to get the desired form.

Remark. The inequality $\mu(m)^q(\mu(A_\alpha(m)))^{-(q-1)} \le \mu(m)$ for $q > 1$ implies that $\Psi_q(\alpha)$ takes only real values.

Then, for any chain Γ and any $q > 1$ the set $\Psi_q(\Gamma)$ is a totally disconnected real set of null Lebesgue measure, and if Γ is closed, then $\Psi_q(\Gamma)$ is compact.

Now we shall analyze some consequences of Theorem 2.2 for the special case $q = 2$. In particular, we shall get some new information on the structure of maximal chains of partitions. This is related to problems set in [**1**].

Proposition 2.1. *Let Γ be a maximal chain of partitions and μ any probability measure with support μ. Then $\mathbf{E}_\mu^{\mathcal{W}} = \mathbf{E}_\mu^{\mathcal{N}} + \sum_{\alpha\in\Gamma^-}(\mathbf{E}_\mu^{\alpha^+} - \mathbf{E}_\mu^{\alpha})$.*

If Γ' is another maximal chain commuting with Γ, i.e., for all $\alpha' \in \Gamma'$, $\alpha \in \Gamma : \mathbf{E}_\mu^{\alpha'}\mathbf{E}_\mu^{\alpha} = \mathbf{E}_\mu^{\alpha}\mathbf{E}_\mu^{\alpha'}$, then for all $\alpha' \in \Gamma'^-$ there exists an $\alpha \in \Gamma^-$ such that $\mathbf{E}_\mu^{\alpha^+} - \mathbf{E}_\mu^{\alpha} = \mathbf{E}_\mu^{\alpha'+} - \mathbf{E}_\mu^{\alpha'}$.

Proof. From Theorem 2.2, $\Psi_2(\mathcal{W}) = \Psi_2(\mathcal{N}) + \sum_{\alpha\in\Gamma^-}(\Psi_2(\alpha^+) - \Psi_2(\alpha))$. So $\sum_{m\in\Omega} ||1_{\{m\}}||_2^2 = \sum_{m\in\Omega} ||\mathbf{E}_\mu^{\mathcal{N}} 1_{\{m\}}||_2^2 + \sum_{\alpha\in\Gamma^-}\sum_{m\in\Omega} ||(\mathbf{E}_\mu^{\alpha^+} - \mathbf{E}_\mu^{\alpha})1_{\{m\}}||_2^2$. From the orthogonality of the family of projections $\{\mathbf{E}_\mu^{\mathcal{N}}, \mathbf{E}_\mu^{\alpha^+} - \mathbf{E}_\mu^{\alpha}\}$ we get $||1_{\{m\}}||_2^2 = ||\mathbf{E}_\mu^{\mathcal{N}} 1_{\{m\}}||_2^2 + \sum_{\alpha\in\Gamma^-} ||(\mathbf{E}_\mu^{\alpha_2^+} - \mathbf{E}_\mu^{\alpha_2})1_{\{m\}}||_2^2$. Then $\mathbf{E}_\mu^{\mathcal{W}} = \mathbf{E}_\mu^{\mathcal{N}} + \sum_{\alpha\in\Gamma^-}(\mathbf{E}_\mu^{\alpha^+} - \mathbf{E}_\mu^{\alpha})$.

So, for any $\alpha' \in \Gamma'^- : (\mathbf{E}_\mu^{\alpha'^+} - \mathbf{E}_\mu^{\alpha'}) = \sum_{\alpha\in\Gamma^-}(\mathbf{E}_\mu^{\alpha} - \mathbf{E}_\mu^{\alpha})(\mathbf{E}_\mu^{\alpha'^+} - \mathbf{E}_\mu^{\alpha'})$. By maximality (see (1.1)) the operators $\mathbf{E}_\mu^{\alpha'^+} - \mathbf{E}_\mu^{\alpha'}$, $\mathbf{E}_\mu^{\alpha^+} - \mathbf{E}_\mu^{\alpha}$ project into a one-dimensional subspace, so the commutativity relation among these projections implies that for any $\alpha' \in \Gamma'^-$ there exists $\alpha \in \Gamma^-$ such that $\mathbf{E}_\mu^{\alpha'^+} - \mathbf{E}_\mu^{\alpha'} = \mathbf{E}_\mu^{\alpha^+} - \mathbf{E}_\mu^{\alpha}$. □

