ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 2, Spring 1996

ULTRAFILTERS OVER N AND OPERATORS ON L^1

MINOS A. PETRAKIS

Introduction. Let X be a Banach space. B_X denotes the unit ball of X and X^* denotes the dual of X. Let (Ω, Σ, μ) be a probability space. We denote by $L^1(\mu)$ the Banach space of all μ-integrable functions with the usual norm. L^1 denotes the space of Lebesgue integrable functions on the unit interval $[0,1]$. For a nonnegligible subset A of Ω, $P(A)$ is the set $\{f \in L^1(\mu) : f \geq 0, \operatorname{supp}(f) \subseteq A \text{ and } \int f\, d\mu = 1\}$ of all probability densities supported in A.

A *tree* in X is a bounded family $(x_{n,k})$, $n = 0, 1, \dots,$ $k = 1, 2, \dots 2^n$ of elements of X verifying $x_{n,k} = (x_{n+1,2k-1} + x_{n+1,2k})/2$ for each $n = 0, 1, 2, \dots,$ $k = 1, 2, 3, \dots, 2^n$.

A *δ-tree* is a tree verifying $||x_{n+1,2k-1} - x_{n+1,2k}|| \geq \delta$ for each $n = 0, 1, 2, \dots,$ $k = 1, 2, \dots, 2^n$.

A *δ-Rademacher tree* is a tree $(x_{n,k})$ verifying $|| \sum_{k=1}^{2^n} (-1)^{k+1} x_{n,k}|| \geq \delta 2^n$.

Let $I_{n,k} = [(k-1)/2^n, k/2^n]$, $n = 0, 1, 2, \dots, k = 1, 2, \dots, 2^n$ and $h_{n,k} = 2^n \cdot \mathbf{X}_{n,k}$ where $\mathbf{X}_{n,k}$ is the characteristic function of the dyadic interval $I_{n,k}$. If $T : L^1 \to X$ is a (bounded) operator it is clear that $(T(h_{n,k}))$, $n = 0, 1, 2, \dots,$ $k = 1, 2, \dots, 2^n$ is a tree in X. Conversely, every tree $(x_{n,k})$ in X produces an operator $T : L^1 \to X$ such that $T(h_{n,k}) = x_{n,k}$, $n = 0, 1, \dots,$ $k = 1, 2, \dots 2^n$.

An operator $T : L^1(\mu) \to X$ is called *Dunford-Pettis* if it maps weakly convergent sequences in $L^1(\mu)$ into norm convergent sequences in X.

A Banach space X has the *complete continuity property* if every operator from L^1 into X is Dunford-Pettis.

In [**2**] it is proved that if $T : L^1 \to X$ is an operator and $||T(r_n)|| > 2\varepsilon$ for some L^∞-bounded sequence (r_n) and some $\varepsilon > 0$, then there exists a set A of positive Lebesgue measure such that $\limsup_n ||T(r_n \cdot f)|| \geq \varepsilon$ for all f in $P(A)$. This result was used in [**2**] and [**4**] in the construction

Received by the editors on April 1, 1994.
AMS *Subject Classification.* 46B20, 47B38, 46B22, 47B07.

of certain trees in Banach spaces. Later many authors [**12, 16, 10**] used variations of the above constructions. In this paper we present some more variations on the same theme. We are led naturally to the following result:

Theorem 7. *If the Banach space X fails the* CCP*, then there is a subspace Z of X such that Z has a finite dimensional decomposition and contains a δ-tree which is also a δ-Rademacher tree. In particular, Z fails the* CCP.

This result answers a problem posed to the author by Professor S. Argyros in May, 1992. This result was also obtained by M. Girardi and W.B. Johnson.

In Proposition 1 and Corollaries 2 and 3, we use ultrafilters over the set $\mathbf{N}$ of the positive integers. For anything on ultrafilters, we refer to [**7, 13, 14**]. All other notation and terminology are as in [**8**].

Proposition 1 (and its proof) is similar to Proposition 5 in [**2**].

