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EMBEDDING DERIVATIVES OF $\mathcal{M}$-HARMONIC HARDY SPACES $\mathcal{H}^p$ INTO LEBESGUE SPACES, $0 < p < 2$

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ABSTRACT. A characterization is given of those positive measures μ on B, the open unit ball in C^n, such that differentiation of order k maps the $\mathcal{M}$-harmonic Hardy space $\mathcal{H}^p$ boundedly into $L^p(\mu)$, $0 < p < 2$.

1. Introduction. In [**9**], D. Luecking determined the conditions on positive measure μ on U, the upper half space in R^{n+1}, so that a partial derivative $D^\beta f$ of f of order $|\beta| = k$ belongs to $L^q(\mu)$ whenever $f \in H^p$, the harmonic Hardy space. In this paper we consider the corresponding problem for the unit ball in C^n with the k-fold gradient $\partial^k f$ replacing $D^\beta f$, and by modifying a technique of Luecking we show that the result is very similar to the one for U.

Let B be the open ball in C^n, $n \geq 1$, with normalized volume measure m, and let S denote its boundary. If $\alpha > 0$ and $\xi \in S$, the Koranyi approach regions are defined by

$$D_\alpha(\xi) = \{z = r\eta \in B : |1 - \langle \eta, \xi\rangle| < \alpha(1 - r)\}.$$

(Note that the regions $D_\alpha(\xi)$ are equivalent to the usual admissible approach regions $\{z \in B : |1 - \langle z, \xi\rangle| < 2^{-1}\beta(1 - |z|^2)\}$, $\beta > 1$. For each $E \subset S$ we define the α-tent over E to be $\hat{E}_\alpha = (\cup_{\xi \notin E} D_\alpha(\xi))^C$, the complement being taken in B. If $\alpha = 1$, we will write $\hat{E}$ and $D(\xi)$ instead of $\hat{E}_1$ and $D_1(\xi)$.

Following Coifman, Mayer and Stein [**3**], and Luecking [**9**], we define tent spaces T_r^s for $0 < r$, $s \leq \infty$. Thus, if ν is a positive measure on B, finite on compact sets, and if $r < \infty$, let

$$A_{r,\nu}(f)(\xi) = \left(\int_{D(\xi)} |f|^r \, d\nu \right)^{1/r}$$

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and

$$C_{r,\nu}(f)(\xi) = \left(\sup_{\xi \in Q} \frac{1}{\sigma(Q)} \int_{\hat{Q}} |f(z)|^r (1-|z|)^n \, d\nu(z) \right)^{1/r},$$

where $Q = Q(\eta, \delta) = \{\zeta \in S : |1 - \langle \zeta, \eta \rangle| < \delta\}$ are nonisotrophic balls in S and σ, the rotation invariant probability measure on S.

If $r = \infty$, let

$$C_{\infty,\nu}(f)(\xi) = A_{\infty,\nu}(f)(\xi) = \nu - \operatorname*{ess\,sup}_{z \in D(\xi)} |f(z)|.$$

The tent space $T_r^s(\nu)$ is defined to be the space of ν-equivalence of functions f such that

(i) $A_{r,\nu}(f) \in L^s(\sigma)$, if $0 < r \le \infty$, $0 < s < \infty$,

(ii) $C_{r,\nu}(f) \in L^\infty(\sigma)$, if $0 < r \le \infty$, $s = \infty$.

In case $d\nu(z) = d\tau(z) = (1 - |z|^2)^{-n-1} \, dm(z)$, we omit the subscript ν.

We note that the use of approach regions of "aperture" 1 in the definition of T_r^s is merely a convenience: approach regions of any other aperture would yield the same class of functions with an equivalent norm.

Let $\tilde{\Delta}$ be the invariant Laplacian on B. That is, $(\tilde{\Delta} f)(z) = \Delta(f \circ \varphi_z)(0)$, $f \in C^2(B)$, where Δ is the ordinary Laplacian and φ_z the standard automorphism of B taking 0 to z (see [**10**]). A function f defined on B is $\mathcal{M}$-harmonic, $f \in \mathcal{M}$, if $\tilde{\Delta}(f) = 0$.

