ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 1, Winter 1996

PERTURBATION ANALYSIS OF A SEMILINEAR PARABOLIC PROBLEM WITH NONLINEAR BOUNDARY CONDITIONS

A.A. LACEY, J.R. OCKENDON, J. SABINA AND D. SALAZAR

ABSTRACT. We consider a diffusion model in which a distributed nonlinear absorption mechanism competes with a nonlinear boundary source. By assuming both these nonlinearities to be weak, a formal asymptotic approximation can be constructed to describe the magnitude and stability of the different responses that can occur in different parameter regimes.

1. Introduction. Our objective is the study of the behavior of positive solutions of the nonlinear, parabolic, initial-boundary value problem:

$$u_t = u_{xx} - \lambda u^p, \qquad x \in (0,1),\ t > 0, \tag{1.1a}$$

$$u_x(0,t) = 0, \qquad u_x(1,t) = u^q(1,t), \qquad t > 0, \tag{1.1b}$$

$$u(x,0) = u_0(x), \tag{1.1c}$$

where λ, p, q are constants $\lambda > 0$, $p > 1$, $q > 1$, and $u_0 > 0$.

The problem (1.1) is a generalization of some other nonlinear diffusion problems. One example is the question of global existence of positive solutions to the heat equation $u_t = \Delta u$, $(x,t) \in \Omega \times \mathbf{R}^+$, Ω a bounded domain in $\mathbf{R}^N$, subject to the nonlinear boundary condition $\partial u/\partial \nu = f(u)$ on $\partial\Omega$ (ν being the exterior unit normal to $\partial\Omega$) and with a nonnegative initial condition $u(x,0) = u_0(x)$. The main feature of this problem is the general tendency of positive solutions to blow up in finite time provided that $f = f(u)$ is a superlinear function. This blow-up property was first proved for this problem in [**4**] for $f(u)$ a power u^q, $q > 1$, and certain large enough initial data u_0 (see also [**7**] for early related but complementary global existence results). In [**5**] the blow-up property was proved for all nonnegative initial data u_0 provided that $f(u) = u^q$ and either $q > 1$, $N = 1, 2$, or $1 < q < N/(N-2)$ for $N > 2$

Received by the editors on October 7, 1993, and in revised form on March 28, 1994.

and it can thence be shown that blow-up in fact occurs for $q > 1$ and arbitrary N. On the other hand, blow-up was proved and found to be located on the boundary in [**2**] for every nonnegative nonzero u_0, both for all two-dimensional simply-connected domains and for balls in $\mathbf{R}^N$, provided $1/f$ is integrable at infinity. Thus, the general picture is that an explosion of $u(x,t)$ is caused by the nonlinear flux law $\partial u/\partial\nu = f(u)$ [**4**]. It is natural then to study what happens when some "sink" effect is introduced in the problem, and this is the role played by the term $-\lambda u^p$ in (1.1a). Indeed, our problem may be thought of as a model for the competition between, say, a nonlinear endothermic chemical reaction taking place in the bulk of some material $0 < x < 1$ and a nonlinear exothermic reaction taking place at the boundary $x = 1$. Observe that we are only dealing with symmetric $(u(x,t) = u(-x,t))$ solutions to the one-dimensional problem. Our asymptotic methods can in fact be applied to more general problems, both with asymmetry and in higher dimensions, but we address the one-dimensional symmetric case here for simplicity; a crucial first step is the study of the stationary solutions to (1.1) and their stability properties.

The stationary solutions to (1.1a), (1.1b) and their stability have been recently studied in [**1, 3**] and [**6**] by using phase space, variational techniques and comparison methods for weak solutions. We shall now complement those results here by providing a framework in which to describe the evolution of $u = u(x,t)$ from arbitrary initial data in the parameter regime of p and q are near unity. This enables us to exploit the solution of the linear problem, $p = 1$, $q = 1$, for which it is easy to show the following results:

Let μ^* be the unique positive root of $\mu^* \tanh \mu^* = 1$. Then the only steady solution of (1.1a), (1.1b) is $u = 0$ unless $\lambda = \lambda^* \equiv (\mu^*)^2$, in which case there exists a one-parameter family of eigensolutions

$$u = A \cosh \mu^* x, \qquad A = \text{constant}. \tag{1.2}$$

Moreover, the zero solution is globally stable for $\lambda > \lambda^*$ and unstable for $\lambda < \lambda^*$. We have the bifurcation diagram shown in Figure 1; note that $\lambda^* > \mu^* > 1$.

In particular, for the linear case, $u = u(x,t)$ solves $u_t = u_{xx} - \lambda u$ with $u_x = 0$ on $x = 0$, $u_x = u$ on $x = 1$, $u = u_0$ at $t = 0$ and can be

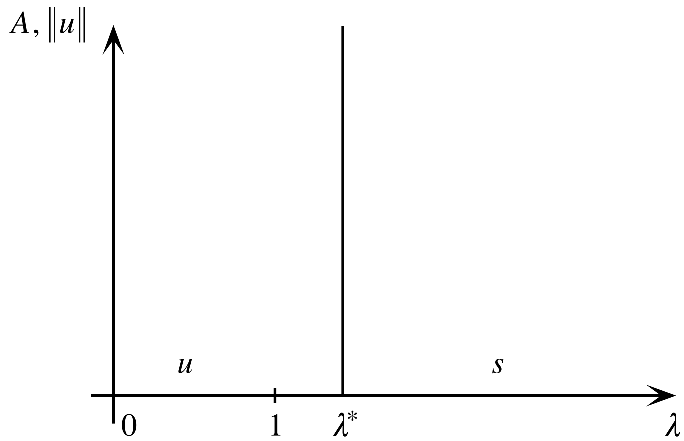

FIGURE 1. Response diagram for the steady states of the linear problem (1.1a), (1.1b) with $p = q = 1$; s and u denote stable and unstable branches.

found explicitly as a Fourier series:

$$u(x,t) = A_0 e^{(\lambda^* - \lambda)t} \cosh \mu^* x + \sum_1^\infty A_n e^{-(\lambda + \mu_n^2)t} \cos \mu_n x \tag{1.2a}$$

where μ_n is the nth positive root of $\mu_n \tan \mu_n + 1 = 0$, $(n - 1/2)\pi < \mu_n < n\pi$. The Fourier coefficients $A_0, A_1, A_2, \ldots$ are determined from the initial condition u_0. In particular,

$$A_0 = 2 \int_0^1 u_0(x) \cosh \mu^* x \, dx / (1 + \sinh^2 \mu^*) > 0.$$

Hence as $t \to \infty$, $u(x,t) \sim A_0 e^{(\lambda^* - \lambda)t} \cosh \mu^* x$. For $\lambda = \lambda^*$, $u = u(x,t)$ evolves to a steady state of the form given by (1.2) with $A = A_0$. Also, u grows or decays exponentially in time for $\lambda < \lambda^*$ or $\lambda > \lambda^*$, respectively.

