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INTERPOLATION THEOREM FOR UNBOUNDED OPERATORS

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ABSTRACT. We prove the following unbounded generalization of the strong interpolation theorem [**2**, Corollary 3.16] under some extra hypotheses:

1. If h and k are self-adjoint operators on a Hilbert space H, k is bounded, $h \geq k$ and h_-, k_+ are compact, then there is a compact operator a such that $k \leq a \leq h$.

2. If h and k are self-adjoint operators on H, $h \geq k$ and h_-, k_+ are compact, then for all $\varepsilon > 0$ there is a compact operator a such that $k - \varepsilon 1 \leq a \leq h$.

Let H denote an infinite dimensional Hilbert space and $\mathcal{K}$ the space of compact operators on H. It is well known that there is one-to-one correspondence between bounded below self-adjoint operators, h, on H and densely defined closed quadratic forms, q_h, which are bounded from below given by $q_h(v) = (hv, v)$ on $D(q_h) = D([h - \lambda 1]^{1/2})$, $\lambda \leq h$. Given two densely defined quadratic forms q_1 and q_2, we write $q_1 \leq q_2$ if $D(q_1) \supset D(q_2)$ and $q_1(v) \leq q_2(v)$ for all $v \in D(q_2)$. This ordering defines an ordering of bounded below self-adjoint operators, i.e., we write $h \geq k$ if and only if $q_h \geq q_k$ where q_h and q_k are quadratic forms corresponding to h and k, respectively. As an extension of this ordering, if h is bounded below and k is bounded above self-adjoint operators on H, we will write $h \geq k$ if and only if $q_h(v) \geq q_k(v)$ for all v in $D(q_h) \cap D(q_k)$.

Now we introduce the notion of semi-continuous operators. Let A be a C^*-algebra and A^{**} its enveloping von Neumann algebra. For $M \subset A^{**}$, M_{sa} denotes the set of self-adjoint elements of M and M_{sa}^m the limits of increasing nets of elements of M_{sa}. We also denote by $\tilde{A}$ the C^*-algebra generated by A and the unit 1 of A^{**} and $Q(A)$ the quasi-state space of A, i.e., $Q(A) = \{\phi \in A^* \mid \phi \geq 0, ||\phi|| \leq 1\}$. Equipped with the weak* topology inherited from A^*, $Q(A)$ is a compact convex set. Recall that the evaluation map $\widehat{\ }$ on A_{sa}^{**} given by $\widehat{x}(\phi) = \phi(x)$ for x in A_{sa}^{**} and ϕ in $Q(A)$ is an order preserving isometry of A_{sa}^{**} onto the

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set of bounded affine real valued functions on $Q(A)$ vanishing at zero. C.A. Akemann and G.K. Pedersen [**1**] showed the following theorem, and we refer the readers to [**1, 2**] for further theory of semi-continuous operators.

Theorem 1. [**1**, Theorem 2.1]. *Let h be an element of A_{sa}^{**}. The following conditions are equivalent.*

(1) *h is strongly* lsc*, that is, $h \in \overline{A_{sa}^m}$.*

(2) *$\widehat{h}$ is weak* lower semi-continuous on $Q(A)$.*

(3) *There is an increasing net $(h_i + \alpha_i 1)$ in $\tilde{A}_{sa}$ with limit h such that $h_i \in A_{sa}$, $\alpha_i \in \mathbf{R}$, and $\alpha_i \nearrow 0$.*

Then the following theorem shows that if h is strongly lsc and k is strongly usc such that $h \geq k$ then we can interpolate them by a continuous (both lsc and usc) operator.

Theorem 2. (Strong interpolation theorem) [**2**, Corollary 3.16]. *If $k \leq h$, $k \in [(A_{sa})_m]^-$ and $h \in \overline{A_{sa}^m}$, then there exists an $a \in A_{sa}$ such that $k \leq a \leq h$.*

Remark. If $A = \mathcal{K}$, then $A^{**} = B(H)$ and $\overline{A_{sa}^m} = \{h \in B(H)_{sa} \mid h_- \in \mathcal{K}\}$ (see [**2**, Example 5.A]). So the above theorem says that if $h, k \in B(H)_{sa}$, $h \geq k$ and h_-, $k_+ \in \mathcal{K}$, then there exists an $a \in \mathcal{K}$ such that $h \geq a \geq k$.