Proposition 2.2 [**2**]. *Let Ω be countable and μ a probability measure on it. Then for any maximal chain of partitions the family of projections $\{\mathbf{E}_\mu^{\alpha} : \alpha \in \Gamma\} \cup \{0\}$ is of simple spectrum on $L^2(\mu)$.*

Proof. Recall that an increasing and left continuous family of projections $(P_t : t \in [a, b])$ defined in a Hilbert space H is said to be of simple spectrum if there exists some vector $h \in H$ such that the closed linear subspace generated by $(P_t H : t \in [a, b])$ is equal to H. Take any $h \in L^2(\mu)$ such that: $\mathbf{E}_\mu^{\mathcal{N}} h \neq 0$ and $(\mathbf{E}_\mu^{\alpha^+} - \mathbf{E}_\mu^{\alpha})h \neq 0$ for any $\alpha \in \Gamma^-$. From Proposition 2.1, and since it is easy to parametrize $(\mathbf{E}_\mu^{\alpha} : \alpha \in \Gamma)$ in an increasing and left continuous way, we deduce the result. □

3. Example. Let Ω be countably infinite. It is easy to construct countable maximal chains. In fact, on $\Omega = \mathbf{N}$ consider the chain of partitions $\Gamma_0 = \{\alpha_n : n \in \mathbf{N} \cup \{-1, +\infty\}\}$ given by $A_{\alpha_n}(i) = \{i\}$ if $i \leq n$, $A_{\alpha_n}(i) = \{i' \in \mathbf{N} : i' > n\}$ if $i > n$. From (1.1) the countable chain Γ_0 is maximal, so also $h_\mu(\Gamma_0)$ is a countable set for any μ.

But also there exist closed chains Γ which are perfect sets (i.e., without isolated points) and such that, for some probability measure μ, the set $h_\mu(\Gamma)$ has strictly positive fractional dimension. For proving that those properties hold in a particular example we shall use a result from Palis-Takens on Hausdorff dimension. In this context recall that if (X, d) is a compact metric space then its Hausdorff dimension is $HD(X) = \inf\{p > 0 : \lim_{\varepsilon\to 0} \inf_{\{\mathcal{U}(\varepsilon)\}}(\sum_{U\in\mathcal{U}(\varepsilon)} \delta(U)^p) < \infty\}$, where $\mathcal{U}(\varepsilon)$ is the class of open coverings of X such that the diameter $\delta(U)$ of any $U \in \mathcal{U}(\varepsilon)$ is $\leq \varepsilon$.

Now let X be a Cantor real set. Consider $X^- = \{x \in X : x < x^+\}$. Obviously $a = \inf X$ and $b = \sup X$ do not belong to X^-. The set of gaps of X is given by $\{\Delta_x = (x, x^+) : x \in X^-\}$. Denote by $|\Delta_x| = (x^+ - x)$ the length of the gap.

For $x \in X^-$ write $y_+(x) = \inf\{y \in X^- : y > x, |\Delta_x| \leq |\Delta_y|\}$ and $y_-(x) = \sup\{z \in X^+ : z < x, |\Delta_x| \leq |\Delta_z|\}$ with the convention $\inf\phi = b$, $\sup\phi = a$. Define the thickness at the right of Δ_x and at the left of Δ_x by

$$\tau_+(x) = |\Delta_x|^{-1}(y_+(x) - x^+), \qquad \tau_-(x) = |\Delta_x|^{-1}(x - y_-(x)). \tag{3.1}$$

The thickness of the Cantor set X is

$$\tau(X) = \inf\{\tau_+(x), \tau_-(x) : x \in X^-\}.$$

In [**4**] Palis and Takens proved that $HD(X) \geq \log 2/(\log(2 + 1/\tau(x)))$.