We now proceed formally with a perturbation analysis taking $p = 1 + \varepsilon\alpha$ and $q = 1 + \varepsilon\beta$ where ε is a small positive quantity, and we will seek the dependence of the solution on the positive parameters α, β assumed of $O(1)$ as $\varepsilon \to 0$.

With ε so small so that nonlinear terms can be neglected, u is proportional to an exponential of time for large, but not too large t. Indeed, for $\lambda < \lambda^*$ or $\lambda > \lambda^*$, u becomes exponentially large or small in $1/\varepsilon$; thus, the linear approximation fails when $u^{\alpha\varepsilon}$ and $u^{\beta\varepsilon}$, which were previously neglected, become $O(1)$, i.e., for $t = O(1/\varepsilon)$. We must then study what happens over time scales of $O(1/\varepsilon)$.

2. Preliminary asymptotic analysis. We write $\tau = \varepsilon t$ to obtain:

$$\begin{aligned} \varepsilon u_\tau = u_{xx} - \lambda u^{1+\alpha\varepsilon}, \qquad 0 < x < 1,\ \tau > 0; \\ u_x(0,\tau) = 0, \qquad u_x(1,\tau) = u^{1+\beta\varepsilon}(1,\tau). \end{aligned} \tag{2.1}$$

The solution of (2.1) must match with the $t = O(1)$ solution:

$$u \sim A_0 e^{(\lambda^* - \lambda)\tau/\varepsilon} \cosh \mu^* x \quad \text{as } \tau \to 0+. \tag{2.2}$$

The form of the boundary value problem (2.1) and the matching condition (2.2) both suggest the use of a WKB expansion of the form

$$u(x,\tau) = w(x,\tau;\varepsilon) e^{v(\tau)/\varepsilon}, \tag{2.3}$$

where $w \sim w_0(x,\tau) + w_1(x,\tau)\varepsilon + \dots$ as $\varepsilon \to 0$.

On substituting (2.3) into (2.1), the two leading order terms give:

$$\begin{aligned} w_{0_{xx}} - (v' + \lambda e^{\alpha v}) w_0 = 0 \\ w_{0_x} = 0 \quad \text{on } x = 0, \qquad w_{0_x} = e^{\beta v} w_0 \quad \text{on } x = 1; \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} w_{1_{xx}} - (v' + \lambda e^{\alpha v}) w_1 = w_{0\tau} + \alpha\lambda e^{\alpha v} w_0 \ln w_0 \\ w_{1_x} = 0 \quad \text{on } x = 0, \qquad w_{1_x} = e^{\beta v} w_1 + \beta e^{\beta v} w_0 \ln w_0 \quad \text{on } x = 1. \end{aligned} \tag{2.5}$$

Setting $\mu^2 = v' + \lambda e^{\alpha v}$, with $\mu > 0$, we readily see that (2.4) admits positive solutions provided that μ is the unique positive root of the equation

$$\mu \tanh \mu = e^{\beta v}. \tag{2.6}$$

Thus the dominant behavior of the solution to (2.1), (2.2), i.e. the order of magnitude of u, is now given by the solution to

$$(v' + \lambda e^{\alpha v})^{1/2} \tanh(v' + \lambda e^{\alpha v})^{1/2} = e^{\beta v} \tag{2.7}$$

subject to the initial condition

$$v(0) = 0, \tag{2.8}$$

obtained by matching with (1.2a) as $\tau \to 0$. To find u to leading order we must still determine w_0. To do this, we notice that the solution to (2.4) has the form

$$w_0 = A(\tau)\cosh(\mu(\tau)x)$$

where $A(\tau) > 0$ satisfies a differential equation, given by the solvability condition that (2.5) has a solution w_1 satisfying an initial condition provided by matching with (1.2a) as $\tau \to 0$. Thus, multiplying (2.5) by the eigenfunction $\cosh \mu x$ and integrating, we obtain the solvability condition

$$\int_0^1 (w_{0\tau} + \alpha\lambda e^{\alpha v} w_0 \ln w_0) \cosh \mu x \, dx = \beta e^{\beta v} (w_0 \ln w_0)|_{x=1} \cosh \mu$$

(this is, of course, the Fredholm alternative) which is to say

$$\begin{aligned} A' \int_0^1 \cosh^2 \mu x \, dx + A\mu' \int_0^1 x \cosh \mu x \sinh \mu x \, dx & \\ + \alpha\lambda A e^{\alpha v} \int_0^1 \cosh^2 \mu x \ln \cosh \mu x \, dx & \\ + \alpha\lambda (A \ln A) e^{\alpha v} \int_0^1 \cosh^2 \mu x \, dx & \\ = A\beta e^{\beta v} \cosh^2 \mu \ln \cosh \mu + (A \ln A)\beta e^{\beta v} \cosh^2 \mu. & \end{aligned}$$

The equation for A is thus of the form

$$A' I_1 + (I_2 \mu' + I_3)A + I_4 A \ln A = 0 \tag{2.9}$$

where the integrals $I_1 - I_4$ can be regarded as functions of either v or μ, all being elementary functions except that the evaluation of

$\int_0^1 \cosh^2 \mu x \ln(\cosh \mu x)\, dx$, which appears in I_3, involves the dilogarithm. The derivative μ' can be obtained from the relation (2.6) between μ and v and the differential equation (2.7):

$$\mu' = \beta e^{\beta v} \frac{\mu^2 - \lambda e^{\alpha v}}{\tanh \mu + \mu \operatorname{sech}^2 \mu}. \tag{2.10}$$

Our task is now to investigate the solution of (2.9) subject to the initial condition, given by matching

$$A(0) = A_0, \tag{2.11}$$

as well as, more importantly, to find the solution v of (2.7) and (2.8).

