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ON RANDOM WALKS AND LEVELS OF THE FORM n^α

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1. Introduction. Let $\{X_i, i = 1, 2, 3, \dots\}$ be a collection of independent and identically distributed random variables with finite expectation μ and finite variance σ^2. Define $S_n = \sum_{i=1}^n X_i$ for $n = 1, 2, 3, \dots$. It follows from results such as the Hartman-Wintner law of the iterated logarithm (see [**10**, p. 293]) that if $\alpha > 1/2$, then $\lim_{n\to\infty}(S_n - n\mu)/n^\alpha = 0$ a.s. (The case $\alpha = 1$ is the standard Strong Law of Large Numbers.) An equivalent statement is that for each number $\varepsilon > 0$ the inequality

$$|S_n - n\mu| > \varepsilon n^\alpha$$

is satisfied for only finitely many n.

Consider the sequence of events $\{A_n, n = 1, 2, 3, \dots\}$ where $A_n = (|S_n - n\mu| > \varepsilon n^\alpha)$. Let $I(A)$ denote the indicator function of the event A. We consider the random variable $N(\varepsilon)$ where $N(\varepsilon)$ is defined by

$$N(\varepsilon) = \sum_{n=1}^{\infty} I(A_n). \tag{1.1}$$

$N(\varepsilon)$ represents the number of indices for which A_n occurs. For notational convenience we suppress, in the notation, the dependence of $N(\varepsilon)$ on α. For $\alpha > 1/2$, this is a finite-valued random variable. Conditions related to the existence of various moments of $N(\varepsilon)$ for a variety of boundaries and settings have been investigated by Lai and Lan [**5**], Slivka and Severo [**9**], Slivka [**8**], Stratton [**11**], and Griffiths and Wright [**4**].

In Section 2 of this paper we consider the expected value of $N(\varepsilon)$ for the special case when X_i is normally distributed. In Slivka and

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Severo [**9**, Theorem 2], explicit upper and lower bounds are derived in this setting where $\alpha = 1$. Using a simpler method we derive analogous upper and lower bounds for $E(N(\varepsilon))$ for the more general case $\alpha > 1/2$ (Theorem 2.3), which includes the aforementioned bounds as a special case. Theorem 2.4 describes the asymptotic behavior of $E(N(\varepsilon))$ as $\varepsilon \to 0$. This generalizes a result indicated in Slivka and Severo [**9**] for the case $\alpha = 1$.

In part 3 we consider the number of times the random walk $\{S_n, n = 1, 2, 3, \dots\}$ lies above a linear boundary. More precisely, we define the random variable $N_+(\varepsilon)$ as

$$N_+(\varepsilon) = \sum_{n=1}^{\infty} I(B_n) \tag{1.2}$$

where B_n is the event defined by $B_n = (S_n > n(\mu + \varepsilon))$ and $I(\cdot)$ is the indicator function defined earlier. This variable has been investigated in Razanadrakoto and Severo [**6**] where expressions generating the exact distribution of $N_+(\varepsilon)$ are derived. Section 3.1 begins with the derivation of close upper and lower bounds for the expected value of $N_+(\varepsilon)$ in the special case when X_i is normally distributed (Corollary 3.1). We also note an analogous result under the same conditions on the variance of $N_+(\varepsilon)$ (inequality (3.3)).

The primary results in Section 3 concern the first index, if one exists, for which $(S_n > n(\mu + \varepsilon))$ occurs. We define the variable $T(\varepsilon)$ by

$$T(\varepsilon) = \begin{cases} \inf\{n \geq 1 : S_n > n(\mu + \varepsilon)\} & \text{if such an } n \text{ exists} \\ \infty & \text{otherwise.} \end{cases} \tag{1.3}$$

Under the hypothesis that $\{X_i, i = 1, 2, 3, \dots\}$ are independent and identically distributed random variables with finite expectation, we derive a general recursive expression generating the exact distribution of $T(\varepsilon)$ (Theorem 3.3).

The final results in Section 3 center around the conditional expectation of $T(\varepsilon)$ given that $T(\varepsilon)$ is finite. Theorem 3.4 describes a relationship between this conditional expectation and the expected value of $N_+(\varepsilon)$. We then investigate the asymptotic behavior of the conditional expectation as $\varepsilon \to 0$ in the case where the common distribution function of $\{X_i, i = 1, 2, 3, \dots\}$ is symmetric and continuous (Theorem

3.5) and in the special case where the common distribution function of $\{X_i, i = 1, 2, 3, \dots\}$ is normal (Theorem 3.6).