We shall call $\mathcal{H}^p = \mathcal{M} \cap T_\infty^p$, $0 < p < \infty$, $\mathcal{M}$-harmonic Hardy space. For $f \in \mathcal{M}$, let $\partial f(z) = (\partial f/\partial z_1, \dots, \partial f/\partial z_n, \partial f/\partial \overline{z_1}, \dots, \partial f/\partial \overline{z_n})$ and for any positive integer k we write $\partial^k f(z) = (\partial^\alpha \bar{\partial}^\beta f(z))_{|\alpha|+|\beta|=k}$ and $|\partial^k f(z)|^2 = \sum_{|\alpha|+|\beta|=k} |\partial^\alpha \bar{\partial}^\beta f(z)|^2$ where

$$\partial^\alpha \bar{\partial}^\beta f(z) = \frac{\partial^{|\alpha|+|\beta|} f(z)}{\partial z_1^{\alpha_1}, \dots, \partial z_n^{\alpha_n}, \partial \overline{z_1}^{\beta_1}, \dots, \partial \overline{z_n}^{\beta_n}},$$

α and β are multiindices.

Let μ be a positive measure on B and consider the problem of determining what conditions on μ imply $|\partial^k f| \in L^q(\mu)$ whenever

$f \in \mathcal{H}^p$. A standard application of the closed graph theorem leads to the following equivalent problem.

Characterize the μ for which there exists a constant C satisfying

$$\left(\int_B |\partial^k f|^q \, d\mu \right)^{1/q} \le C||f||_{\mathcal{H}^p} = C||A_\infty(f)||_{L^p(\sigma)}.$$

The purpose of this paper is to present a solution of this problem in the case $0 < p = q < 2$. Other previously known cases $2 \le p = q < \infty$ and $0 < p < q < \infty$ will be discussed briefly in Section 4. It seems that the solutions for the remaining two cases: $0 < q < \min\{2, p\}$, $2 \le q < p$, are also similar to the one for the upper half space U.

For $z \in B$ and ε, $0 < \varepsilon < 1$, $E_\varepsilon(z) = \{w \in B : |\varphi_z(w)| < \varepsilon\}$. In discussions where the actual value of ε is irrelevant, it may be omitted from the subscripts.

Constants will be denoted by C which may indicate a different constant from one occurrence to the next.

Theorem 1. *Let $0 < p < 2$. For a positive measure μ on B and a positive integer k, a necessary and sufficient condition for*

$$\left(\int_B |\partial^k f|^p \, d\mu \right)^{1/p} \le C||f||_{\mathcal{H}^p} \tag{1.1}$$

is that the function $g(z) = \mu(E(z))/(1 - |z|)^{kp+n}$ belongs to $T^\infty_{2/(2-p)}$.

2. Proof of sufficiency. The following three lemmas will be needed in the proof of sufficiency of Theorem 1.

Lemma 2.1 [**8**]. *Let $k \ge l$ be nonnegative integers, $0 < p < \infty$ and $0 < \varepsilon < 1$. There exists a constant $K = K(k, l, p, \varepsilon, n)$ such that if $f \in \mathcal{M}$, then*

$$|\partial^k f(w)|^p \le K(1 - |w|)^{(l-k)p} \int_{E_\varepsilon(w)} |\partial^l f(z)|^p \, d\tau(z),$$

for all $w \in B$.

Lemma 2.2. *Let $1 < r < \infty$. The following inequality holds whenever $f \in T^1_r(\nu)$ and $g \in T^\infty_{r'}(\nu)$, $r^{-1} + r'^{-1} = 1$,*

$$\int_B |f(z)g(z)|(1-|z|)^n \, d\nu(z) \le C \int_S A_{r,\nu}(f)(\eta) C_{r',\nu}(g)(\eta)\, d\sigma(\eta).$$