Now, for $a_1 \in X^+ \cup \{a\}$, $b_1 \in X^- \cup \{b\}$ with $a_1 < b_1$ the set $X_1 = [a_1, b_1] \cap X$ is a Cantor set and $HD(X_1) \leq HD(X)$. On the other hand, $X_1^+ = (X_1 \cap X^+) - \{a_1\}$, $X_1^- = (X_1 \cap X^-) - \{b_1\}$ so Δ_x is a gap of X_1 if and only if Δ_x is a gap of X and $\Delta_x \subset (a_1, b_1)$. Hence, a sufficient condition for $HD(X) \geq \log 2/\log(2 + 1/\delta)$ is the existence for any $0 < \varepsilon < \delta$ of $a_1(\varepsilon) \in X^+ \cup \{a\}$, $b_1(\varepsilon) \in X^- \cup \{b\}$, $a_1(\varepsilon) < b_1(\varepsilon)$ such that $\tau([a_1(\varepsilon), b_1(\varepsilon)] \cap X) \geq \delta - \varepsilon$.

Theorem 3.1. *Let Ω be a countable infinite set. Then there exists a closed chain Γ which is a Cantor set with respect to the topology $\mathcal{T}(\Gamma)$. Furthermore, there exists a probability measure μ on Ω such that $h_\mu(\Gamma)$ is a real Cantor set with strictly positive Hausdorff dimension* (*at least* $\geq 1/2$).

Proof. Let $\Omega = \mathcal{D}$ be the set of dyadic numbers in (0,1). Each $m \in \mathcal{D}$ can be uniquely expanded in the form: $m = \sum_{i \in \mathbf{N}^*} m(i) 2^{-i}$, $m(i) \in \{0,1\}$, $m(i) = 0$ for $i \geq i_0(m)$. For $m \in \mathcal{D}$ we identify m with $(m(i) : i \in \mathbf{N}^*)$ and denote $\mathcal{L}(m) = \sup\{i \in \mathbf{N}^* : m(i) = 1\}$.

Now consider the chain of partitions $\Gamma' = \{\alpha_m : m \in \mathcal{D}\}$ defined by

$$\begin{aligned} A_{\alpha_m}(m') &= \{m'\} \quad \text{if } m' \leq m, \\ A_{\alpha_m}(m') &= \{m'' > m\} \quad \text{if } m' > m. \end{aligned} \tag{3.2}$$

The order of Γ' is compatible with the order of $\mathcal{D}$ (recall that the order of $\mathcal{D}$ as a real subset is the same as the lexicographical order as a subset of $\{0,1\}^{\mathbf{N}^*}$).

Let $m \in \mathcal{D}$; we have $\alpha_m^- = \sup\{\alpha_{m'} : m' \in \mathcal{D}, m' < m\}$. Then $A_{\alpha_m^-}(m') = \{m\}$ if $m' < m$, $A_{\alpha_m^-}(m') = \{m'' \geq m\}$ if $m' \geq m$. For $r \in (0,1)\backslash\mathcal{D}$ define α_r by $A_{\alpha_r}(m) = \{m\}$ if $m < r$, $A_{\alpha_r}(m) = \{m' > r\}$ if $m > r$. It is easy to verify that $\alpha_r = \sup\{\alpha_m : m \in \mathcal{D}, m < r\} = \inf\{\alpha_m : m \in \mathcal{D}, m > r\} = \inf\{\alpha_{m^-} : m \in \mathcal{D}, m > r\}$.

Define $\alpha_0 = \mathcal{N}(\mathcal{D})$, $\alpha_1 = \mathcal{W}(\mathcal{D})$ and write $\alpha_m^- = \alpha_{m^-}$. Denote by Γ the smallest closed chain containing Γ'. It can be shown that it verifies $\Gamma = \{\alpha_t : t \in [0,1]\backslash\mathcal{D}\} \cup \{\alpha_{m^-}, \alpha_m : m \in \mathcal{D}\}$ and it is maximal by property (1.1). Let $\mathcal{D}^- = \{m^- : m \in \mathcal{D}\backslash\{0,1\}\}$. The set $\mathcal{M} = [0,1] \cup \mathcal{D}^-$ is totally ordered by the canonical order in $[0,1]$ and by defining for $t \in [0,1]$, $m, m_1 \in \mathcal{D} : m^- < m_1^-$ if $m < m_1$, $t < m^-$ if $t < m$ and $t > m^-$ if $t \geq m$. So $\Gamma = \{\alpha_t : t \in \mathcal{M}\}$ is ordered by the order of $\mathcal{M}$.