2. The expected value of $N(\varepsilon)$. The following lemmas are employed at several points in the development.

Lemma 2.1. *If Y is a nonnegative random variable, then*

$$\sum_{n=1}^{\infty} P(Y \geq n) \leq EY \leq 1 + \sum_{n=1}^{\infty} P(Y \geq n).$$

Proof. See Ash [**1**, p. 275]. □

Lemma 2.2. *If the random variable Z has a standard normal distribution, then for $p > -1$,*

$$E|Z|^p = \frac{2^{p/2}}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right).$$

Proof. The result follows immediately using an elementary change of variables. □

Theorem 2.3. *Let $\{X_i, i = 1, 2, 3, \dots\}$ be a sequence of independent random variables, normally distributed with mean μ and variance σ^2. Let $\alpha > 1/2$ and $N(\varepsilon)$ be as defined in (1.1). For $\varepsilon > 0$,*

(2.1)

$$\rho^{-2/(2\alpha-1)} \frac{2^{1/(2\alpha-1)}\Gamma((2\alpha+1)/(2(2\alpha-1)))}{\sqrt{\pi}} - 1 \leq EN(\varepsilon) \leq \rho^{-2/(2\alpha-1)} \frac{2^{1/(2\alpha-1)}\Gamma((2\alpha+1)/(2(2\alpha-1)))}{\sqrt{\pi}}$$

where $\rho = \varepsilon/\sigma$.

Proof. It follows from the definition of $N(\varepsilon)$ that

$$\begin{aligned} EN(\varepsilon) &= \sum_{n=1}^{\infty} P(|S_n - n\mu| > \varepsilon n^{\alpha}) \\ &= \sum_{n=1}^{\infty} P(|Z| > \rho n^{\alpha-1/2}) \\ &= \sum_{n=1}^{\infty} P(\rho^{-2/(2\alpha-1)}|Z|^{2/(2\alpha-1)} > n) \end{aligned}$$

where Z has a standard normal distribution. The application of Lemmas 2.1 and 2.2 to the latter sum yields inequality (2.1). □

Remarks. 1) We note that, as expected, the bounds on $E(N(\varepsilon))$ are based on ε and σ through $\rho = \varepsilon/\sigma$.

2) For the case $\alpha = 1$, Theorem 2.3 reduces to Theorem 2 in Slivka and Severo [**9**].

Examples. The following examples demonstrate the relative closeness of the upper and lower bounds in Theorem 2.3 when ρ is small.

1) Consider the case $\sigma = 1$ and $\alpha = 1$. If $\varepsilon = .5$ Theorem 2.3 yields the inequality $3 \leq E(N(\varepsilon)) \leq 4$. For $\varepsilon = .1$, the inequality $99 \leq E(N(\varepsilon)) \leq 100$ is obtained from Theorem 2.3.

2) Consider the case $\sigma = 1$ and $\alpha = 3/4$. For $\varepsilon = 1$, Theorem 2.3 yields $2 \leq E(N(\varepsilon)) \leq 3$. For $\varepsilon = .5$, it follows that $47 \leq E(N(\varepsilon)) \leq 48$. For $\varepsilon = .1$, we obtain the estimate that $29,999 \leq E(N(\varepsilon)) \leq 30,000$.

Theorem 2.4 describes an asymptotic approximation for $E(N(\varepsilon))$ as $\rho \to 0$ for the general case $\alpha > 1/2$. This includes as a special case a result noted in Slivka and Severo [**9**, p. 732] for $\alpha = 1$. We apply a method of proof suggested in their paper.

Theorem 2.4. *Assume the same hypotheses as in Theorem* 2.3. *Then*

(2.2)
$$\lim_{\rho\to 0}\left|E(N(\varepsilon)) - \left\{\rho^{-2/(2\alpha-1)}\frac{2^{1/(2\alpha-1)}\Gamma((2\alpha+1)/2(2\alpha-1)))}{\sqrt{\pi}} - \frac{1}{2}\right\}\right| = 0$$

where $\rho = \varepsilon/\sigma$.