Proof. The idea of proof is taken from [**4**, pp. 148, 149]. In this connection see also [**3**]. We define the truncated Koranyi approach region $D^h(\xi)$, $0 < h \le 1$, by

$$D^h(\xi) = \{z \in B : z \in D(\xi), 1-h < |z| < 1\}$$

and set

$$A_{r,\nu}(f|_h)(\xi) = \left(\int_{D^h(\xi)} |f(z)|^r \, d\nu(z) \right)^{1/r}.$$

Now let Q be any nonisotropic ball of radius δ, and let cQ be the ball of the same center as Q of radius $c\delta$, where c is an absolute constant such that $D^\delta(\eta) \subset \widehat{(cQ)}$, for every $\eta \in Q$. By using Fubini's theorem and the definition of $C_{r',\nu}(g)$ we find that

$$\begin{aligned} (2.1)\qquad & \frac{1}{\sigma(Q)} \int_Q [A_{r',\nu}(g|_\delta)]^{r'}(\eta)\, d\eta(\eta) \\ &\le \frac{C}{\sigma(cQ)} \int_Q \left(\int_{D^\delta(\eta)} |g(z)|^{r'} \, d\nu(z) \right) d\sigma(\eta) \\ &\le \frac{C}{\sigma(cQ)} \int_{\widehat{cQ}} |g(z)|^{r'} \, d\nu(z) | \int_Q \chi_{D^\delta(\eta)}(z)\, d\sigma(\eta) \\ &\le \frac{C}{\sigma(cQ)} \int_{\widehat{cQ}} |g(z)|^{r'} (1-|z|)^n \, d\nu(z) \\ &\le C(n) \inf_{\eta \in Q} [C_{r',\nu}(g)(\eta)]^{r'}. \end{aligned}$$

Let M be a positive constant such that $M^{r'} > 2C(n)$. For every g we define $h(\eta)$ as

$$h(\eta) = \sup_{h>0} \{A_{r',\nu}(g|_h)(\eta) \le M C_{r',\nu}(g)(\eta)\}.$$

From (2.1) we see that $\sigma(\{\eta \in Q : h(\eta) < \delta\}) \leq \sigma(Q)/2$. Hence, $\sigma(\{\eta \in Q : h(\eta) \geq \delta\}) \geq C\delta^n$. Using this, Fubini's theorem and Hölder's inequality we find that

$$\begin{aligned}\int_B |f(z)|\,|g(z)|(1-|z|)^n\,d\nu(z) \\ &\leq C^{-1}\int_S \left(\int_{D^{h(\eta)}(\eta)} |f(z)|\,|g(z)|\,d\nu(z)\right) d\sigma(\eta) \\ &\leq C^{-1}\int_S A_{r,\nu}(f|_{h(\eta)})(\eta)A_{r',\nu}(g|_{h(\eta)})(\eta)\,d\sigma(\eta) \\ &\leq MC^{-1}\int_S A_{r,\nu}(f)(\eta)C_{r',\nu}(g)(\eta)\,d\sigma(\eta).\end{aligned}$$

This finishes the proof of the lemma. □

Lemma 2.3. *For $f \in \mathcal{M}$ the following are equivalent* (*with an aperture α fixed*):

(i) $f \in \mathcal{H}^p$,

(ii) $\int_{D_\alpha(\eta)} |\partial^1 f(z)|^2(1-|z|)^{1-n}\,dm(z) \in L^p(\sigma)$,

(iii) *For some $k \geq 1$, $\int_{D_\alpha(\eta)} |\partial^k f(z)|^2(1-|z|)^{2k-n-1}\,dm(z) \in L^p(\sigma)$,*

(iv) *Same as* (iii) *but for every $k \geq 1$.*

Proof. The equivalence (i) ⇔ (ii) can be found in [**5**]. By Lemma 2.1 we have

$$\begin{aligned}|\partial^k f(z)|^2(1-|z|)^{2k-n-1} \\ &\leq C\int_{E_\varepsilon(z)} |\partial^1 f(w)|^2(1-|w|^2)^{1-n}\,dm(w), \quad \varepsilon > 0.\end{aligned}$$

Integrating over $D_\alpha(\eta)$ and applying Fubini on the right gives

$$\begin{aligned}\int_{D_\alpha(\eta)} |\partial^k f(z)|^2(1-|z|^2)^{2k-n-1}\,dm(z) \\ &\leq C\int_{D_\beta(\eta)} |\partial^1 f(z)|^2(1-|z|^2)^{1-n}\,dm(z),\end{aligned}$$

where $\beta > \alpha$.