The equalities $\Gamma^- = \{\alpha_{m^-} : m \in \mathcal{D}\}$, $\Gamma^+ = \{\alpha_m : m \in \mathcal{D}\}$, imply that Γ has no isolated points, so it is a Cantor set as well as $\Psi(\Gamma)$ for any strictly increasing continuous function $\Psi : \Gamma \to \mathbf{R}$.

Now define the following probability measure on $\mathcal{D}$

$$\mu(m) = 2^{-2\mathcal{L}(m)+1}, \quad m \in \mathcal{D}. \tag{3.3}$$

Denote $X = h_\mu(\Gamma)$. We have $0 = \inf X$, $h_\mu(\mathcal{W}) = 3\log 2 = \sup X$. The set of gaps of the Cantor set X is $\{\Delta_m = (h_\mu(\alpha_m^-), h_\mu(\alpha_m)) : m \in \mathcal{D}\}$ and the length of the gap Δ_m is $|\Delta_m| = -\sum_{A \in \alpha_{m^-}} \sum_{A' \in \alpha_m : A' \subset A} \mu(A') \log \mu(A'/A)$.

Define the function $\varphi(\delta, x) = \delta \log(1 + x/\delta) + x \log(1 + \delta/x)$ for $\delta > 0$, $x > 0$. It is symmetric in δ and x, it increases in δ and x and verifies

$\varphi(\delta,x)=\delta\varphi(1,\delta^{-1}x)=\gamma\varphi(\gamma^{-1}\delta,\gamma^{-1}x)$ for any $\gamma>0$. Also, it can be shown that it verifies

$$\text{(3.4)}\quad \begin{array}{l}\text{for any } R\geq 1 \text{ and } x_0>0 \text{ there exists } \delta(R,x_0)>0 \text{ such that}\\ \forall\, r\in[1,R], \delta\leq\delta(R,x_0), x\geq x_0 : \varphi(\delta,x)\leq\varphi(r\delta,r^{-1}x)\end{array}$$

Set $\eta(m)=\sum_{m'>m}\mu(m')$. It can be shown that $|\Delta_m|=\mu(m)\log(1+\eta(m)/\mu(m))+\eta(m)\log(1+\mu(m)/\eta(m))=\varphi(\mu(m,\eta(m))$.

Now fix $0<\varepsilon<1/2$. We shall prove that there exists a point $m_1\in\mathcal{D}$ such that $HD([0,b_1]\cap X)\geq 1/2-\varepsilon$ where $b_1=h_\mu(\alpha_{m_1^-})$. Then the result will follow. The point $m_1\in\mathcal{D}$ is of the form $m_1(i)=0$ if $i\neq l_0$, $m_1(l_0)=1$; where l_0 is an integer >1 (so $m_1<1/2$) satisfying two conditions.

First we take $l_0>1$ in order that $\mu(m_1)\leq\eta(m_1)$ (it exists because $\mu(m)\rightarrow_{m\rightarrow 0}0$, $\eta(m)\rightarrow_{m\rightarrow 0}1$). Remark that if $m\in(0,m_1]\cap\mathcal{D}$, then $\mathcal{L}(m)\geq l_0$ so $\mu(m)\leq\mu(m_1)$. Since $\eta(m)$ increases if m decreases we obtain that the choice of l_0 implies that for any $m\in(0,m_1]\cap\mathcal{D}$: $\mu(m)\leq\mu(m_1)=2^{-2l_0+1}\leq\eta(m_1)\leq\eta(m)$.