Proof. Let $\Phi(\cdot)$ denote the cumulative distribution function for the standard normal distribution. Then, as in the proof of Theorem 2.3, we obtain

$$E(N(\varepsilon)) = 2\sum_{n=1}^{\infty} \Phi(-\rho n^{\alpha-1/2}).$$

Define the function f by $f(x) = \Phi(-\rho x^{\alpha-1/2})$ and the function ξ by $\xi(x) = \int_0^x ([t] - t + 1/2)\,dt$ for $x \geq 0$. We apply a standard form of the Euler-MacLaurin summation formula to the latter sum and obtain

$$E(N(\varepsilon)) = 2\int_1^{\infty} f(x)\,dx + f(1) + \lim_{n\to\infty} f(n) + 2\int_1^{\infty} f''(x)\xi(x)\,dx.$$

Since $\xi(x) \leq 1/8$ for $x > 0$ and $\lim_{n\to\infty} f(n) = 0$, it follows that
(2.3)

$$\left|E(N(\varepsilon)) - 2\int_0^{\infty} f(x)\,dx + 2\int_0^1 f(x)\,dx - \Phi(-\rho)\right| \leq \frac{1}{4}\int_1^{\infty} |f''(x)|\,dx.$$

We apply the change of variables $y = \rho x^{\alpha-1/2}$ and integration by parts to the integral $\int_0^\infty f(x)\,dx$ with the result
(2.4)

$$\begin{aligned}\int_0^{\infty} f(x)\,dx &= \int_0^{\infty} \Phi(-\rho x^{\alpha-1/2})\,dx \\ &= \frac{2}{2\alpha-1}\rho^{-2/(2\alpha-1)} \int_0^{\infty} [1-\Phi(y)]y^{(3-2\alpha)/(2\alpha-1)}\,dy \\ &= \rho^{-2/(2\alpha-1)}\Bigg\{\lim_{y\to\infty}[1-\Phi(y)]y^{2/(2\alpha-1)} \\ &\qquad + \int_0^{\infty} y^{2/(2\alpha-1)}\phi(y)\,dy\Bigg\}\end{aligned}$$

where $\phi(\cdot)$ denotes the density function of the standard normal distribution. Using standard bounds on the tail probabilities of the standard normal distribution (see Feller [**2**, p. 175]) or other methods, it is easily shown that

$$\lim_{y\to\infty}[1-\Phi(y)]y^{2/(2\alpha-1)} = 0.$$

Applying Lemma 2.2 with $p = 2/(2\alpha - 1)$ to the last integral in (2.4) results in

(2.5)
$$\int_0^\infty f(x)\,dx = \rho^{-2/(2\alpha-1)} 2^{(2-2\alpha)/(2\alpha-1)} \pi^{-1/2} \Gamma((2\alpha+1)/(2(2\alpha-1))).$$

Since $\Phi(\cdot)$ is bounded, the Lebesgue dominated convergence theorem can be used to obtain

$$\lim_{\rho\to 0} \int_0^1 \Phi(-\rho x^{\alpha-1/2})\,dx = \int_0^1 \lim_{\rho\to 0} \Phi(-\rho x^{\alpha-1/2})\,dx = \frac{1}{2}. \tag{2.6}$$

We now show that

$$\lim_{\rho\to 0} \int_1^\infty |f''(x)|\,dx = 0. \tag{2.7}$$

Elementary calculations show that there exist constants $C_1 > 0$ and $C_2 > 0$ depending only upon α so that

(2.8)
$$|f''(x)| \le C_1 \rho x^{\alpha-5/2} \exp\left(-\frac{\rho^2 x^{2\alpha-1}}{2}\right) + C_2 \rho^3 x^{3\alpha-7/2} \exp\left(-\frac{\rho^2 x^{2\alpha-1}}{2}\right).$$

Consider $\int_1^\infty x^{\alpha-5/2} \exp(-\rho^2 x^{2\alpha-1}/2)\,dx$. In the case $\alpha < 3/2$, it is clear that $\int_1^\infty x^{\alpha-5/2} \exp(-\rho^2 x^{2\alpha-1}/2)\,dx < C$ for some positive constant C independent of ρ. Therefore, for $\alpha < 3/2$, it follows that

$$\lim_{\rho\to 0} \rho \int_1^\infty x^{\alpha-5/2} \exp\left(-\frac{\rho^2 x^{2\alpha-1}}{2}\right) dx = 0. \tag{2.9}$$

Consider the case $\alpha > 3/2$. The change of variables $y = \rho^2 x^{2\alpha-1}/2$ yields the existence of a constant C independent of ρ so that

$$\int_0^\infty x^{\alpha-5/2} \exp(-\rho^2 x^{2\alpha-1}/2)\,dx = C\rho^{(3-2\alpha)/(2\alpha-1)} \int_0^\infty y^{-(2\alpha-1)/2(2\alpha-1)} \exp(-y)\,dy.$$

It follows from this that (2.9) holds for all $\alpha > 3/2$.