We may choose $\varepsilon > 0$ so small that if $z \in D_\alpha(\eta)$ then $E_\varepsilon(z) \subset D_\beta(\eta)$. Thus (ii) $\Rightarrow$ (iii). To get (iii) $\Rightarrow$ (ii), the approach in [**2**] is easily modified to show that

$$\int_{D_\alpha(\eta)} |\partial^1 f(z)|^2 (1-|z|^2)^{1-n}\, dm(z) \leq C \int_{D_\beta(\eta)} |\partial^k f(z)|^2 (1-|z|)^{2k-n-1}\, dm(z), \beta > \alpha.$$

Hence, (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof of sufficiency. Let $f \in \mathcal{H}^p$. By Lemma 2.1, we have

$$|\partial^k f(z)|^p \leq C \int_{E(z)} |\partial^k f(w)|^p\, d\tau(w),$$

where C depends on the radius ε of $E(z) = E_\varepsilon(z)$ and on p. From this we get

$$\begin{aligned}
&\int_B |\partial^k f(z)|^p\, d\mu(z) \\
&\quad \leq C \int_B (1-|z|)^{kp} |\partial^k f(z)|^p g(z) \frac{dm(z)}{1-|z|} \\
&\quad \leq C \int_S A_{2/p}((1-|z|)^{kp}|\partial^k f(z)|^p)(\eta) C_{2/(2-p)}(g)(\eta)\, d\sigma(\eta) \\
&\quad \leq C ||g||_{T^\infty_{2/(2-p)}} \\
&\qquad \times \int_S \left(\int_{D(\eta)} (1-|z|)^{2k-n-1} |\partial^k f(z)|^2\, dm(z) \right)^{p/2} d\sigma(\eta) \\
&\quad \leq C ||g||_{T^\infty_{2/(2-p)}} ||f||^p_{\mathcal{H}^p},
\end{aligned}$$

by Lemma 2.2 and Lemma 2.3. □

3. Proof of necessity. Define an $\alpha - T^1_r$-atom, $1 < r < \infty$, as a function $a(z)$ on B, supported in $\widehat{Q_\alpha}$ for some ball Q in S, and satisfying $\int_{\widehat{Q_\alpha}} |a(z)|^r (1-|z|)^{-1}\, dm(z) \leq \sigma(Q)^{1-r}$. In case $r = \infty$, $a(z)$ must satisfy $|a(z)| \leq \sigma(Q)^{-1}$.

An atomic decomposition of the space T^1_∞ is obtained in [**1**]. Analogously, we have the following atomic decomposition of spaces T^1_r, $1 < r < \infty$.

Lemma 3.1. *For each $\alpha > 0$ there is a constant $C = C(\alpha)$ such that, for every $f : B \to C$ such that $\int_S (\int_{D_\alpha(\xi)} |f(z)|^r \, d\tau(z))^{1/r} \, d\sigma(\xi) < \infty$, there are nonnegative α-atoms a_k and nonnegative numbers λ_k such that $|f(z)| \leq \sum_{k\geq 1} \lambda_k a_k(z)$ and $\sum_{k\geq 1} \lambda_k \leq C \int_S (\int_{D_\alpha(\xi)} \times |f(z)|^r \, d\tau(z))^{1/r} \, d\sigma(\xi)$.*

Lemma 3.2. *If $0 < s < 1$ and $\lambda > 1$, then there is a constant $C = C(s, \lambda, n, \alpha)$ such that for any positive finite measure ν on B,*

$$\int_S \left(\int_B \left(\frac{1 - |z|}{|1 - \langle z, \xi \rangle|} \right)^{\lambda n} d\nu(z) \right)^s d\sigma(\xi) \leq C \int_S (\nu(D_\alpha(\xi)))^s \, d\sigma(\xi). \tag{3.1}$$