As a starting point, we need to investigate the existence and stability of steady states. We observe that any stationary solution is given by $v' = 0$ so $v = V(\lambda) = (1/\alpha)\ln[s(\lambda)^2/\lambda]$ where $s = s(\lambda)$ is a positive solution of

$$s \tanh s = (s^2/\lambda)^{\beta/\alpha},$$

i.e.,

$$s^{2-\alpha/\beta}/(\tanh s)^{\alpha/\beta} = \lambda. \tag{2.12}$$

Equivalently, from (2.6) and (2.10), $\mu = s(\lambda)$ is a steady state of

$$\mu' = \beta \mu \tanh \mu \frac{\mu^2 - \lambda(\mu \tanh \mu)^{\alpha/\beta}}{\tanh \mu + \mu \operatorname{sech}^2 \mu}. \tag{1.15}$$

The stability of μ, and hence of V, is thus determined by the sign of $(\partial/\partial\mu)[\mu^2 - \lambda(\mu \tanh \mu)^{\alpha/\beta}]|_{\mu=s}$. Noting that $s^2 - \lambda(s \tanh s)^{\alpha/\beta} = 0$, we find that

$$\frac{ds}{d\lambda} \frac{\partial}{\partial s}\bigg|_\lambda [s^2 - \lambda(s \tanh s)^{\alpha/\beta}] - (s \tanh s)^{\alpha/\beta} = 0$$

and hence that $\mu = s(\lambda)$ and $V = (1/\alpha)\ln[s(\lambda)^2/\lambda]$ are asymptotically stable if $s(\lambda)$ is decreasing; $\mu = s(\lambda)$ and $V = (1/\alpha)\ln[s(\lambda)^2/\lambda]$ are unstable if $s(\lambda)$ is increasing.

It is clear from this that the steady state of (2.1) is unstable if $s(\lambda)$ is increasing. Before we can assert stability in the case when $s(\lambda)$ is decreasing, we also need to check on the local behavior of A. Fixing $\mu = s(\lambda)$ as a constant, the reduced version of equation (2.9),

$$I_1 A' + (I_3 + I_4 \ln A)A = 0,$$

should have the stable steady state $a(\lambda) \equiv e^{-I_3/I_4}$. Now for μ at its equilibrium value $s(\lambda)$,

$$\begin{aligned} I_4 &= \alpha\lambda e^{\alpha v} \int_0^1 \cosh^2 sx\, dx - \beta e^{\beta v} \cosh^2 s \\ &= \frac{1}{2}\alpha\lambda e^{\alpha v}\left(\frac{\sinh 2s}{2s} + 1\right) - \beta e^{\beta v}\cosh^2 s \\ &= -\frac{\beta e^{\beta v}\cosh^2 s}{2s}\left[2s - \frac{\lambda\alpha}{\beta} e^{(\alpha-\beta)v}(\tanh s + s\,\mathrm{sech}^2 s)\right], \\ &= -\frac{\beta e^{\beta v}\cosh^2 s}{2s}\left[2s - \frac{\lambda\alpha}{\beta}(s\tanh s)^{\alpha/\beta - 1}(\tanh s + s\,\mathrm{sech}^2 s)\right] \\ &= -\frac{\beta}{2s} e^{\beta v}\cosh^2 s \left.\frac{\partial}{\partial s}\right|_\lambda [s^2 - (s\tanh s)^{\alpha/\beta}\lambda], \end{aligned}$$

where we have replaced v by its equilibrium value given by $e^{\beta v} = s\tanh s$.

Noting that $I_1 = \int_0^1 \cosh^2 \mu x\, dx > 0$ we see that the steady state $a(\lambda)$ is asymptotically stable for $(\partial/\partial s)|_\lambda[s^2 - (s\tanh s)^{\alpha/\beta}\lambda] < 0$ and unstable where $(\partial/\partial s)|_\lambda[s^2 - (s\tanh s)^{\alpha/\beta}\lambda] > 0$. Thus, in general, the steady state in which $A = a(\lambda)$ is stable if and only if $s(\lambda)$, the corresponding steady solution for μ, is also stable. This should be expected as the alternative expansion, $u \sim \exp[v_0(\tau)/\varepsilon + v_1(\tau) + \varepsilon v_2(x,\tau) + \cdots]\cosh(\mu(\tau)x)$, gives, to leading order, $u \sim e^{(v_0/\varepsilon + v_1)}\cosh \mu x$, with $v_1 = \ln A$, and the steady state will be stable if and only if $v(\lambda)/\varepsilon + \ln a(\lambda)$ is stable as a steady state of $\tilde{v} = v_0/\varepsilon + v_1$.

All the above arguments implicitly assume that $s(\lambda)$ is a smooth function, but we will shortly encounter the possibility of turning points where $d\lambda/ds$ vanishes; then I_4 becomes zero and our expansions become invalid. The stability near such a turning point can only be considered by using a revised asymptotic expansion for $a(\lambda)$, but that is beyond the scope of the present paper.

It should also be remarked that inspection of (2.9) shows that the only way A can either become unbounded or reach zero in a finite time is if $v \to \pm\infty$ as t approaches that time.

We can make no further general remarks about the solution u to (2.1) (or the solution of (1.1)) until we have studied the properties of equation (2.7) for v; this is done in the following section.

3. Steady states and possible asymptotic behavior of v. We summarize the results concerning the differential equation for v in the following theorem. We shall denote the value corresponding to the trivial steady state, $u \equiv 0$, by $V_\infty = -\infty$. It is clear that $u = 0$ will be asymptotically stable or unstable according to whether $v' < 0$ or $v' > 0$ for large negative values of v.

Theorem. *The equation* (2.7) *has the following properties according to the values of the positive parameters α, β and λ*

(a) *If $0 < \alpha < \beta$ (2.7) has a unique steady state $V(\lambda) = (1/\alpha)\ln[s(\lambda)^2/\lambda]$ for every $\lambda > 0$. Here $s(\lambda)$ is a smooth, positive, increasing function such that $s(\lambda) \sim \lambda^{\beta/2(\beta-\alpha)}$ as $\lambda \to 0$ and $s(\lambda) \sim \lambda^{\beta/(2\beta-\alpha)}$ as $\lambda \to \infty$. The steady solution $V(\lambda)$ is unstable whereas V_∞ is asymptotically stable. If $v > V(\lambda)$, then v blows up at a finite time. If $v < V(\lambda)$ then $v \to -\infty$ as $t \to \infty$.*

(b) *If $\alpha = \beta > 0$ (2.7) has a unique steady state $V(\lambda) = (1/\lambda)\ln[s(\lambda)^2/\lambda]$ if and only if $\lambda > 1$. Here $s(\lambda)$ is a smooth, positive, increasing function such that $s(1) = 0$ and $s(\lambda) \sim \lambda$ as $\lambda \to \infty$. The stationary solution $V(\lambda)$ is unstable. For $0 < \lambda \leq 1$, V_∞ is unstable but for $\lambda > 1$ V_∞ is asymptotically stable. For $0 < \lambda \leq 1$ all solutions of* (2.7) *blow up. For $\lambda > 1$ only those starting greater than $V(\lambda)$ blow up while if $v < V(\lambda)$ then $v \to -\infty$ as $t \to \infty$.*