For $\alpha = 3/2$, the same change of variables results in the equality

$$\int_1^\infty x^{\alpha-5/2} \exp(-\rho^2 x^{2\alpha-1}/2)\, dx = \frac{1}{2}\int_{\rho^2/2}^\infty y^{-1} \exp(-y)\, dy. \tag{2.10}$$

The latter can be written as

$$\int_{\rho^2/2}^1 y^{-1} \exp(-y)\, dy + \int_1^\infty y^{-1} \exp(-y)\, dy. \tag{2.11}$$

Clearly, $\lim_{\rho\to 0}(1/2)\rho \int_1^\infty y^{-1}\exp(-y)\, dy = 0$. We also observe that

$$\int_{\rho^2/2}^1 y^{-1} \exp(-y)\, dy < \int_{\rho^2/2}^1 y^{-1}\, dy = \ln 2 - 2\ln \rho. \tag{2.12}$$

Since $\lim_{\rho\to 0} \rho \ln \rho = 0$, (2.10)–(2.12) yield the validity of (2.9) for the case $\alpha = 3/2$ and hence for all $\alpha > 1/2$.

We now consider the integral $\int_1^\infty x^{3\alpha-7/2}\exp(-\rho^2 x^{2\alpha-1}/2)\, dx$. For $1/2 < \alpha < 5/6$, it is clear that there exists a constant C independent of ρ so that $\int_1^\infty x^{3\alpha-7/2}\exp(-\rho^2 x^{2\alpha-1}/2)\, dx < C$. Therefore,

$$\lim_{\rho\to 0} \rho^3 \int_1^\infty x^{3\alpha-7/2} \exp(-\rho^2 x^{2\alpha-1}/2)\, dx = 0. \tag{2.13}$$

The case $\alpha > 5/6$ follows as before. With the change of variables $y = \rho^2 x^{2\alpha-1}/2$, we obtain

$$\begin{aligned}\int_1^\infty & x^{3\alpha-7/2} \exp(-\rho^2 x^{2\alpha-1}/2)\, dx \\ &< \int_0^\infty x^{3\alpha-7/2} \exp(-\rho^2 x^{2\alpha-1}/2)\, dx \\ &= C\rho^{(5-6\alpha)/(2\alpha-1)} \int_0^\infty y^{(2\alpha-3)/(2(2\alpha-1))} \exp(-y)\, dy\end{aligned} \tag{2.14}$$

for some constant C independent of ρ. It is easily seen that the last integral converges. Therefore, (2.14) is valid for $\alpha > 5/6$.

Consider the case $\alpha = 5/6$. We obtain

$$\int_1^\infty x^{3\alpha-7/2} \exp(-\rho^2 x^{2\alpha-1}/2)\, dx = \frac{3}{2}\int_{\rho^2/2}^\infty y^{-1} \exp(-y)\, dy. \tag{2.15}$$

Write the latter integral as in (2.11). Clearly,

$$\lim_{\rho\to 0}\rho^3\int_1^\infty y^{-1}\exp(-y)\,dy = 0. \tag{2.16}$$

From (2.14)–(2.16) and the fact that $\lim_{\rho\to 0}\rho^3\ln\rho = 0$, it follows that (2.13) holds for $\alpha = 5/6$ and hence for all $\alpha > 1/2$. Thus, (2.7) holds for all $\alpha > 1/2$.

Combining (2.5)–(2.7) proves (2.2). □

Examples. 1) In the case $\alpha = 1$, Theorem 2.4 demonstrates that

$$\lim_{\rho\to 0}|E(N(\varepsilon)) - (\rho^{-2} - 1/2)| = 0.$$

This is the same result described in Slivka and Severo [**9**, p. 732]. (Note there is a misprint in that paper.)

2) In the case $\alpha = 3/4$, it follows from Theorem 2.4 that

$$\lim_{\rho\to 0}|E(N(\varepsilon)) - (3\rho^{-4} - 1/2)| = 0.$$

3. The variable $N_+(\varepsilon)$. Recall the definition of $N_+(\varepsilon)$ as given in (1.2). This variable was studied in Razanadrakoto and Severo [**6**] where, among other things, they derived an expression for the probability generating function of $N_+(\varepsilon)$.

3.1. *The expected value and variance of* $N_+(\varepsilon)$. We begin this section with two results that complement those in Section 2 of the present paper for the special case $\alpha = 1$.

Corollary 3.1. *Let $\{X_i, i = 1,2,3,\dots\}$ be a sequence of i.i.d. random variables, normally distributed with mean μ and variance σ^2. For $\varepsilon > 0$,*

$$(\rho^{-2} - 1)/2 \le EN_+(\varepsilon) \le \rho^{-2}/2 \tag{3.1}$$

where $\rho = \varepsilon/\sigma$.