Now let $K>0$ be an integer such that $2^{-K}<\varepsilon$. Fix $x_0=\eta(1/2)$, $R=2^{2K}$, and take $\delta=d(2^{2K},x_0)$ be given by property (3.4). The second condition to be verified by the integer $l_0>1$ is $2^{-2l_0+1}<\delta(2^{2K},x_0)$. Since $\mu(m)\leq 2^{-2l_0+1}$ and $\eta(m)\geq x_0$ for any $m\in(0,m_1]\cap\mathcal{D}$ we obtain that:

$$\text{(3.5)}\quad \text{for any } r\in[1,2^{2K}] \text{ and } m\in(0,m_1)\cap\mathcal{D} : \\ \varphi(\mu(m),\eta(m))\leq\varphi(r\mu(m),r^{-1}\eta(m))$$

Now let $m\in(0,m_1)\cap\mathcal{D}$. Denote $l=\mathcal{L}(m)$ and $s=\sup\{i\leq l : m(i)=0\}$. Associate to m the following point $\bar{m}\in(m,m_1]\cap\mathcal{D}$: $\bar{m}(i)=m(i)$ if $i<s$, $\bar{m}(s)=1$, $\bar{m}(i)=0$ if $i>s$. Any dyadic $m'\in(m,\bar{m})$ is of the form $m'(i)=m(i)$ if $i\leq l$, $\mathcal{L}(m')>l$. Then $\mu(m')<\mu(m)$. Since $m'>m$ we also have $\eta(m')<\eta(m)$. From the strictly increasing property of φ we get $|\Delta_{m'}|<|\Delta_m|$. Hence the thickness at the right of the gap Δ_m in the set $[0,b_1]\cap X$ satisfies

$$\text{(3.6)}\qquad \tau_+(m)\geq|\Delta_m|^{-1}\sum_{m'\in(m,\bar{m})}|\Delta_{m'}|.$$

Denote $\mathcal{K}(m') = \mathcal{L}(m') - l$. For any $k \geq 1$ there are 2^{k-1} different points $m' \in (m, \bar{m})$ satisfying $\mathcal{K}(m') = k$ and any one of them verifies $\mu(m') = 2^{-2k}\mu(m)$. We have

$$\begin{aligned}\eta(m) &= \eta(m') + \mu(m)\bigg(\sum_{k=1}^{\infty} 2^{-2k}2^{k-1}\bigg)\\ &= \eta(m') + \frac{1}{2}\mu(m) \leq \eta(m') + \frac{1}{2}\eta(m).\end{aligned}$$

So $\eta(m') > (1/2)\eta(m)$. Then

$$\begin{aligned}\sum_{m' \in (m,\bar{m})} |\Delta_{m'}| &> \sum_{m' \in (m,\bar{m})} \varphi(2^{-2\mathcal{K}(m')}\mu(m), (1/2)\eta(m))\\ &= \sum_{k=1}^{\infty} 2^{k-1}2^{-2k}\varphi(\mu(m), 2^{2k-1}\eta(m))\\ &\geq \sum_{k=1}^{\infty} 2^{k-1}2^{-2k}\varphi(\mu(m), \eta(m))\\ &= \frac{1}{2}|\Delta_m|.\end{aligned}$$

From (3.6) we conclude that for any gap Δ_m in $[0, b_1] \cap X : \tau_+(m) \geq 1/2$.

Let $\underline{m} \in \mathcal{D}$ be such that $\underline{m}(i) = m(i)$ for $i < l$, $\underline{m}(i) = 0$ for $i \geq l$. Any dyadic $m' \in (\underline{m}, m)$ is of the form $m'(i) = \underline{m}(i)$ for $i < l$, $\mathcal{L}(m') > l$. Then $\mu(m') < \mu(m)$. As before, denote $\mathcal{K}(m') = \mathcal{L}(m') - l$. Also, in this case we have that for any $k \geq 1$ there are 2^{k-1} different points $m' \in (\underline{m}, m)$ with $\mathcal{K}(m') = k$, any one of them verifying $\mu(m') = 2^{-2k}\mu(m)$. We have $\eta(m') < \eta(m) + \mu(m) + \sum_{m'' \in (\underline{m}, m)} \mu(m'') = \eta(m) + \mu(m) + \sum_{k=1}^{\infty} 2^{-2k}2^{k-1}\mu(m) = \eta(m) + (3/2)\mu(m) \leq (5/2)\eta(m) < 4\eta(m)$. So $2^{2k}\mu(m') = \mu(m)$, $2^{-2k}\eta(m') < 2^{-2(k-1)}\eta(m) \leq \eta(m)$. Now consider only those $m' \in (\underline{m}, m)$ such that $\mathcal{K}(m') \leq K$. From (3.5) we deduce