(c) *If $0 < \beta < \alpha < 2\beta$ (2.7) has stationary solutions if and only if $\lambda \geq \lambda_c$ where $\lambda_c(\alpha, \beta)$ satisfies $1 < \lambda_c \leq \lambda^*$. For $\lambda > \lambda_c$ there are two steady states given by $V_j(\lambda) = (1/\alpha)\ln[s_j(\lambda)^2/\lambda]$, $j = 1, 2$ where $s_1(\lambda) < s_2(\lambda)$ are smooth, positive functions for $\lambda > \lambda_c$; $s_1(\lambda)$ is decreasing while $s_2(\lambda)$ is increasing with $s_1(\lambda) \sim \lambda^{-\beta/2(\alpha-\beta)}$ and $s_2(\lambda) \sim \lambda^{-\beta/(2\beta-\alpha)}$ as $\lambda \to \infty$. For $\lambda = \lambda_c$ there is a unique steady state $V_c = (1/\alpha)\ln[s_c^2/\lambda]$ and $s_j(\lambda) \to s_c$ as $\lambda \to \lambda_{c^+}$. The steady states V_∞, V_c, and V_2* (*where they exist*) *are all unstable but* (*for $\lambda > \lambda_c$*) *V_1*

is stable. Indeed, for $\lambda < \lambda_c$ *all solutions* v *to* (2.7) *blow up*; *for* $\lambda = \lambda_c$, v *blows up if initially* $v > V_c$, *otherwise* $v \to V_{c^-}$ *as* $t \to \infty$; *for* $\lambda > \lambda_c$ *if* $v(0) > V_2$, *then again* v *blows up whereas* $v(0) < V_2$ *gives* $v \to V_1$ *as* $t \to \infty$.

(d) *If* $0 < \alpha = 2\beta$ (2.7) *has a steady state if and only if* $\lambda > 1$. *This unique solution* $V(\lambda) = (1/\alpha)\ln[s(\lambda)^2/\lambda]$ *where* $s(\lambda)$ *is a smooth, decreasing, positive function with* $s(\lambda) \to \infty$ *as* $\lambda \to 1+$ *and* $s(\lambda) \sim \lambda^{-1/2}$ *as* $\lambda \to \infty$. *The stationary solution* $V(\lambda)$ *is stable for all* $\lambda > 1$ *whereas* V_∞ *is always unstable. For* $\lambda < 1$ *all solutions to* (2.7) *blow up, and for* $\lambda > 1$, $\nu \to V(\lambda)$ *as* $t \to \infty$.

(e) *If* $\alpha > 2\beta > 0$ (2.7) *has a unique stationary solution* $V(\lambda) = (1/\alpha)\ln[s(\lambda)^2/\lambda]$ *for all* $\lambda > 0$. *Here* $s(\lambda)$ *is a smooth, positive, decreasing function such that* $s(\lambda) \sim \lambda^{-\beta/(\alpha-2\beta)}$ *as* $\lambda \to 0$ *and* $s(\lambda) \sim \lambda^{-\beta/2(\alpha-\beta)}$ *as* $\lambda \to \infty$. *The steady state* $V(\lambda)$ *is stable while* V_∞ *is unstable. All solutions of* (2.7) *exist for all time* t *and satisfy* $v \to V(\lambda)$ *as* $t \to \infty$.

We can make some further remarks about the behavior of $s(\lambda)$ when it becomes large. We note that if $\sigma = \beta/(2\beta - \alpha)$, then $s''(\lambda) \simeq \sigma(\sigma-1)\lambda^{\sigma-2}$ as $\lambda \to \infty+$ in the case (a), $\lambda \to +\infty$ and $s = s_2(\lambda)$ in the case (c), $\lambda \to 0+$ in the case (e). In the case (b) $s''(\lambda) \simeq -8(\lambda-1)e^{-2\lambda}$ as $\lambda \to +\infty$. Finally, $s''(\lambda) \simeq (3\lambda - 1)/[\lambda^{3/2}(\lambda-1)^2]$ as $\lambda \to 1+$ in the case (d). Thus, $s(\lambda)$ is concave near infinity in the cases (a) and (b), and it is convex in the cases (c) with $s = s_2(\lambda)$, (d) and (e). Similarly, when $s(\lambda)$ converges to zero we note that if $\theta = \beta/(2(\beta - \alpha))$, then $s''(\lambda) \simeq \theta(\theta - 1)\lambda^{\theta-2}$ when $\lambda \to +\infty$ in the cases (c) with $s = s_1(\lambda)$, (d) and (e). Since $\theta < 0$, $s(\lambda)$ is convex near zero in these cases. In the case (b), where $\beta = \alpha$, $s''(\lambda) \simeq -1/s(\lambda)$ as $\lambda \to 1+$, so $s(\lambda)$ is concave. In the case (a), $s''(\lambda) \simeq \theta(\theta - 1)\lambda^{\theta-2}$ as $\lambda \to 0+$ except when $\theta = 1$ where $s''(\lambda) \simeq -\lambda$ as $\lambda \to 0+$. Thus, in the case (a), $s(\lambda)$ is convex near zero if $\alpha < \beta < 2\alpha$ and concave in the remaining cases.

The possible types of qualitative behavior of $s(\lambda)$, which is given by (2.12), are shown in Figure 2, which we shall now explain. It is clear that response diagrams showing $||u||_\infty$, as computed from the approximation $u \sim a(\lambda)e^{V(\lambda)/\varepsilon}\cosh(s(\lambda)x)$, will be very similar to those of s, since $s\tanh s = e^{\beta V}$ (V increases with s), i.e., $u = O((s\tanh s)^{1/\varepsilon\beta})$. In particular, $V_\infty = -\infty$ corresponds to $s = 0$ and