Proof. From the definition of $N_+(\varepsilon)$ in (1.2) we note that

$$EN_+(\varepsilon) = \sum_{n=1}^{\infty} P(S_n > n(\mu+\varepsilon)). \tag{3.2}$$

The result (3.1) follows from Theorem 2.3 and the symmetry of the random variables $\{X_i, i = 1, 2, 3, \dots\}$. □

With the hypotheses and notation of Theorem 3.1, the following result on the variance of $N_+(\varepsilon)$ holds:

$$3\rho^{-4}/4 - 1/8 \le \mathrm{Var}\,[N_+(\varepsilon)] \le 3\rho^{-4}/4 + 1/8. \tag{3.3}$$

The proof, which we omit, makes use of a series representation for the variance of $N_+(\varepsilon)$ derived in Razanadrokoto and Severo [**6**, p. 181] and follows along the lines of the proof of Theorem 2.4.

3.2. *On the distribution and conditional expectation of the random variable* $T(\varepsilon)$. Let $\{X_i, i = 1, 2, 3, \dots\}$ be a collection of independent and identically distributed random variables with finite expectation μ. For Theorem 3.3 and Theorem 3.4 we make no further assumptions concerning the distribution of the random variables.

Recall that the random variable $T(\varepsilon)$, defined by (1.3), describes the first index, if one exists, for which the event $(S_n > n(\mu+\varepsilon))$ occurs. It follows from Razanadrokoto and Severo [**6**] that $T(\varepsilon)$ is a defective random variable, i.e., $P(T(\varepsilon) < \infty) < 1$. As in their work, we consider the equivalent event that $(S_n' > 0)$ for $S_n' = \sum_{i=1}^n X_i'$ with $X_i' = X_i - (\mu+\varepsilon)$.

For arbitrary $\varepsilon \ge 0$, define τ_n and a_n for $n = 1, 2, 3, \dots$, by

$$\tau_n = P(T(\varepsilon) = n) \tag{3.4}$$

and

$$a_n = P(S_n > n(\mu+\varepsilon)). \tag{3.5}$$

Let $\tau(s)$ represent the generating function of $\{\tau_n\}$. That is, we define $\tau(s)$ by

$$\tau(s) = \sum_{n=1}^{\infty} \tau_n s^n. \tag{3.6}$$

Applying a result in Feller [**3**, Theorem 1, p. 413], it is immediate that

$$
\begin{aligned}
\log \frac{1}{1-\tau(s)} &= \sum_{n=1}^{\infty} \frac{s^n}{n} P(S'_n > 0) \\
&= \sum_{n=1}^{\infty} \frac{s^n}{n} P(S_n > n(\mu+\varepsilon)) \qquad (3.7)\\
&= \sum_{n=1}^{\infty} \frac{a_n}{n} s^n.
\end{aligned}
$$

In order to determine the distribution of $T(\varepsilon)$ from its generating function, we use the following lemma which is used for an analogous purpose in Razanadrokoto and Severo [**6**].

Lemma 3.2. *Let* $\{u_i, i = 0, 1, 2, \dots\}$ *and* $\{v_i, i = 0, 1, 2, \dots\}$ *be sequences of real numbers, and assume that* $U(s) = \sum_{j=0}^{\infty} u_j s^j$ *and* $V(s) = \sum_{j=0}^{\infty} v_j s^j$ *are such that* $U(s), V(s)$*, and their nth derivatives exist for* $0 < s < \eta$ *for some* $\eta > 0$*. If*

$$U(s) = \exp(V(s)) \quad \text{for } 0 < s < \eta,$$

then

$$u_n = n^{-1} \sum_{j=1}^{n} j v_j u_{n-j}, \qquad n = 1, 2, \dots .$$

Proof. See Razanadrakoto and Severo [**6**, p. 180]. □

Theorem 3.3 provides a recursive formula that generates the exact distribution of the defective random variable $T(\varepsilon)$.

Theorem 3.3. *Let* $\{X_i, i = 1, 2, 3, \dots\}$ *be independent and identically distributed random variables with finite expectation* μ*. Let* $\varepsilon \geq 0$*. Then, with the notation defined in* (3.4) *and* (3.5),

$$\tau_n = -n^{-1} \sum_{j=1}^{n} a_j \tau_{n-j} \quad \text{for } n = 1, 2, \dots , \qquad (3.8)$$

where τ_0 is defined as $\tau_0 = -1$.