Proof. It suffices to show (3.1) for $d\nu(z) = f(z) \, d\tau(z)$, where $f(z) \geq 0$. Let $g(z)^r = f(z)$, where $rs = 1$. We need to show that

$$\int_S \left(\int_B \left(\frac{1 - |z|}{|1 - \langle z, \xi \rangle|} \right)^{\lambda n} g(z)^r \, d\tau(z) \right)^{1/r} d\sigma(\xi) \leq C ||g||_{T^1_r}, \tag{3.2}$$

with the T^1_r norm based on $D_\beta(\xi)$, $\beta > \alpha$, (see [**9**]). Because of Lemma 3.1, it suffices to find an upper bound for the left side of (3.2) when $g(z) = a(z)$, a $\beta - T^1_r$-atom.

Without loss of generality, we may suppose that the atom $a(z)$ is supported in $\hat{Q}_\beta$, with $Q = Q(e_1, \delta)$ and that $\int_{\hat{Q}_\beta} |a(z)|^r (1 - |z|)^{-1} \, dm(z) \leq 1$. In this case we divide the outer integral in (3.2) into two parts: the integral I_1 over $|1 - \xi_1| > 2\delta$ and the integral I_0 over $|1 - \xi_1| \leq 2\delta$.

Since

$$\frac{(1 - |z|)^{\lambda n - n}}{|1 - \langle z, \xi \rangle|^{\lambda n}} \leq \frac{C}{|1 - \xi_1|^n}$$

when $|1-\xi_1| > 2\delta$ and $z \in \hat{Q}_\beta$, we see that

$$I_1 \le C \int_{|1-\xi_1|>2\delta} \frac{d\sigma(\xi)}{|1-\xi_1|^{n/r}} \left(\int_{\hat{Q}_\beta} \frac{|a(z)|^r}{1-|z|}\, dm(z) \right)^{1/r} \le C$$

(see [**10**, p. 17]).

By Holder's inequality followed by Fubini's theorem,

$$I_0 \le C \bigg(\int_{\hat{Q}_\beta} \int_{|1-\xi_1|\le 2\delta} \frac{(1-|z|)^{\lambda n - n}}{|1-\langle z,\xi\rangle|^{\lambda n}} |a(z)|^r \times (1-|z|)^{-1}\, d\sigma(\xi)\, dm(z) \bigg)^{1/r} |\sigma(Q)|^{1-1/r}.$$

Since $\lambda > 1$ we have

$$\int_{|1-\xi_1|\le 2\delta} \frac{d\sigma(\xi)}{|1-\langle z,\xi\rangle|^{\lambda n}} \le \frac{C}{(1-|z|)^{\lambda n - n}} \qquad \text{(see [\textbf{10}, p. 17])}.$$

Thus,

$$I_0 \le C \int_{\hat{Q}_\beta} |a(z)|^r (1-|z|)^{-1}\, dm(z) \le C.$$

If $\{z_k\}$ is a sequence in B, we say that it is separated if there is an $\varepsilon \in (0,1)$ such that the balls $E_\varepsilon(z_k)$ are disjoint. When $\nu = \sum_k \delta_{z_k}$ (where δ_z denotes a unit mass at z) we will write $T_r^s\{z_k\}$ instead of $T_r^s(\nu)$. □

Lemma 3.3. *Let $0 < p < 2$, $t > 0$ and $\lambda = n+1+t$. Then $S_\lambda(b_k)(z) = \sum_{k\ge 1} b_k((1-|z_k|)/(1-\langle z,z_k\rangle))^\lambda$ is a bounded map from $T_2^p\{z_k\}$ into H^p whenever the sequence $\{z_k\}$ is separated.*

Recall that H^p is a subspace of $\mathcal{H}^p$ consisting of holomorphic functions.