As a generalization of the above remark, we may ask if we can still find a compact operator a which interpolates h and k where h and k are unbounded self-adjoint operators such that $h \geq k$ and h_-, $k_+ \in \mathcal{K}$. The following example gives negative answer for this question.

Example. Let H be a separable infinite dimensional Hilbert space. Consider the following two sequences of 2×2 matrices (h_n) and (k_n)

given by

$$h_n = \begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix} \begin{pmatrix} -\beta_n & 0 \\ 0 & \alpha_n \end{pmatrix} \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{pmatrix}$$
$$k_n = \begin{pmatrix} -\alpha_n & 0 \\ 0 & \beta_n \end{pmatrix}$$

where $1 \le \alpha_n \nearrow \infty$, $1 \ge \beta_n \searrow 0$ and $\theta_n = \cos^{-1}(2\sqrt{\alpha_n\beta_n}/(\alpha_n+\beta_n))$. We construct operators h and k on H by taking h_n and k_n in order as their diagonal block submatrices and 0 elsewhere. It is easy to see that $h_n \ge k_n$ for all n (hence $h \ge k$) and $h_-, k_+ \in \mathcal{K}$. If there exists an operator a such that $h \ge a \ge k$, then a little computation shows that a cannot be in $\mathcal{K}$ unless $\sqrt{\alpha_n\beta_n} \to 0$.

So, we cannot generalize it as we like to have. However, the following two theorems, which are the main result of this paper, give affirmative answers under some extra hypotheses for the question. We let $f_\delta(x) = x/(1+\delta x)$ on $(-1/\delta, \infty)$ if $\delta > 0$ on $(-\infty, -1/\delta)$ if $\delta < 0$. Note that f_δ is an operator monotone function on its domain such that $f_\delta \circ f_{-\delta} = f_{-\delta} \circ f_\delta = \mathrm{id}$ and $f_\delta \circ f_\varepsilon = f_{\delta+\varepsilon}$ where defined. For a net (h_i) of bounded below self-adjoint operators, we write $h_i \nearrow h$ if and only if $f_\delta(h_i) \nearrow f_\delta(h)$ for any $\delta > 0$ such that $-1/\delta < h_i, h$. Note that this is equivalent to $q_{h_i} \nearrow q_h$.

Theorem 3. *Let h and k be self-adjoint operators on H such that $h \ge k$ and $h_-, k_+ \in \mathcal{K}$. If either h or k is bounded, then there exists an $a \in \mathcal{K}$ such that $k \le a \le h$.*

Proof. Assume $||k|| < n_0 \in \mathbf{N}$. Then, for $n \ge n_0$, $f_{1/n}(h)$ and $f_{1/n}(k)$ are bounded and $f_{1/n}(h)_-$, $f_{1/n}(k)_+ \in \mathcal{K}$. Let B be the c_0-direct sum of countably many copies of $\mathcal{K}$, $B = \oplus_{n=n_0}^\infty \mathcal{K}_n$, and let

$$\tilde{h} = (\tilde{h}_n) = \left(\frac{1}{n} f_{1/n}(h)\right)_{n=n_0}^\infty$$

and

$$\tilde{k} = (\tilde{k}_n) = \left(\frac{1}{n} f_{1/n}(k)\right)_{n=n_0}^\infty.$$

Then $\tilde{h}, -\tilde{k} \in \overline{B_{sa}^m}$ by Brown [**2**, Proposition 2.11]. Since $\tilde{k}_n \le \tilde{h}_n$ for all $n \ge n_0$, we have $\tilde{k} \le \tilde{h}$ so that we can apply the strong interpolation theorem [**2**, Corollary 3.16]. Therefore, there is b in B_{sa} such that $\tilde{k} \le b \le \tilde{h}$, that is, there exists a $b_n \in \mathcal{K}$ for all $n \ge n_0$ such that $||b_n|| \to 0$ and $(1/n)f_{1/n}(k) \le b_n \le (1/n)f_{1/n}(h)$. We fix an integer $m \ge n_0$ such that $||b_m|| < 1/2$. Then $f_{1/m}(k) \le mb_m \le f_{1/m}(h)$ and $||mb_m|| < m/2$. Hence $f_{-1/m}(mb_m) \in \mathcal{K}$ and $k \le f_{-1/m}(mb_m) \le h$.