Proposition 1. *Let $T : L^1(\mu) \to X$ be a bounded operator $\varepsilon > 0$ and (r_n), $n = 1, 2, \ldots,$ a bounded sequence in $L^\infty(\mu)$ such that $||T(r_n)|| > 2\varepsilon$. Let $\mathbf{U}$ be a free ultrafilters over the set $\mathbf{N}$ of the positive integers. Then there exists a set A in Σ with $\mu(A) > 0$ such that $\lim_{\mathbf{U}} ||T(r_n f)|| \geq 4\varepsilon/3$ for every f in $P(A)$.*

Proof. Let (x_n^*), $n = 1, 2, \ldots,$ be a sequence in X^* such that $||x_n^*|| = 1$ and $x_n^*(T(r_n)) = ||T(r_n)||$ for each n in $\mathbf{N}$. Consider the set $\mathcal{K} = \{f \in L^1(\mu) : f \geq 0 \text{ and } \lim_{\mathbf{U}} x_n^*(T(r_n)) \leq (4/3)\varepsilon \cdot ||f||_1\}$.

$\mathcal{K}$ is a closed convex cone, and the constant function 1 on Ω does not belong to $\mathcal{K}$. By separation there exists g in $L^\infty(\mu)$ such that $\int g > \int gf$ for each f in $\mathcal{K}$. Since 0 belongs to $\mathcal{K}$, we have that $\int g > 0$ and therefore the set $A = \{x \in \Omega : g(x) > 0\}$ has positive μ-measure. If $f \in \mathcal{K}$, then $n \cdot f$ belongs to $\mathcal{K}$ for each n in $\mathbf{N}$ so $\int fg < \int (1/n) g$. This means that $\int fg \leq 0$ for all f in $\mathcal{K}$. It is clear that if f is a probability density in $P(A)$, then f does not belong to $\mathcal{K}$ and $\lim_{\mathbf{U}} x_n^*(T(r_n f)) \geq 4\varepsilon/3$. Therefore, $\lim_{\mathbf{U}} ||T(r_n f)|| \geq (4/3)\varepsilon$ for every f in $P(A)$. $\square$

Corollary 2. *Under the assumptions of Proposition* 1 *given any finite number* $f_1, f_2, \dots, f_d$ *of elements of* $P(A)$*, there is an infinite subset* N' *of* $\mathbf{N}$ *such that* $||T(r_n f_i)|| \geq \varepsilon$ *for every* n *in* N' *and for all* $i = 1, 2, 3, \dots, d$*. Moreover, if the space* $L^1(\mu)$ *is separable, there is a subsequence* (r'_n) *of* (r_n) *such that* $\liminf_n ||T(r'_n f)|| \geq \varepsilon$ *for every* f *in* $P(A)$*.*

Proof. Let $U_i = \{n \in \mathbf{N} : ||T(r_n f_i)|| \geq \varepsilon\}$, $i = 1, 2, \dots, d$. Every U_i belongs to the ultrafilter $\mathbf{U}$ and therefore the set $N' = U_1 \cap U_2 \cap \cdots \cap U_d$ is an element of $\mathbf{U}$. Since $\mathbf{U}$ is a free ultrafilter, the set N' is infinite. To prove the second statement, let (φ_i), $i = 1, 2, \dots$, be dense in an $L^1(\mu)$ norm sequence in $P(A)$. Set $V_i = \{n \in \mathbf{N} : ||T(r_n \varphi_k)|| \geq 4\varepsilon/3$ for all $k \leq i\}$, $i = 1, 2, \dots$. Note that each V_i belongs to the ultrafilter $\mathbf{U}$ and $V_{i+1} \subseteq V_i$, $i = 1, 2, \dots$. A diagonal argument produces an infinite subset N'' of $\mathbf{N}$ such that, for every i in $\mathbf{N}$, all but finitely many elements of N'' belong to V_i. Suppose that $N'' = \{k_1 < k_2 < \cdots < k_n < \cdots\}$, and let $r'_n = r_{k_n}$, $n = 1, 2, \dots$. It is clear that $||T(r'_n \varphi_i)|| \geq 4\varepsilon/3$ for all but finitely many n in $\mathbf{N}$. Let $f \in P(A)$. We can find φ_1 so that $||f - \varphi_i|| < (\varepsilon/3M)||T||^{-1}$ where M is a uniform bound of the sequence (r_n). Now $||T(r'_n f)|| \geq ||T(r'_n \varphi_i)|| - ||T((\varphi_i - f)r'_n)|| \geq \varepsilon$ for all but finitely many n in $\mathbf{N}$. We have proved that $\liminf_n ||T(r'_n f)|| \geq \varepsilon$ for all f in $P(A)$. □