Proof. For $t > 0$ and $k > n+1$ we define linear operator $\mathcal{R}_t^k$ on H^∞ (the space of bounded analytic functions on B) by

$$\mathcal{R}_t^k f(z) = \gamma_t \int_B \frac{(1-|w|^2)^t f(w)\, dm(w)}{(1-\langle z,w\rangle)^{n+1+k+t}},$$

where $\gamma_t = \Gamma(n+t+1)/(\Gamma(n+1)\Gamma(t+1))$. Let $K(z,w) = 1/(1-\langle z,w\rangle)^{n+1}$ and set $K_w = K(\cdot,w)$. Then $\mathcal{R}_t^k(K_{z_k}^{1+t/(n+1)}) = K_{z_k}^{1+(k+t)/(n+1)}$. Using this and Holder's inequality, we find that

$$|\mathcal{R}_t^k S_\lambda(b_k)(z)|^2 \le \sum_{j\ge 1} |b_j|^2 \frac{(1-|z_j|)^\lambda}{|1-\langle z,z_j\rangle|^{\lambda+k}} \sum_{j\ge 1} \frac{(1-|z_j|)^\lambda}{|1-\langle z,z_j\rangle|^{\lambda+k}}.$$

Fix $\varepsilon \in (0,1)$ so that $E_\varepsilon(z_j)$ are disjoint. Then

$$\begin{aligned}\sum_{j\ge 1} \frac{(1-|z_j|)^\lambda}{|1-\langle z,z_j\rangle|^{\lambda+k}} &\le C\int_B \frac{(1-|w|)^t\,dm(w)}{|1-\langle w,z\rangle|^{\lambda+k}}\\ &\le \frac{C}{(1-|z|)^k} \qquad \text{(see [}\mathbf{10}\text{])}.\end{aligned}$$

Thus,

$$|\mathcal{R}_t^k S_\lambda(b_k)(z)|^2 \le C(1-|z|)^{-k} \sum_{j\ge 1} |b_j|^2 \frac{(1-|z_j|)^\lambda}{|1-\langle z,z_j\rangle|^{\lambda+k}}.$$

To get the H^p norm of $\mathcal{R}_t^k S_\lambda(b_k)(z)$, we integrate this over $D_\alpha(\eta)$ with respect to $(1-|z|)^{2k-n-1}\,dm(z)$ to obtain

$$\begin{aligned}&\int_{D_\alpha(\eta)} |\mathcal{R}_t^k(S_\lambda(b_k))(z)|^2(1-|z|)^{2k-n-1}\,dm(z)\\ &\qquad\le C\sum_{j\ge 1}|b_j|^2(1-|z_j|)^\lambda \int_{D_\alpha(\eta)} \frac{(1-|z|)^{k-n-1}\,dm(z)}{|1-\langle z,z_j\rangle|^{\lambda+k}}\\ &\qquad\le C\sum_{j\ge 1}|b_j|^2(1-|z_j|)^\lambda \int_{D_\alpha(\eta)} \frac{dm(z)}{|1-\langle z,z_j\rangle|^{2n+2+t}}.\end{aligned}$$

An integration in polar coordinates shows that

$$\int_{D_\alpha(\eta)} \frac{dm(z)}{|1-\langle z,z_j\rangle|^{2n+2+t}} \le \frac{C}{|1-\langle z_j,\eta\rangle|^{n+1+t}}.$$

Hence,

$$\begin{aligned}&\int_{D_\alpha(\eta)} |\mathcal{R}_t^k(S_\lambda(b_k))(z)|^2(1-|z|)^{2k-n-1}\,dm(z)\\ &\qquad\le C\sum_{j\ge 1}|b_j|^2\left(\frac{1-|z_j|}{|1-\langle z_j,\eta\rangle|}\right)^\lambda.\end{aligned}$$

Raising to the $p/2$ power, integrating in η and applying Theorem 3.1 and Corollary 3.7 in [**2**] and Lemma 2.2, yields

$$||S_\lambda(b_k)||_{H^p} \leq C||\{b_k\}||_{T_2^p\{z_k\}}.$$

Remark. In [**2**] a characterization of H^p is given in terms of the radial derivative operators $\mathcal{R}^k$, but it is easily seen that the same characterization continues to hold for the operators $\mathcal{R}_t^k$.