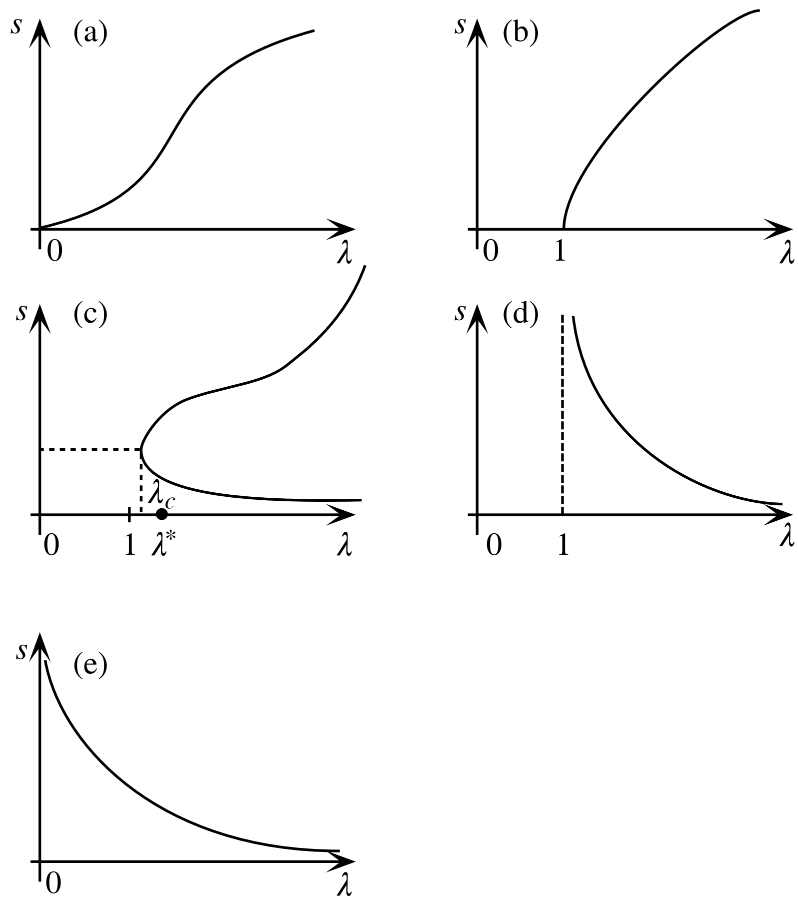

FIGURE 2. Plots of $s(\lambda)$ for the cases (a) $\alpha < \beta$, (b) $\alpha = \beta$, (c) $\beta < \alpha < 2\beta$, (d) $\alpha = 2\beta$, and (e) $\alpha > 2\beta$.

to $u = 0$.

The stability properties of $V(\lambda) = (1/\alpha)\ln[s(\lambda)^2/\lambda]$ follow immediately from the sign of $ds/d\lambda$ (which coincides with that of $dV/d\lambda$), as discussed above. The stability of the trivial steady state is given by the sign of v', as in (2.7), for v large and negative; since

$$\begin{aligned} v' &= \mu^2 - \lambda e^{\alpha v} \quad \text{with } \mu \tanh \mu = e^{\beta v} \\ &\sim e^{\beta v} - \lambda e^{\alpha v} \quad \text{for } -v \gg 1. \end{aligned}$$

Thus, for $0 < \alpha < \beta$ or $\lambda > 1$ with $\alpha = \beta > 0$ we have $v' < 0$ when $-v \gg 1$ and the trivial steady state is asymptotically stable. However, for $\alpha > \beta > 0$ or $\lambda < 1$ with $\alpha = \beta > 0$ we obtain $v' > 0$ when $-v \gg 1$ and the trivial steady state is unstable. The critical case of $\lambda = 1$, $\alpha = \beta$ has to be looked at more carefully: as $v \to -\infty$, $\mu \tanh \mu \sim \mu^2(1 - \mu^2/3 + \cdots) \sim e^{\beta v}$ so $\mu^2 \sim e^{\beta v}(1 + (1/3)e^{\beta v} + \cdots)$. Then $v' \sim (1/3)e^{2\beta v}$ so the trivial steady state is again unstable.

To check on which values of λ admit real values of $s(\lambda)$, and hence $V(\lambda)$, and find the response curves, we need only study (2.12), which is more conveniently written as

$$\lambda = \frac{s^2}{(s \tanh s)^{\alpha/\beta}} \quad \text{or} \quad \Lambda \equiv \lambda^{-\beta/a} = s^{1-2\beta/\alpha} \tanh s. \tag{3.1}$$

Now

$$\begin{aligned} s^{2\beta/\alpha} \frac{d\Lambda}{ds} &= \left(1 - \frac{2\beta}{\alpha}\right) \tanh s + s \operatorname{sech}^2 s \\ &= \left[s + \left(1 - \frac{2\beta}{\alpha}\right) \sinh s \cosh s\right] \operatorname{sech}^2 s. \end{aligned}$$

It is immediately apparent that for $1 - 2\beta/\alpha \geq 0$, i.e., $\alpha \geq 2\beta$, $d\Lambda/ds > 0$ so $s(\lambda)$ is decreasing and $V(\lambda)$ is asymptotically stable wherever they are defined. Moreover, using $\sinh s > s$ and $\cosh s > 1$ for $s > 0$, for $-2\beta/\alpha + 1 \leq -1$, i.e., $\alpha \leq \beta$, $d\Lambda/ds < 0$ so $s(\lambda)$ is increasing and $V(\lambda)$ is unstable wherever they are defined. It is also clear that:

(i) As $s \to \infty$, $\Lambda \sim s^{1-2\beta/\alpha}$ and $\lambda \sim s^{2-\alpha/\beta}$, i.e., $s \sim \lambda^{\beta/2(\beta-\alpha)} \to \infty$ as $\lambda \to \infty$ for $\alpha < 2\beta$ ($s = s_2$ for $\beta < \alpha < 2\beta$), $s \sim \lambda^{-\beta/(\alpha-2\beta)} \to \infty$ as $\lambda \to 0$ for $\alpha > 2\beta$ and $s \to \infty$ as $\lambda \to 1+$ for $\alpha = 2\beta$.

(ii) As $s \to 0$, $\Lambda \sim s^{2-2\beta/\alpha}$ and $\lambda \sim s^{2(1-\alpha/\beta)}$, i.e., $s \sim \lambda^{\beta/2(\beta-\alpha)} \to 0$ as $\lambda \to 0$ for $\alpha < \beta$, $s \sim \lambda^{-\beta/2(\alpha-\beta)} \to 0$ as $\lambda \to \infty$ for $\alpha > \beta$ ($s = s_1$ for $\beta < \alpha < 2\beta$) and $s \to 0$ as $\lambda \to 1+$ for $\alpha = \beta$.