Proof. With τ_0 as defined in the hypotheses, the relation described in (3.7) can be rewritten as

$$\sum_{n=0}^{\infty}(-\tau_n)s^n = \exp\left[-\sum_{n=1}^{\infty}\frac{a_n}{n}s^n\right]. \tag{3.9}$$

The result (3.8) is obtained by applying Lemma 3.2 with $u_n = -\tau_n$ and $v_n = -a_n/n$. □

Example. Let $\{X_i, i = 1, 2, 3, \dots\}$ be a sequence of independent standard normal random variables. The following probabilities are obtained (correct to the number of decimal places shown) from Theorem 3.3:

n	$P(T(1) = n)$	$P(T(.5) = n)$	$P(T(.01) = n)$
1	.158655	.308538	.499601
2	.0267391	.0722773	.124917
3	.00830393	.032322	.0624605
4	.00318112	.0179333	.0390383
5	.00135864	.0111152	.027327
6	.000620403	.00737263	.0204954
7	.000296445	.00511974	.0161035
8	.000146376	.00367504	.0130842
9	.0000740955	.00270495	.0109035
10	.0000382454	.0020304	.00926796

We have noted earlier that the variable $T(\varepsilon)$ is a defective random variable. Therefore, it is of interest to consider the conditional expectation of $T(\varepsilon)$ given that $T(\varepsilon)$ is finite.

Theorem 3.4. *Let $\{X_i, i = 1, 2, 3, \dots\}$ be independent and identically distributed random variables with finite expectation μ. Let $\varepsilon > 0$.*

Then

$$E[T(\varepsilon)|T(\varepsilon)<\infty]=\frac{P(N_+(\varepsilon)=0)}{P(N_+(\varepsilon)>0)}EN_+(\varepsilon). \tag{3.10}$$

Note. The probabilities appearing in (3.10) can be expressed as $P(N_+(\varepsilon)=0)=P(T(\varepsilon)=\infty)$ and $P(N_+(\varepsilon)>0)=P(T(\varepsilon)<\infty)$.

Proof. Since

$$P(T(\varepsilon)=n|T(\varepsilon)<\infty)=\frac{\tau_n}{P(T(\varepsilon)<\infty)},$$

it follows that

$$E[T(\varepsilon)|T(\varepsilon)<\infty]=\left(\sum_{n=1}^{\infty}\tau_n\right)^{-1}\sum_{n=1}^{\infty}n\tau_n. \tag{3.11}$$

Let $\tau(s)$ represent the generating function of $\{\tau_n, n=1,2,3,\dots\}$ as defined in (3.6). Then (3.11) can be rewritten as

$$E[T(\varepsilon)|T(\varepsilon)<\infty]=\frac{\tau'(1)}{\tau(1)}. \tag{3.12}$$

A simple calculation using (3.7) leads to

$$\frac{\tau'(s)}{1-\tau(s)}=\sum_{n=1}^{\infty}P(S_n>n(\mu+\varepsilon))s^{n-1}. \tag{3.13}$$

Recall that

$$E(N_+(\varepsilon))=\sum_{n=1}^{\infty}P(S_n>n(\mu+\varepsilon)). \tag{3.14}$$

Therefore, (3.13) and (3.14) imply that

$$\tau'(1)=(1-\tau(1))EN_+(\varepsilon). \tag{3.15}$$

We apply (3.15), (3.12), and the note following the statement of Theorem 3.4 to obtain

$$\begin{aligned} E[T(\varepsilon)|T(\varepsilon) < \infty] &= \frac{1-\tau(1)}{\tau(1)} EN_+(\varepsilon) \\ &= \frac{P(T(\varepsilon) = \infty)}{P(T(\varepsilon) < \infty)} EN_+(\varepsilon) \\ &= \frac{P(N_+(\varepsilon) = 0)}{P(N_+(\varepsilon) > 0)} EN_+(\varepsilon). \qquad \square \end{aligned}$$

Example. For $\{X, i = 1, 2, 3, \dots\}$, standard normal random variables, Corollary 3.1 provides close upper and lower bounds on $E(N_+(\varepsilon))$. Estimates for $P(N_+(\varepsilon) = 0)$ can be obtained from the work of Razanadrakoto and Severo [**6**]. Theorem 3.4 can then be applied to obtain upper and lower bounds for $E[T(\varepsilon)|T(\varepsilon) < \infty]$. For example, with $\varepsilon = .5$, Theorem 3.4 establishes that $1.68 \le E[T(.5)|T(.5) < \infty] \le 2.25$.

We consider the behavior of $E[T(\varepsilon)|T(\varepsilon) < \infty]$ as $\varepsilon \to 0$ where the common distribution function of the random variables $\{X_i, i = 1, 2, 3, \dots\}$ is continuous and symmetric.