Lemma 3.4. *For $1 < r < \infty$ the dual of $T_r^1(\nu)$ is $T_{r'}^\infty(\nu)$, $r^{-1} + r'^{-1} = 1$. The pairing is $\langle f, g\rangle = \int_B f(z)g(z)(1-|z|)^n \, d\nu(z)$.*

Proof. From Lemma 2.2 we see that $T_{r'}^\infty(\nu)$ is contained in the dual of $T_r^1(\nu)$. Conversely, let L be a continuous linear functional on $T_r^1(\nu)$. Let

$$L^\infty L^{r'}(d\nu\, d\sigma) = \left\{ f(z,\xi) : B \times S \to C, \sup_{\xi \in S} \left(\int_B |f(z,\xi)|^{r'} \, d\nu(z) \right)^{1/r'} < \infty \right\}.$$

Clearly, $T_{r'}^\infty(\nu)$ embeds in $L^\infty L^{r'}(d\nu\, d\sigma)$ by $f(z) \to f(z)\chi_{D(\xi)}(z)$. Since $(L^1L^r)^* = L^\infty L^{r'}$ by the Hahn-Banach theorem, there is a function $g(z,\xi) \in L^\infty L^{r'}(d\nu\, d\sigma)$ such that

$$L(f) = \int_S \int_{D(\xi)} g(z,\xi) f(z) \, d\nu(z) \, d\sigma(\xi),$$

with $||L|| = ||g||_{\infty, r'}$. By Fubini's theorem,

$$L(f) = \int_B f(r\eta) \int_{\{\xi : |1 - \langle \xi, \eta\rangle| < 1 - r\}} g(r\eta, \xi) \, d\sigma(\xi) \, d\nu(r\eta).$$

It now suffices to show that

$$P^0 g(\rho\eta) = (1-\rho)^{-n} \int_{\{\xi : 1 - \langle \xi, \eta\rangle| < 1 - \rho\}} g(\rho\eta, \xi) \, d\sigma(\xi)$$

defines a bounded linear operator from $L^\infty L^{r'}(d\nu\, d\sigma)$ to $T^\infty_{r'}(\nu)$.

Let Q be a nonisotropic ball in S, and consider

$$\begin{aligned}\frac{1}{\sigma(Q)}\int_{\hat{Q}} |P^0 g(z)|^{r'}(1-|z|)^n\, d\nu(z) &\le \frac{1}{\sigma(Q)}\int_{\hat{Q}}\int_{\{\xi:1-\langle\xi,\eta\rangle|<1-\rho\}} |g(\rho\eta,\xi)|^{r'}\, d\sigma(\xi)\, d\nu(\rho\eta)\\ &= \frac{1}{\sigma(Q)}\int_S\int_{\hat{Q}\cap D(\xi)} |g(z,\xi)|^{r'}\, d\nu(z)\, d\sigma(\xi)\\ &\le \frac{1}{\sigma(Q)}\int_Q\int_B |g(z,\xi)|^{r'}\, d\nu(z)\, d\sigma(\xi)\\ &\le ||g||^{r'}_{r',\infty}.\end{aligned}$$

Thus $||C_{r',\nu}(P^0 g)||_\infty \le ||g||_{r',\infty}$.

Proof of necessity. Let μ be a positive measure on B satisfying $(\int_B |\partial^k f(z)|^p\, d\mu(z))^{1/p} \le C||f||_{\mathcal{H}^p}$, $f \in \mathcal{H}^p$. Then we also have $\int_B |\mathcal{R}^k_t f(z)|^p\, d\mu(z) \le C||f||^p_{H^p}$, for every $f\in H^p$. Let f be set equal to $f(z) = S_\lambda(b_k)(z) = \sum b_k((1-|z_k|)/(1-\langle z,z_k\rangle))^\lambda$, $\lambda = n+1+t$, $t>0$, for some $\{b_k\}\in T^p_2\{z_k\}$ and some separated sequence $\{z_k\}$. Then by Lemma 3.3, we get

$$\int_B \left|\sum_{k\ge1} b_k\frac{(1-|z_k|)^\lambda}{(1-\langle z,z_k\rangle)^{\lambda+k}}\right|^p d\mu(z) \le C||\{b_k\}||^p_{T^p_2\{z_k\}}.$$