If h is bounded, by a symmetric argument, we can find $b \in \mathcal{K}$ such that $k \le b \le h$. □

Theorem 4. *Let h and k be self-adjoint operators on H such that $h \ge k$ and $h_-, k_+ \in \mathcal{K}$. Then for all $\varepsilon > 0$, there exists $a, b \in \mathcal{K}$ such that $k - \varepsilon 1 \le a \le h$, $k \le b \le h + \varepsilon 1$.*

Proof. Fix an integer m such that $m > \max\{||h_-||, ||k_+||\}$. Then, for $n \ge m$, note that $f_{1/n}(h), f_{1/n}(-k) \in \overline{\mathcal{K}_{sa}^m}$. Hence $f_{1/n}(h) + f_{1/n}(-k) \in \overline{\mathcal{K}_{sa}^m}$, and $(f_{1/n}(h) + f_{1/n}(-k))^\wedge$ is a weak* lower semicontinuous function on $Q(\mathcal{K})$ by [**1**, Theorem 2.1]. Since $f_{1/n}(h) \nearrow h$ and $f_{1/n}(-k) \nearrow (-k)$, $\{(f_{1/n}(h) + f_{1/n}(-k))^\wedge\}$ forms an increasing sequence. For any $\phi_v : a \to (av, v)$ in $Q(\mathcal{K})$,

$$\begin{aligned}(f_{1/n}(h) + f_{1/n}(-k))^\wedge(\phi_v) &= \phi_v(f_{1/n}(h)) + \phi_v(f_{1/n}(-k)) \\ &\nearrow \phi_v(h) + \phi_v(-k) \\ &= q_h(v) - q_k(v) \\ &\ge 0 \quad \text{if } v \in D(q_h) \cap D(q_k).\end{aligned}$$

By Dini's theorem, we have $-\varepsilon 1 \le f_{1/n}(h) + f_{1/n}(-k)$ for sufficiently large n, and hence $-f_{1/n}(-k) - \varepsilon 1 \le f_{1/n}(h) \le h$. Applying the previous theorem for $-f_{1/n}(-k) - \varepsilon 1$ and h we can find a in $\mathcal{K}$ such that

$$k - \varepsilon 1 \le -f_{1/n}(-k) - \varepsilon 1 \le a \le h.$$

By a symmetric argument we can find $b \in \mathcal{K}$ such that $k \le b \le h + \varepsilon 1$. □

Remark. Since we are more interested in densely defined operators we restricted our attention to that case. However, we may proceed without the restriction. If h and k in the above theorem are not densely defined, then we assign $+\infty$ and $-\infty$, respectively, for $q_h(v)$ and $q_k(w)$ for $v \notin D(q_h)$, $w \notin D(q_k)$, that is, we regard h and k as $h \oplus \infty(1-p_h)$ and $k \oplus (-\infty)(1-p_k)$ where p_h and p_k denote the projections on $\overline{D(h)}$ and $\overline{D(k)}$, respectively. Then we have no difficulty to proceed as the above thanks to the theory of quadratic forms (see [**4, 5**]).

REFERENCES

1. C.A. Akemann and G.K. Pedersen, *Complications of semicontinuity in C^*-algebra theory*, Duke Math. J. **40** (1973), 785–795.

2. L.G. Brown, *Semicontinuity and multipliers of C^*-algebras*, Canad. J. Math. **40** (1988), 865–988.

3. Hyoungsoon Kim, *Semicontinuity for unbounded operators affiliated with operator algebras*, Ph.D. dissertation, Purdue University, 1992.

4. B. Simon, *Lower semicontinuity of positive quadratic forms*, Proc. Roy. Soc. Edinburgh Sect. A **79** (1977), 267–273.

5. ———, *A canonical decomposition for quadratic forms with applications to monotone convergence theorems*, J. Funct. Anal. **28** (1978), 377–385.

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