Remarks. (i) It has been noted by several people that the second statement of Corollary 2 follows from Proposition 5 in [**2**] and a diagonal argument. We note that the set N'' in the proof above can be taken to belong to $\mathbf{U}$ in case the ultrafilter $\mathbf{U}$ is a p-point. Under Martin's axiom p-points (in fact, Ramsey ultrafilters over $\mathbf{N}$) exist (see [**14**, pp. 257–259]).

(ii) Let $(X)_\mathbf{U}$ be the ultraproduct of the Banach space X. If $x = (x_i)$, $i = 1, 2, \dots$, is an element of $(X)_\mathbf{U}$ the norm of x is the quantity $\lim_\mathbf{U} ||x_i||$ (see [**13**]). Let T be as in Proposition 1. Consider the operator $S : L^1(\mu) \to (X)_\mathbf{U}$ defined by $S(f) = (T(r_n f))$, $f \in L^1$. Proposition 1 says that $||S(f)|| \geq 4\varepsilon/3$ for every f in $P(A)$.

If K is a subset of the Banach space X, $x^* \in X^*$ and $a > 0$, we denote by $S(x^*, K, a)$ the slice $\{x \in K : x^*(x) > a\}$ of K.

Corollary 3. *Let $T : L^1(\mu) \to X$ be an operator. Consider the set $K = T(P(\Omega))$ of the images of the probability densities of Ω. Suppose that $x_n^* \in B_X^*$, $n = 1, 2, \dots,$ and $a > 0$, $\varepsilon > 0$.*

Let (S_n), $n = 1, 2, \dots, S_n = S(x_n^*, K, a)$ be a sequence of slices of K such that $\cap S(x_n^*, K, a + \varepsilon) \neq \varnothing$. Then there exist a subset A of Ω with $\mu(A) > 0$ and an infinite subset N' of $\mathbf{N}$ such that for all f in $P(A), T(f)$ is contained in S_n for all but finitely many elements $n \in N'$.

Proof. Suppose $h \in P(\Omega)$ such that $T(h) \in S_n(x_n^*, K, a + \varepsilon)$ for all n in $\mathbf{N}$. This means that $x_n^*(T(h)) > a + \varepsilon$ for $n = 1, 2, \dots$. Let $\mathbf{U}$ be a free ultrafilter over $\mathbf{N}$. Consider the cone $\mathcal{K} = \{f \in L^1(\mu) : f \geq 0$ such that $\lim_{\mathbf{U}} x_n^*(T(f)) \leq a \cdot ||f||_1\}$. Note that h does not belong to $\mathcal{K}$ since $\lim_{\mathbf{U}} x_n^*(T(h)) > a$. There exists $g \in L^\infty(\mu)$ such that $\int hg > \int fg$ for every f in $\mathcal{K}$. We can proceed now as in the proof of Proposition 1. □

Remark. Note that Corollary 3 implies that every slice of $T(P(\Omega))$ contains $T(P(A))$ for some nonnegligible subset A of Ω. This result is Lemma 1.2 of [**3**] and therefore Corollary 3 can be considered as a variation of it. (See also [**11**]).

It is known [**2**] that if a Banach space fails the complete continuity property, then X contains a δ-tree. It is also known [**9, 10**] that separated trees (see [**9**]) for a definition) and δ-Rademacher trees grow in any space that fails the CCP. We have shown in [**15**] that a slight variation of the arguments in [**16**] gives that if X fails the CCP, then there exist a $\delta > 0$ and a closed bounded subset K of X such that every convex combination of slices of K contains a δ-tree. The next proposition is related to all the results above.