$$\forall\, k \leq K : \varphi(\mu(m'), \eta(m')) \leq \varphi(2^{2k}\mu(m'), 2^{-2k}\eta(m')) < \varphi(\mu(m), \eta(m)).$$

Hence, the thickness at the left of the gap Δ_m in the set $[0, b_1] \cap X$ verifies

$$\tau_-(m) \geq |\Delta_m|^{-1} \sum_{\substack{m' \in (\underline{m}, m)\\ \mathcal{K}(m') \leq K}} |\Delta_{m'}|. \tag{3.7}$$

For any $m' \in (\underline{m}, m)$ we have $\eta(m') \geq \eta(m)$, so $|\Delta_{m'}| \geq \varphi(2^{-2\mathcal{K}(m')}\mu(m), \eta(m))$. Then

$$\sum_{\substack{m' \in (\underline{m}, m) \\ \mathcal{K}(m') \leq K}} |\Delta_{m'}| \geq \sum_{k=1}^{K} 2^{k-1} \varphi(2^{-2k}\mu(m), \eta(m))$$

$$= \sum_{k=1}^{K} 2^{k-1} 2^{-2k} \varphi(\mu(m), 2^{2k}\eta(m))$$

$$\geq \sum_{k=1}^{K} 2^{-k-1} \varphi(\mu(m), \eta(m))$$

$$= \frac{1}{2}(1 - 2^{-K})\varphi(\mu(m), \eta(m)).$$

Hence we have shown that for any gap Δ_m in $[0, b_1] \cap X : \tau_-(m) \geq (1/2)(1 - 2^{-K})$. Then the thickness of $X_1 = [0, b_1] \cap X$ verifies: $\tau(X_1) \geq (1/2)(1 - 2^{-K})$. Then we conclude the result $HD(X) \geq 1/2$. □

Remark. 1. In the example the whole set of entropy values $h_\mu(\mathcal{A}(\mathcal{D}))$ contains the interval $[0, \log 2]$. In fact any $u \in [0, \log 2]$ can be written $u = -(t \log t + (1-t)\log(1-t))$ with $t \in [0, 1]$. From the definition of μ there exists a partition $\alpha = (A, \mathcal{D} - A)$ such that $\sum_{m \in A} \mu(m) = t$, so $h_{\mu(\alpha)} = u$. So the null Lebesgue property holds for the set of entropy values of chains of partitions and not for the entropy values of the set of all partitions.

2. In the example we have developed, it can be shown that we have $HD(X) = 1/2$.

3. Consider the functionals Ψ_q for $q > 1$. Then it can be proved that the sets $X_q = \Psi_q(\Gamma)$ are regular Cantor sets (see [**3**]) verifying $HD(X_q) = 1/2$.

REFERENCES

1. C. Dellacherie, *Filtrations: Commutation, maximalité, atomes*, preprint.

2. P. Dartnell, S. Martínez and J. San Martín, *Opérateurs filtrés et chaînes de tribus invariantes sur un espace probabilisé dénombrable*, L.N.M. **1321** (1988), 197–213.

3. G. Michon, *Arbre, cantor, dimension*, Prép. Lab. Top. Univ. Bourgogne, 1988.

4. J. Palis and F. Takens, *Homoclinic bifurcations: Hyperbolicity, fractional dimension and infinitely many attractors*, Cambridge Univer. Press, to appear.

Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 170-3, Correo 3, Santiago, Chile