To complete the results on the qualitative behavior of $s(\lambda)$, and the attendant stability, it remains to show that s can take precisely two real values, $s_1 < s_2$, for $\lambda > \lambda_c$ in the case of $\beta < \alpha < 2\beta$; since $\lambda \to \infty$ both as $s \to 0$ and as $s \to \infty$, it is clear that a minimum positive value λ_c of λ exists above which $s(\lambda)$ is real. Now $ds/d\lambda$ has the opposite sign to $s + (1 - 2\beta/\alpha)\sinh s \cosh s$ which has negative second derivative. This indicates that $ds/d\lambda$ changes sign at most twice. Since, in the interval $\beta < \alpha < 2\beta$, $s + (1 - 2\beta/\alpha)\sinh s \cosh s$ is positive if s is small and negative if s is large, $ds/d\lambda$ changes sign precisely once: $d\lambda/ds < 0$ for $s < s_c$ ($s = s_1(\lambda)$ is decreasing), $d\lambda/ds > 0$ for $s > s_c$ ($s = s_2(\lambda)$ is increasing). Regarding the size of λ_c we note that $V = 0$ for $s = \mu^* > 1$ ($\mu^* \tanh \mu^* = 1$), that is, $\lambda = \lambda^* = (\mu^*)^2 > 1$, and, whatever positive values α and β take, real values of $s(\lambda^*)$ and $V(\lambda^*)$ can be defined. Thus, for $\beta < \alpha < 2\beta$, $\lambda_c \le \lambda^*$. Moreover, using (3.1) and writing $S = s\tanh s$, $\lambda = s^2/S^{\alpha/\beta}$ so for $S < 1$, i.e., $s < \mu^*$, $\lambda > s^2/S = s/\tanh s > 1$ taking $\alpha > \beta$, while for $S > 1$, i.e., $s > \mu^*$, $\lambda > s^2/S^2 = 1/(\tanh s)^2 > 1$ taking $\alpha < 2\beta$. Hence, $\lambda_c > 1$.

We should also note that at $s = \mu^*$, $\lambda = \lambda^*$ and ($V = 0$), $d\lambda/ds$ has sign opposite to that of $s + (1 - 2\beta/\alpha)\sinh s \cosh s$, which gives at $s = \mu^*$

$$\begin{aligned}\mu^*\left[1 + \left(1 - \frac{2\beta}{\alpha}\right)\sinh^2 \mu^*\right] &= \mu^*[\cosh^2 \mu^* - (2\beta/\alpha)\sinh^2 \mu^*] \\ &= \mu^*((\mu^*)^2 - 2\beta/\alpha)\sinh^2 \mu^*,\end{aligned}$$

and

$$\begin{array}{llllll} d\lambda/ds < 0 & \text{at } s = s^* & \text{so } s_c > \mu^* & \text{and} & \lambda_c < \lambda^* & \text{for } \alpha > 2\beta/\lambda^*, \\ d\lambda/ds = 0 & \text{at } s = s^* & \text{so } s_c = \mu^* & \text{and} & \lambda_c = \lambda^* & \text{for } \alpha = 2\beta/\lambda^*, \\ d\lambda/ds > 0 & \text{at } s = s^* & \text{so } s_c < \mu^* & \text{and} & \lambda_c < \lambda^* & \text{for } \alpha < 2\beta/\lambda^*. \end{array}$$

Of course, $\lambda^* > 1$ so $2\beta/\lambda^* < 2\beta$. Moreover, $\lambda^* = 1 + 1/\sinh^2 \mu^*$ where $\mu^* > 1$ so $\sinh \mu^* > \sinh 1 > 1$, i.e., $\lambda^* < 2$ and $2\beta/\lambda^* > \beta$. Hence, there is a critical value of α/β, namely,

$$\frac{\alpha}{\beta} = \frac{2}{\lambda^*}, \tag{3.2}$$

such that s_c crosses μ^* from below as α/β increases through this value. At the turning point $V = V_c < 0$ for $\alpha/\beta < 2\lambda^*$, $V_c = 0$ for $\alpha/\beta = 2/\lambda^*$,

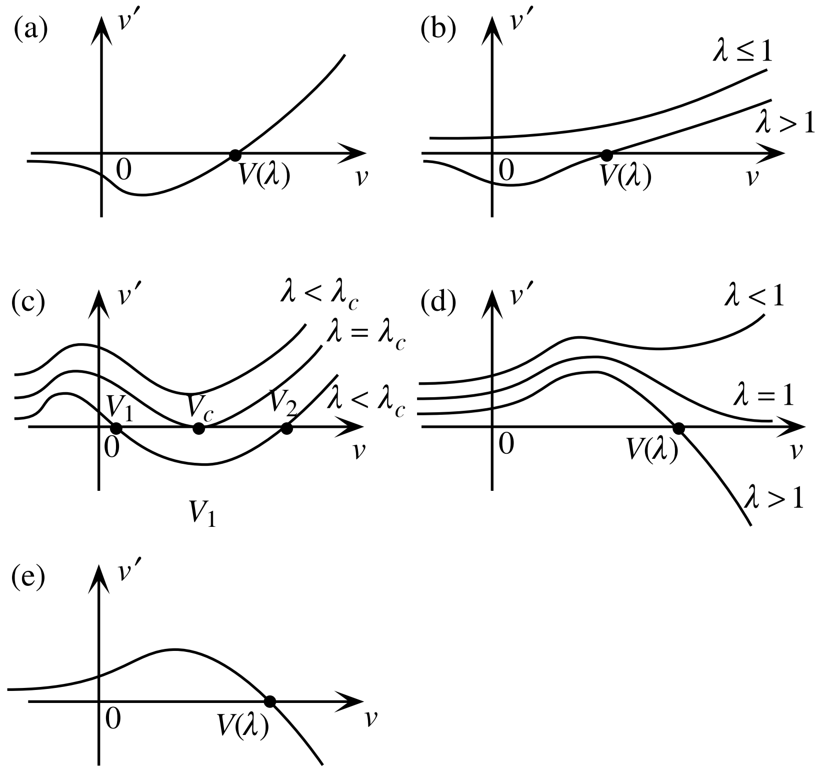

FIGURE 3. Graphs of $v' = \mu^2 - \lambda e^{\alpha v}$ against v for the five cases (a) $\alpha < \beta$, (b) $\alpha = \beta$, (c) $\beta < \alpha < 2\beta$, (d) $\alpha = 2\beta$, and (e) $\alpha > 2\beta$.

and $V_c > 0$ for $\alpha/\beta > 2/\lambda^*$ since $V = 0$ at $s = \mu^*$ and V is an increasing function of s.

The behavior of the solution to an initial value problem for v follows immediately from consideration of $v' = \mu^2 - \lambda e^{\alpha\nu}$ as a function of v with v and μ related by (2.6) (for fixed α, β and λ). Using knowledge of the zeros $V(\lambda)$ of v', the behavior for large negative v, and

$$v' \sim e^{2\beta v} - \lambda e^{\alpha v} \quad \text{for } v \gg 1 \quad \text{as } (\lambda, \alpha) \neq (1, 2\beta),$$
$$v' \sim 4\exp(-e^{\beta v} + 2\beta v) \quad \text{for } v \gg 1 \quad \text{as } (\lambda, \alpha) = (1, 2\beta),$$

then the five cases give graphs as shown in Figure 3.