Proposition 4. *Let X be a Banach space that fails the* CCP. *Then there exist a closed convex bounded subset K of X and a $\delta > 0$ such that every convex combination of slices of K contains a δ-tree which is also a δ-Rademacher tree. In particular, any weakly open set V so that $V \cap K \neq \varnothing$ contains such a tree.*

Proof. Since X fails the CCP, there is an operator $T : L^1 \to X$ and an $\varepsilon > 0$ such that $||T(r_n)|| > 2\varepsilon$ for all n in $\mathbf{N}$, where now (r_n) is a weakly null sequence in L^1. We may assume (see [**2, 16**]) that each r_n is a simple function and $|r_n| \le 1$ for all n in $\mathbf{N}$. The construction of the tree in X is similar, of course, to the standard construction in [**2**].

By Corollary 2 there is a subsequence of (r'_n) of (r_n) and a set $A \subseteq [0,1]$ with $m(A) > 0$ (here m denotes the Lebesgue measure on $[0,1]$) so that $\liminf_n ||T(r'_n f)|| > \varepsilon$ for all f in $P(A)$. Let $K = \overline{T(P(A))}$. Let $S_1, S_2, \dots, S_d$ be (nonempty) slices of K and $c_i \ge 0$ such that $\sum_{i=1}^d c_i = 1$. We claim that the set $\sum_{i=1}^d c_i S_i$ contains a δ-tree which is also a δ-Rademacher tree. For each $i = 1, 2, \dots, d$ there exists a set $A_i \subseteq [0,1]$, $m(A_i) > 0$ such that $T(P(A_i)) \subseteq S_i$ (see the Remark after Corollary 3). We may assume that the A_i's are disjoint subsets of the set A. Let $w_{0,1} = \sum_{i=1}^d c_i \cdot p^i$ where $p^i = \mathbf{X}_{A_i}/m(A_i)$. Clearly $T(w_{0,1})$ belongs to $\sum_{i=1}^d c_i \cdot S_i$. Suppose now that $w_{n,k}$ for some n in $\mathbf{N}$ and all $k = 1, 2, \dots, 2^n$ has been constructed such that $w_{n,k} = \sum_{i=1}^d c_i \cdot p^i_{n,k}$, where $p^i_{n,k}$, $i = 1, 2, \dots, d$, $k = 1, 2, \dots, 2^n$, are simple functions in $P(A_i)$. For every $\eta > 0$ there exists a function r'_m in the sequence (r'_n) such that $||T(w_{n,k} r'_m)|| > \varepsilon$ for all $k = 1, 2, \dots, 2^n$, $\int p^i_{n,k} r_m < \eta$ and $||T((\sum_{k=1}^{2^n} w_{n,k}) \cdot r'_m)|| \ge 2^n \varepsilon$. Although the integral $\int p^i_{n,k} r'_m$ can be made as small as we wish, it might not be equal to 0. However, a perturbation argument in [**16**] shows that we can replace r'_m by a function g_m close to r'_m in L^∞ norm such that $\int p^i_{n,k} g_m = 0$. (If $\mathbf{A}_n$ is the finite algebra generated by the simple functions $\{p^i_{n,k}\}$, $i = 1, 2, \dots, d$, $k = 1, 2, \dots, 2^n$, as in [**16**], one can set $g_m = r'_m - E(r'_m \mid \mathbf{A}_n)$. It follows that $||E(r'_m | A_n)||_\infty \to 0$ as $m \to \infty$.) Assume for simplicity that $\int p^i_{n,k} r'_m = 0$, $i = 1, 2, \dots, d$, $k = 1, 2, \dots, 2^n$. Now set $w_{n+1,2k-1} = w_{n,k}(1 + (1/3) r'_m)$ and $w_{n+1,2k} = w_{n,k}(1 - (1/3) r'_m)$, $k = 1, 2, \dots, 2^n$. Note that both the $w_{n+1,2k-1}$ and $w_{n=1,2k}$ are of the form $\sum_{i=1}^d c_i p^i$ where each p^i is a simple function in $P(A_i)$, $i = 1, 2, \dots, d$. Clearly, the system $(T(w_{n,k}))$, $n = 0, 1, \dots,$ $k = 1, 2, \dots, 2^n$, is a tree inside $\sum_{i=1}^d c_i \cdot S_i$. Since $||T(w_{n=1,2k-1} - w_{n+1,2k})|| \ge 2\varepsilon/3$, we see that $(T(w_{n,k}))$ is a δ-tree for $\delta = 2\varepsilon/3$. Now notice that

$$\left\| \sum_{k=1}^{2^n} T(w_{n+1,2k} - w_{n+1,2k}) \right\|$$
$$= \left\| T\left(\left(\sum_{k=1}^{2^n} (2/3) w_{n,k} \right) \cdot r'_m \right) \right\| \geq (2/3)\varepsilon 2^n.$$

Therefore, the δ-tree $(T(w_{n,k}))$ is also a δ-Rademacher tree.