Theorem 3.5. *Let $\{X_i, i = 1, 2, 3, \dots\}$ be independent and identically distributed random variables with common mean μ and distribution function which is continuous and symmetric. Then, for $\varepsilon > 0$,*

$$\lim_{\varepsilon \to 0} E[T(\varepsilon)|T(\varepsilon) < \infty] = \infty.$$

Proof. In order to emphasize the dependence of the probabilities a_n and τ_n on ε, denote a_n as $a_n(\varepsilon)$ and τ_n as $\tau_n(\varepsilon)$. Then for each fixed index n, $n = 1, 2, 3, \dots,$

$$\lim_{\varepsilon \to 0} a_n(\varepsilon) = 1/2 \tag{3.16}$$

and

$$\lim_{\varepsilon \to 0} \tau_n(\varepsilon) = \tau_n^* \tag{3.17}$$

where $\{\tau_n^*, n = 1, 2, 3, \dots\}$ is the solution to the recursive relation described in (3.8) with a_j equal to 1/2.

Due to the symmetry and continuity of the common distribution of the random variables $\{X_i, i = 1, 2, 3, \dots\}$, τ_n^* can be equivalently characterized by $\tau_n^* = P(S_1' \leq 0, S_2' \leq 0, \dots, S_{n-1}' \leq 0, S_n' > 0)$ where $S_n' = \sum_{i=1}^n X_i'$ with $X_i' = X_i - \mu$. As a result, the probability generating function for $\{\tau_n^*, n = 1, 2, 3, \dots\}$, denoted here by $\tau^*(s)$, satisfies relation (3.7) (replace τ with τ^*) with a_j equal to 1/2. It follows from this that

$$\tau^*(s) = 1 - \sqrt{1-s}. \tag{3.18}$$

Therefore,

$$\sum_{n=1}^{\infty} n\tau_n^* = \lim_{s \uparrow 1} \tau^{*\prime}(s) = \infty.$$

Theorem 3.5 follows from this and (3.12). □

For the case where $\{X_i, i = 1, 2, 3, \dots\}$ are normally distributed random variables, we derive a result characterizing the asymptotic behavior of $E[T(\varepsilon)|T(\varepsilon) < \infty]$ as ε approaches 0. Specifically, we show in Theorem 3.6 that $E[T(\varepsilon)|T(\varepsilon) < \infty]$ increases to ∞ asymptotically as ρ^{-1} where $\rho = \varepsilon/\sigma$.

Theorem 3.6. *Let $\{X_i, i = 1, 2, 3, \dots\}$ be independent and identically distributed random variables, normally distributed with mean μ and variance σ^2. Then, for $\varepsilon > 0$,*

$$0 < \liminf_{\rho \to 0} \rho E[T(\varepsilon)|T(\varepsilon) < \infty] \leq \limsup_{\rho \to 0} \rho E[T(\varepsilon)|T(\varepsilon) < \infty] < \infty \tag{3.19}$$

where $\rho = \varepsilon/\sigma$.

Proof. Define p_n for $n = 1, 2, 3, \dots$, by $p_n = P(S_k > k(\mu + \varepsilon), k = 1, 2, 3, \dots, n)$, and define p_0 by $p_0 = 1$. Razanadrokoto and Severo [**6**, p. 180] have shown that

$$P(N_+(\varepsilon) = 0) = \left(\sum_{k=0}^{\infty} p_k\right)^{-1}.$$

Combining this result with Theorem 3.4 leads to

$$E[T(\varepsilon)|T(\varepsilon) < \infty] = \left(\sum_{k=1}^{\infty} p_k\right)^{-1} EN_+(\varepsilon). \tag{3.20}$$

We derive (3.19) by obtaining upper and lower bounds on the term $\sum_{k=1}^{\infty} p_k$ and applying Corollary 3.1.

In Razanadrokoto and Severo [**6**, p. 181] it is shown that

$$\sum_{k=1}^{\infty} p_k = -1 + \exp\left[\sum_{k=1}^{\infty} k^{-1} a_k\right] \tag{3.21}$$

where a_k is as defined in (3.5). Let $\Phi(\cdot)$ denote the cumulative distribution function for the standard normal distribution, and let $\phi(\cdot)$ denote its density function. From the monotonicity of the function $f(x) = x\Phi(-\rho\sqrt{x})$ for $x > 0$, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} k^{-1} a_k &= \sum_{k=1}^{\infty} k^{-1}\Phi(-\rho\sqrt{k}) \\ &< \Phi(-\rho) + \int_1^{\infty} x^{-1}\Phi(-\rho\sqrt{x})\,dx. \end{aligned} \tag{3.22}$$