Now if each b_k is replaced by $b_k r_k(t)$ for fixed $t\in[0,1)$, (where $r_k(t)$ are Rademacher functions), the righthand side is unchanged. We can then integrate the resulting equation in t and use the lower bound in Khinchine's inequality to obtain

$$\int_B \left(\sum_{k\ge1}\left|b_k\frac{(1-|z_k|)^\lambda}{(1-\langle z,z_k\rangle)^{\lambda+k}}\right|^2\right)^{p/2} d\mu(z) \le C||\{b_k\}||^p_{T^p_2\{z_k\}}.$$

From this, we get

$$\sum_{j\ge1}|b_j|^p(1-|z_j|)^{-kp}\mu(E(z_j)) \le C||\{b_k\}||^p_{T^p_2\{z_k\}}.$$

Putting $|b_j|^p = c_j$, we get

$$\sum_{j\geq 1} c_j \frac{\mu(E(z_j))}{(1-|z_j|)^{kp+n}}(1-|z_j|)^n \leq C||\{c_j\}||_{T^1_{2/p}\{z_j\}},$$

for any positive $\{c_j\} \in T^1_{2/p}\{z_j\}$. This inequality continues to hold for nonpositive $\{c_j\}$, so we conclude that $\{\mu(E(z_j))/(1-|z_j|)^{kp+n}\} \in (T^1_{2/p}\{z_j\})^* = T^\infty_{2/(2-p)}\{z_j\}$, by Lemma 3.4. □

From this follows a discrete version of Theorem 1, which in turn implies the continuous version stated in Theorem 1 (see [**9**]).

4. Other cases. The solutions for the cases $0 < p < q < \infty$ and $2 \leq p = q < \infty$ were presented in [**6**] and [**7**]. For the reader's convenience, we state them.

Theorem 2. *Let $0 < p < q < \infty$ or $2 \leq p = q < \infty$. For a positive measure μ on B and a positive integer k, a necessary and sufficient condition for* (1.1) *is that there exists a constant K for which*

$$\mu(E(z)) \leq K(1-|z|)^{kq+nq/p}, \quad z \in B.$$

Theorem 1, Theorem 2 and the corresponding results for the upper half-space U lead to the following

Conjecture. *For a positive measure μ on B and a positive integer k, a necessary and sufficient condition for* (1.1) *is that the function $g(z) = \mu(E(z))/(1-|z|)^{kq+n}$ satisfies*

(i) $g \in T^{p/(p-q)}_{2/(2-q)}$, *if* $0 < q < p$, $q < 2$,

(ii) $g \in T^{p/(p-q)}_{\infty}$, *if* $2 \leq q < p$.

REFERENCES

1. P. Ahern, *Exceptional sets for holomorphic Sobolev functions*, Michigan Math. J. **35** (1988), 29–41.

2. P. Ahern and J. Bruna, *Maximal and area integral characterization of Hardy-Sobolev spaces in the unit ball of* C^n, Rev. Math. Iberoamericana **4** (1988), 123–153.

3. R.R. Coifman, Y. Meyer and E.M. Stein, *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal. **62** (1985), 304–335.

4. C. Fefferman and E. Stein, H^p *spaces of several variables*, Acta Math. **129** (1972), 137–193.

5. D. Geller, *Some results on* H^p *theory for the Heisenberg group*, Duke Math. J. **47** (1980), 365–390.

6. M. Jevtić, *Embedding derivatives of* $\mathcal{M}$*-harmonic Hardy spaces into Lebesgue spaces*, Publ. Inst. Math.

7. ———, *Embedding derivatives of* $\mathcal{M}$*-harmonic tent spaces into Lebesgue spaces*, Math. Balcanika, to appear.

8. M. Jevtić and M. Pavlović, *On* $\mathcal{M}$*-harmonic Bloch space*, PAMS **123** (1995), 1385–1393.

9. D. Luecking, *Embedding derivatives of Hardy spaces into Lebesgue spaces*, preprint.

10. W. Rudin, *Function theory in the unit ball of* C^n, Springer Verlag, New York, 1980.

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