It is known (see [**6**] that if V is a weakly open set of X such that $K \cap V \neq \varnothing$ then V contains a convex combination of slices of K. This proves the last statement of the proposition. □

The next proposition is similar to Lemma II.5 in [**12**].

Proposition 5. *Let T be a non Dunford-Pettis operator from L^1 into a Banach space X. Let S be an operator from X into a Banach space Y such that $S \cdot T$ is a Dunford-Pettis operator. Then there exists an operator $W : L^1 \to X$ such that the system $(W(h_{n,k}))$, $n = 0, 1, \dots,$ $k = 1, 2, \dots, 2^n$ is a δ-tree and a δ-Rademacher tree for some $\delta > 0$* (*so in particular W is not Dunford-Pettis*) *and the operator $S \cdot W : L^1 \to X$ is Bochner representable.*

Proof. In [**12**] it is proved that, under the assumptions of Proposition 5, there exists a δ-tree $(x_{n,k})$ in X so that the dyadic martingale corresponding to the tree $(S(x_{n,k}))$ converges almost everywhere. Using Corollary 2 the tree $(x_{n,k})$ can in fact be constructed to also be a δ-Rademacher tree. The operator W is the operator associated with the tree $(x_{n,k})$ by $W(h_{n,k}) = x_{n,k}$, $n = 0, 1, \dots,$ $k = 1, 2, \dots, 2^n$. □

For the next propositions we need some notations: If a Banach space X is contained in a space Y (e_i), $i = 1, 2, \dots,$ is a basis for Y and $\varepsilon > 0$, $p < q$ integers, $x \in X$, we write $x \approx e_i[p, q; \varepsilon]$ if there exist scalars c_i, $i = p, p+1, \dots, q$ so that $||x - \sum_{i=p}^{q} c_i e_i|| < \varepsilon$. If $(x_{n,k})$, $n = 0, 1, \dots,$ $k = 1, 2, \dots, 2^n$ is a tree in X, we denote by $d_{n,k}$ the differences $x_{n+1,2k-1} - x_{n+1,2k}$. These differences are called the *nodes* of the tree.

In [**16**] it is proved that if a Banach space X fails the CCP, then X contains a δ-tree $(x_{n,k})$ such that the sequence $d_{0,1}, d_{1,1}, d_{1,2}, d_{2,1}, \dots$ of the nodes of this tree is a basic sequence. The next proposition is in the same spirit.

Proposition 6. *Suppose that the Banach space X is contained in a space Y with a basis (e_i), $i = 1, 2, \dots$, and that X fails the* CCP. *Let (ε_n), $n = 0, 1, \dots$, be a sequence of positive reals so that $\varepsilon_n \to 0$. Then there exist a δ-tree $(x_{n,k})$ in X, $n = 0, 1, 2, \dots$, $k = 1, 2, \dots, 2^n$ which is also a δ-Rademacher tree and a sequence $p_0 < q_0 < p_1 < q_1 < \cdots$ of positive integers so that for each $n = 0, 1, 2, \dots$, the nodes $d_{n,k}$ have the property that $d_{n,k} \approx e_i[p_n, q_n; \varepsilon_n]$ for all $k = 1, 2, \dots, 2^n$.*

Proof. Let $T : L^1 \to X$ be an operator so that $||T(r_n)|| > 2 \cdot \varepsilon$ for all $n = 1, 2, \dots$, where (r_n) is a weakly null sequence of simple functions in L^1 and $\varepsilon > 0$. We may assume that $|r_n| \leq 1$ for all n in $\mathbf{N}$. By Corollary 2 there is a set $A \subseteq [0, 1]$ with $m(A) > 0$ and a subsequence of (r_n) (which for simplicity is also denoted by (r_n)) so that $\liminf_n ||T(r_n f)|| \geq \varepsilon$ for all f in $P(A)$.