It is easily seen that

$$\begin{aligned} \int_1^{\infty} x^{-1}\Phi(-\rho\sqrt{x})\,dx &= -2\Phi(-\rho)\ln\rho + 2\int_{\rho}^{\infty} (\ln y)\phi(y)\,dy \\ &< -2\Phi(-\rho)\ln\rho + 2\int_0^{\infty} y\phi(y)\,dy \\ &= -2\Phi(-\rho)\ln\rho + C \end{aligned} \tag{3.23}$$

where C is a positive constant independent of ρ. Applying the result described in (3.23) to (3.21), it follows that for $\rho < 1$

$$\sum_{k=1}^{\infty} p_k < -1 + C\rho^{-2\Phi(-\rho)} < -1 + C\rho^{-1} \tag{3.24}$$

for some positive constant C independent of ρ.

The result in (3.23) can be combined with Corollary 3.1, and the result described in (3.20) to obtain the following inequality: For $\rho < 1$, there exists a positive constant C independent of ρ so that

$$E[T(\varepsilon)|T(\varepsilon) < \infty] > (1/2)(\rho^{-2} - 1)(-1 + C\rho^{-1})^{-1}.$$

From this inequality, it follows that

$$0 < \liminf_{\rho \to 0} \rho E[T(\varepsilon)|T(\varepsilon) < \infty]. \tag{3.25}$$

We obtain the righthand inequality in (3.19) in a similar fashion. The monotonicity of the function $f(x) = x\Phi(-\rho\sqrt{x})$ for $x > 0$ leads to

$$\sum_{k=1}^{\infty} k^{-1}\Phi(-\rho\sqrt{k}) > \int_1^{\infty} x^{-1}\Phi(-\rho\sqrt{x})\, dx. \tag{3.26}$$

As before, it is easily shown that
(3.27)

$$\begin{aligned}\int_1^{\infty} x^{-1}\Phi(-\rho\sqrt{x})\, dx &= 2^{-1}\rho \int_1^{\infty} (x^{-1/2}\ln x)\phi(\rho\sqrt{x})\, dx \\ &= (2\sqrt{\pi})^{-1}\bigg\{ \int_{\rho^2/2}^{\infty} (\ln 2y) y^{-1/2} \exp(-y)\, dy \\ &\qquad - \int_{\rho^2/2}^{\infty} 2(\ln \rho) y^{-1/2} \exp(-y)\, dy \bigg\}.\end{aligned}$$

The inequality

$$\int_{\rho^2/2}^{\infty} (\ln 2y) y^{-1/2} \exp(-y)\, dy > \int_0^{\infty} (\ln 2y) y^{-1/2} \exp(-y)\, dy$$

holds for $\rho < 1$. Therefore,

$$\begin{aligned}\int_1^{\infty} x^{-1}\Phi(-\rho\sqrt{x})\, dx &> C - \pi^{-1/2}\ln\rho \int_0^{\infty} y^{-1/2}\exp(-y)\, dy \\ &\qquad + \pi^{-1/2}\ln\rho \int_0^{\rho^2/2} y^{-1/2}\exp(-y)\, dy \\ &= C + (-1 + h(\rho))\ln\rho\end{aligned} \tag{3.28}$$

where C is a constant independent of ρ and $h(\rho) = \pi^{-1/2} \int_0^{\rho^2/2} y^{-1/2} \exp(-y)\, dy$.

We observe that

$$h(\rho) < \pi^{-1/2} \int_0^{\rho^2/2} y^{-1/2}\, dy = \sqrt{(2/\pi)}\, \rho. \tag{3.29}$$

Combining Corollary 3.1 and the results described in (3.20), (3.21), (3.26) and (3.28) yields that for $\rho < 1$,

$$\begin{aligned} E[T(\varepsilon)|T(\varepsilon) < \infty] &< (1/2)\rho^{-2}(-1 + C\rho^{-1+h(\rho)})^{-1} \\ &= (1/2)\rho^{-1}(C\rho^{h(\rho)} - \rho)^{-1} \end{aligned} \tag{3.30}$$

where C is a positive constant independent of ρ.

By (3.29), the inequality $\rho^{h(\rho)} > \rho^{\sqrt{(2/\pi)}\rho}$ holds for $\rho < 1$. It follows from (3.30) that

$$\limsup_{\rho\to 0} \rho E[T(\varepsilon)|T(\varepsilon) < \infty] < \infty. \tag{3.31}$$

Combining the results of (3.25) and (3.31) proves the theorem. □

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