The asymptotic behavior of the solutions v to (2.7) then follows, except that in place of v blowing up or tending to infinity as $t \to \infty$, we only know that v is unbounded above. However, $v' \sim e^{2\beta v}$ as $v \to \infty$ for $\alpha < 2\beta$ and $v' \sim (1-\lambda)e^{2\beta v}$ as $v \to \infty$ for $\lambda < 1$, $\alpha = 2\beta$. Hence, for these two cases any solution that becomes large must blow up. In the remaining case that can admit large solutions, $\lambda = 1$, $\alpha = 2\beta$, $v' \sim 4\exp(-e^{\beta v} + 2\beta v)$ for $v \gg 1$ so $v \to \infty$ as $t \to \infty$, but with comparatively slow growth.

4. Comparison with the linear case. We wish to see how the structure of the bifurcation diagram for the approximate solution to the steady problem,

$$U_{\varepsilon,\lambda} = a(\lambda)e^{V(\lambda)/\varepsilon}\cosh(s(\lambda)x),$$

approaches that for the linear problem as $\varepsilon \to 0$. Let us consider five cases in turn.

(a) $\alpha < \beta$. For all λ there is an unstable, nontrivial, steady state given by $v = V(\lambda)$, with $V(\lambda) > 0$ for $\lambda > \lambda^*$, $V(\lambda) < 0$ for $\lambda < \lambda^*$ ($V(\lambda^*) \equiv 0, V' > 0$ in this case). As $\varepsilon \to 0$, $U \to 0$ for $\lambda < \lambda^*$, i.e., the nontrivial solution approaches the trivial, stable, steady state and $u = 0$ becomes unstable in the limit, while for $\lambda > \lambda^*$, $U \to \infty$, and the basin of attraction of the trivial, steady state becomes unbounded.

(b) $\alpha = \beta$. This is like (a) except that the nontrivial solution that tends to zero only exists for $1 < \lambda < \lambda^*$. For $1 < \lambda < \lambda^*$, $u = 0$ becomes unstable through the approach of U. For $\lambda \leq 1$, $u = 0$ is unstable even for $\varepsilon > 0$.

(c) $\beta < \alpha < 2\beta$. This has to be subdivided into (i) $\beta < \alpha < 2\beta/\lambda^*$, (ii) $\alpha = 2\beta/\lambda^*$, and (iii) $2\beta/\lambda^* < \alpha < 2\beta$.

(i) $\beta < \alpha < 2\beta/\lambda^*$. The turning value $V_c < 0$ is achieved at $\lambda = \lambda_c < \lambda^*$. As $\varepsilon \to 0$, the whole of the lower, stable branch, corresponding to $v = V_1 \leq V_c$, tends to zero. Simultaneously that part of the upper, unstable branch lying in $\lambda_c \leq \lambda < \lambda^*$ also approaches zero whereas for $\lambda > \lambda^*$ then the branch goes off to infinity. For $\lambda < \lambda_c$, $u = 0$ is automatically unstable, for $\lambda_c \leq \lambda < \lambda^*$ we have that $u = 0$ remains unstable due to the approach of the unstable steady state (constraining the stable one beneath it), and for $\lambda > \lambda^*$ we obtain that

$u = 0$ becomes stable, with unbounded basin of attraction as the stable solution approaches zero while the unstable one goes off to infinity.

(ii) $\alpha = 2\beta/\lambda^*$. Here $V_c = 0$ ($\lambda_c = \lambda^*$) and the whole of the lower, stable branch has $V_1 < 0$ and so approaches $u = 0$ as $\varepsilon \to 0$ while all the upper, unstable branch has $V_2 > 0$ and consequently is unbounded as $\varepsilon \to 0$.

(iii) $2\beta/\lambda^* < \alpha < 2\beta$. Now $V_c > 0$ is achieved at $\lambda_c < \lambda^*$. As $\varepsilon \to 0$ the larger, unstable solution, corresponding to $V_2 \geq V_c > 0$, and that part of the smaller, stable solution corresponding to V_1 with $\lambda < \lambda^*$, so $V_1 > 0$, are unbounded. The smaller steady state with $\lambda > \lambda^*$, $v = V_1 < 0$, goes to zero.

(d) $\alpha = 2\beta$. The nontrivial steady state is stable and has $V > 0$, so $U \to \infty$, for $1 < \lambda < \lambda^*$; for $\lambda > \lambda^*$ $V < 0$ so $U \to 0$ as $\varepsilon \to 0$.

(e) $\alpha > 2\beta$. This is like (d) except that the solution going off to infinity applies for all $\lambda < \lambda^*$.

Representative response diagrams for some small value of ε are shown in Figure 4.

In crude terms, we can contrast our results with the linear response as follows:

(i) When $\lambda < \lambda^*$, the nonlinearity permits the existence of stable steady states, but they are either large or just below unstable steady states;

(ii) When $\lambda > \lambda^*$, the nonlinearity permits the existence of unstable steady states, but they are either large or zero and just below stable states.

Thus, the limit problem coincides with the linear, $\varepsilon = 0$, problem, shown in Figure 1.

5. The initial value problem. In order to compute the details of Figure 3, we briefly discuss the implications of the properties of the steady states for the behavior of the initial value problem for v. This follows according to the properties of the differential equation (2.7) subject to the initial condition (2.8). We will only consider the more interesting case in which $\lambda < \lambda^*$.

In cases (a) and (b), $\alpha \leq \beta$, for $\lambda < \lambda^*$ we have that $V(\lambda) < 0$ and