Let $w_{0,1} = \mathbf{X}_A / m(A)$. The sequence $(T(w_{0,1} r_n))$ converges weakly to 0 in X and $||T(w_{0,1} r_n)|| \geq \varepsilon$ for all but finitely many n in $\mathbf{N}$. Choose k_1 in $\mathbf{N}$ and p_0, q_0 in $\mathbf{N}$ such that $T(w_{0,1}; r_{k_1}) \approx e_i[p_0, q_0; \varepsilon_0]$ and $\int w_{0,1} \cdot r_{k_1}$ as small as we wish. In fact, as in [**16**] and in the proof of Proposition 4, we may assume that this integral is actually zero. Define now $w_{1,1} = w_{0,1}(1 + (1/3) r_{k_1})$, $w_{1,2} = w_{0,1}(1 - (1/3) r_{k_1})$, and note that $w_{1,1}, w_{1,2}$ are in $P(A)$. Now find k_2 and p_1, q_1, $p_1 < q_1$, $p_1 > q_0$ in $\mathbf{N}$ so that $T(w_{1,1} \cdot r_{k_2}) \approx e_i[p_1, q_1; \varepsilon_1]$, $T(w_{1,2} \cdot r_{k_2}) \approx e_i[p_1, q_1; \varepsilon_1]$, $||T(w_{1,1} r_{k_2})|| \geq \varepsilon$, $||T(w_{1,2} r_{k_2})|| \geq \varepsilon$, $||T((w_{1,1} + w_{1,2}) r_{k_2})|| \geq 2\varepsilon$ and $\int w_{1,1} r_{k_2} = \int w_{1,2} r_{k_2} = 0$. We can continue in this manner to construct a tree $(w_{n,k})$ in L^1 so that the tree $(T(w_{n,k}))$ has the properties in the statement of Proposition 6. □

A refinement of the arguments in the proof of Proposition 6 gives the following:

Theorem 7. *Suppose the Banach space X fails the* CCP. *Then there exists a subspace Z of X such that Z has a finite dimensional* *decomposition and contains a δ-tree which is also a δ-Rademacher tree. In particular, Z fails the* CCP.

Proof. Assume that X is separable. We consider X as a subspace of the space $C[0,1]$ of the continuous functions on $[0,1]$. Let (e_i), $i = 1, 2, \dots$ be a basis for $C[0,1]$. We denote by (e_i^*), $i = 1, 2, \dots$, the sequence of the biorthogonal functionals associated to the basis (e_i). Let $\varepsilon > 0$ and $T : L^1 \to X$ be an operator so that $||T(r_n)|| > 2\varepsilon$ for al $n = 1, 2, \dots$ where now (r_n) is the sequence of the Rademacher functions on $[0,1]$. Let A be a subset of $[0,1]$ of positive Lebesgue measure and (r'_n) a subsequence of (r_n) such that $\liminf_n ||T(r'_n f)|| \geq \varepsilon$ for f in $P(A)$. We first prove the following claim (see Lemma 18 in [**5**] or Lemma 1.6 in [**1**]): Given p in $\mathbf{N}$, w in L^1, $\varepsilon' > 0$, there exists an s_0 in $\mathbf{N}$ such that if $s > s_0$ there exists a y in L^1 with $||y|| < \varepsilon'$ so that $e_i^*(T(wr_s + y)) = 0$ for all $i = 1, 2, \dots, p$. To prove the claim, consider the map $u : L^1 \to R^p, ||\cdot||^\infty$ given by $u(f) = (e_1^*(T(f)), \dots, e_p^*(T(f)))$, for f in L^1. Let $F = u(L^1)$. There is a subspace E of L^1, $\dim E < \infty$ so that $u(E) = F$. By the open mapping theorem, there is a δ' so that $B(0, \delta') \cap F \subseteq u(E \cap B(0, \varepsilon'))$. Find s_0 large enough such that $|e_i^*(T(w \cdot r_s))| < \delta'$ for all $i = 1, 2, \dots, p$ and $s > s_0$. For every $s > s_0$, there is a y in $E \cap B(0, \varepsilon')$ so that $u(wr_s) = u(-y)$. Therefore, $||y|| < \varepsilon'$ and $e_1^*(T(w \cdot r_s + y)) = 0$ for all $i = 1, 2, \dots, p$.

We construct a tree $(w_{n,k})$, $n = 0, 1, \dots$, $k = 1, 2, \dots, 2^n$ in L^1 and a sequence of finite dimensional subspaces of X so that $F_n = [d_{n,k}, k = 1, 2, \dots, 2^n]$ where $d_{n,k} = T(w_{n+1,2k-1} - w_{n+1,2k})$, $n = 0, 1, 2, \dots$, $k = 1, 2, \dots, 2^n$ are the nodes of the tree $(T(w_{n,k}))$ in X. We give the first few inductive steps of the construction: Let (ε_n), $n = 0, 1, 2, \dots$, be a sequence of poistive reals so that $\sum_{n=0}^\infty \varepsilon_n < 1/(2C)$ where C is the basis constant of the basis (e_i). Set $w_{0,1} = \mathbf{X}_A / m(A)$. Find p_0 in $\mathbf{N}$ so that $||T(w_{0,1}) - \sum_{i=1}^{p_0} e_i^*(T(w_{0,1}))e_i|| < \varepsilon_0$. Find s_0 in $\mathbf{N}$ and $y_{0,1}$ in L^1, $||y_{0,1}|| < \varepsilon/2$ so that $||T(w_{0,1} \cdot r_{s_0} + y_{0,1})|| > \varepsilon/2$ and $e_i^*(T(w_{0,1} r_{s_0} + y_{0,1})) = 0$ for all $i = 1, 2, \dots, p_0$. This is possible by the claim and Corollary 2. Also note that s_0 can be chosen so that by perturbing if necessary the r_{s_0} as is done in [**16**] (and in the Proof of Proposition 4) we may assume that $\int w_{0,1} \cdot r_{s_0} = 0$.

Now set

$$w_{1,1} = w_{0,1}\left(1 + \frac{1}{2} r_{s_0}\right) + \frac{y_0}{2}, \qquad w_{1,2} = w_{0,1}\left(1 - \frac{1}{2} r_{s_0}\right) - \frac{y_0}{2}$$

and note that $||d_{0,1}|| > \varepsilon/2$ and $e_i^*(d_{0,1}) = 0$ for $i = 1, 2, \dots, p_0$. Also note that the positive functions

$$w_{0,1}\left(1 \pm \frac{1}{2} r_{s_0}\right)$$

belong to $P(A)$. We set $F_{-1} = [T(w_{0,1})]$ and $F_0 = [d_{0,1}]$. There is a $p_1 > p_0$ such that if x belongs to the unit ball B_{F_0} of F_0, then $||x - \sum_{i=p_0}^{p_1} e_i^*(x) e_i|| < \varepsilon_1$. Again, by the claim, Corollary 2 and a perturbation argument as in [**16**], we can find $s_1 > s_0$ and $y_{1,1}, y_{1,2}$ in L^1 such that $||y_{0,1}|| + ||y_{1,1}|| + ||y_{1,2}|| < \varepsilon/2$ such that the vectors $T(w_{1,k} \cdot r_{s_1} + y_{1,k})$, $k = 1, 2$, have norms $> \varepsilon/2$ and $e_i^* T(w_{1,k} \cdot r_{s_1} + y_{1,k}) = 0$, $i = 1, 2, \dots p_1$, $k = 1, 2$ and the vector $T((w_{1,1} + w_{1,2}) r_{s_1} + y_{1,1} + y_{1,2})$ has norm $> 2 \cdot \varepsilon/4$. Now defining

$$w_{2,1} = w_{1,1}\left(1 + \frac{1}{2} r_{s_1}\right) + \frac{y_{1,1}}{2}, \qquad w_{2,2} = w_{1,1}\left(1 - \frac{1}{2} r_{s_1}\right) - \frac{y_{1,1}}{2},$$
$$w_{2,3} = w_{1,2}\left(1 + \frac{1}{2} r_{s_1}\right) + \frac{y_{1,2}}{2}, \qquad w_{2,4} = w_{1,2}\left(1 - \frac{1}{2} r_{s_1}\right) - \frac{y_{1,2}}{2},$$