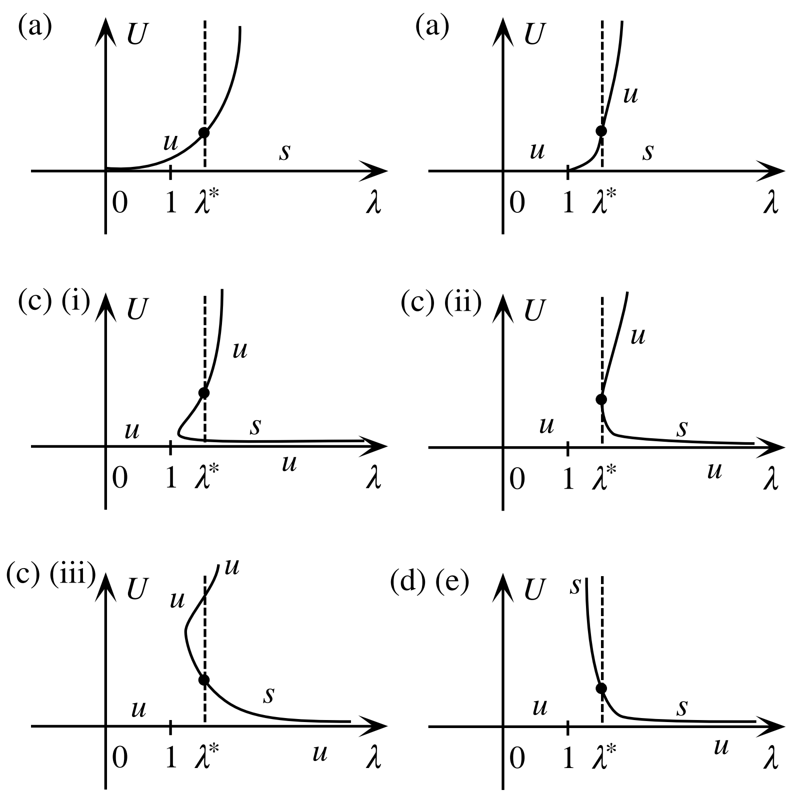

FIGURE 4. Response diagrams for the steady states with $0 < \varepsilon \ll 1$, and so "close" to the linear case of Figure 1, for the seven cases and subcases (a) $\alpha < \beta$, (b) $\alpha = \beta$, (c)(i) $\beta < \alpha < 2\beta/\lambda^*$, (c)(ii) $\alpha = 2\beta/\lambda^*$, (c)(iii) $2\beta/\lambda^* < \alpha < 2\beta$, (d) $\alpha = 2\beta$ and (e) $\alpha > 2\beta$. Subscripts s and u denote stable and unstable branches of solutions, respectively.

so $v(0) = 0$ is larger than any steady state. Consequently v blows up.

For (c), (i) and (ii), $\beta < \alpha \leq 2\beta/\lambda^*$, any steady state $V(\lambda)$ is again negative so v blows up.

However, taking $2\beta/\lambda^* < \alpha < 2\beta$, (c)(iii), the values $\lambda < \lambda_c$ and $\lambda \geq \lambda_c$ must be distinguished. If $\lambda < \lambda_c$ there is no steady state and once again v blows up. Now taking $\lambda_c \leq \lambda < \lambda^*$ there is a steady state greater than $v(0)$, indeed $V(\lambda) > 0$ for these parameter values, and $v(\tau)$ exists for all time, is increasing and bounded above: $v(\tau) \to V_1(\lambda)$

as $\tau \to \infty$ ($V_1 = V_c$ for $\lambda = \lambda_c$).

For (d) $\alpha = 2\beta$, there is a stable, positive, steady state for $1 < \lambda < \lambda^*$ so in this interval, $v(\tau) \to V(\lambda)$ as $\tau \to \infty$. With $\lambda = 1$, $v = v(\tau)$ exists for all time while still being unbounded: $v \to \infty$ as $\tau \to \infty$. For $\lambda < 1$ again there is blow up.

Finally, in case (e), $\alpha > 2\beta$, the stable, positive, steady state $V(\lambda)$ exists for all $\lambda < \lambda^*$ and then $v \to V(\lambda)$ as $\tau \to \infty$.

We thus conjecture that solutions in which v is large exhibit the following behavior:

For $\alpha \leq 2\beta/\lambda^*$, $\lambda < \lambda^*$, then u becomes larger than $O(e^{k/\varepsilon})$ for any k as $t \to \tau^*/\varepsilon$ for some $\tau^* < \infty$.

For $\alpha > 2\beta/\lambda^*$, $\lambda < \lambda_c$ where $1 < \lambda_c < \lambda^*$ if $\alpha < 2\beta$, $\lambda_c = 1$ if $\alpha = 2\beta$, and $\lambda_c = 0$ if $\alpha > 2\beta$, u becomes larger than $O(e^{k/\varepsilon})$ for any k as $t \to \tau^*/\varepsilon$ for some $\tau^* < \infty$. However, for $\lambda_c \leq \lambda < \lambda^*$ with $\alpha < 2\beta$ or $\lambda_c < \lambda < \lambda^*$ with $\alpha = 2\beta$, u approaches a steady state, exponentially large in $1/\varepsilon$, as $t \to \infty$ over a time scale of $O(1/\varepsilon)$. There is a special case $\alpha = 2\beta$, $\lambda = \lambda_c = 1$, for which u becomes much greater than $e^{k/\varepsilon}$ for any k over a time $t \gg 1/\varepsilon$.

6. Conclusions. We have presented a formal asymptotic description of the steady states of (1.1a,b) and their stability when p and q only just exceed unity. This analysis has highlighted the role played by $(p-1)/(q-1)$, especially when this parameter is near one or a half. We expect that our response diagrams will give a reliable guide to the sizes of stationary solutions for larger values of ε. We have also indicated how the steady states can be used to give information on the solution of the unsteady problem.

Acknowledgments. The first and second authors wish to thank the Universidad de La Laguna and the British Council (grant numbers 92/7161 and 93/270) for financial support. The authors also thank a referee for pointing out some estimates and a reference of which they were unaware.

REFERENCES

1. M. Chipot, M. Fila and P. Quittner, *Stationary solutions, blow up and convergence to stationary solutions for semilinear parabolic equations with nonlinear boundary conditions*, Acta Math. Univ. Comenian. **60** (1991), 35–103.

2. J. López-Gómez, V. Márquez and N. Wolanski, *Blow up results and localization of blow up points for the heat equation with a nonlinear boundary condition*, J. Differential Equations **92** (1991), 384–401.

3. ———, *Global behaviour of positive solutions to a semilinear equation with a nonlinear flux condition*, IMA preprint series #810, 1991.

4. H.A. Levine and L.E. Payne, *Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time*, J. Differential Equations **16** (1974), 319–334.

5. M. Fila, *Boundedness of global solutions for the heat equation with nonlinear boundary conditions*, Comment. Math. Univ. Carolin. **30** (1989), 479–484.

6. P. Quittner, *On global existence and stationary solutions for two classes of semilinear parabolic problems*, Comment. Math. Univ. Carolin. **34** (1993), 105–124.

7. W. Walter, *On existence and nonexistence in the large of solutions of parabolic differential equations with a nonlinear boundary condition*, SIAM J. Math. Anal. **6** (1975), 85–90.

Department of Mathematics, Heriot-Watt University, Edingburgh EH14 4AS, UK

Mathematics Institute 24-29 St. Giles, Oxford OX1 3LB, UK

Deptartment of Mathematical Analysis, Univ. La Laguna, La Laguna 38271, Spain

School of Mathematics, Georgia Technical, Atlanta, GA 30332-0160 USA