we may assume that the nonperturbed part of the functions $w_{2,m}$, $m = 1, 2, 3, 4$ lives in $P(A)$. Note that $d_{1,1}, d_{1,2}$ have norms greater than $\varepsilon/2$ and $e_i^*(d_{1,k}) = 0$ for $i = 1, 2, \dots, p_1$ and $k = 1, 2$. Also note that $||d_{1,1} + d_{1,2}|| > 2 \cdot \varepsilon/4$. Let $F_1 = [d_{1,1}, d_{1,2}]$. The compactness of B_{F_1} implies that we can find $p_2 > p_1$ such that $||x - \sum_{i=p_1}^{p_2} e^*(x) e_i|| < \varepsilon/2$ for all x in B_{F_1}. Find $s_2 > s_1$ and $y_{2,k}$, $k = 1, 2, 3, 4$, so that

(i) the vectors $T(w_{2,k} \cdot r_{s_2} + y_{2,k})$ have norm $> \varepsilon/2$,

(ii) $e_i^*(T(w_{2,k} r_{s_2} + y_{2,k})) = 0$, for all $i = 1, 2, \dots, p_2$ and $k = 1, 2, 3, 4$,

(iii) the vector $T((\sum_{k=1}^4 w_{2,k}) r_{s_2} + \sum_{k=1}^4 y_{2,k})$ has norm greater than $2^2 \cdot \varepsilon/4$,

(iv) $||y_{0,1}|| + ||y_{1,1}|| + \cdots + ||y_{2,4}|| < \varepsilon/2$,

(v) the integrals $\int w_{2,k} \cdot r_{s_2} = 0$, $k = 1, 2, 3, 4$.

To ensure the last condition (v) we may have to perturb the r_{s_2} in the sense of [**16**] as we have done in the proof of Proposition 4.

Set

$$\begin{aligned} w_{3,1} &= w_{2,1}\left(1+\frac{1}{2}r_{s_2}\right)+\frac{y_{2,1}}{2}, \\ w_{3,2} &= w_{2,1}\left(1-\frac{1}{2}r_{s_2}\right)-\frac{y_{2,1}}{2}, \\ w_{3,3} &= w_{2,2}\left(1+\frac{1}{2}r_{s_2}\right)+\frac{y_{2,2}}{2}, \dots , \\ w_{3,8} &= w_{2,4}\left(1-\frac{1}{2}r_{s_2}\right)-\frac{y_{2,2}}{2}. \end{aligned}$$

Let $F_2 = [d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}]$. It is clear now how we construct by induction the spaces F_n. The space $Z = \overline{\cup F_n}$ has an F.D.D. and contains an $\varepsilon/2$-tree which is also an $\varepsilon/4$-Rademacher tree. □

Acknowledgments. I wish to express my gratitude to Professor S. Argyros for his constant help and advice. I am particularly grateful to him for the very crucial conversations about this paper. I also wish to thank Dr. A. Kamburelis from whom I learned many things on ultrafilters. Some comments of a referee on an earlier version of this paper have been taken under consideration.

REFERENCES

1. S.A. Argyros and I. Deliyanni, *Representations of convex non-dentable sets*, Pacific J. Math. **155** (1992), 29–70.

2. J. Bourgain, *Dunford-Pettis operators on L^1 and the Radon-Nikodym property*, Israel J. Math. **37** (1980), 34–37.

3. ———, *L'apropriéte de Nikodym*, Publ. Math. Univ. Pierre et Marie Curie **36**, Amer. Math. Soc., Providence, R.I., (1979).

4. ———, *A characterization of non-Dunford-Pettis operators on L^1*, Israel J. Math. **37** (1980), 48–53.

5. ———, *Dentability and finite dimensional decompositions*, Studia Math. **67** (1980), 135–148.

6. J. Bourgain and H. Rosenthal, *Geometrical implications of certain finite-dimensional decompositions*, Bull. Soc. Math. Belg. **3** (1980), 57–82.

7. W.W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Springer-Verlag, Berlin-New York, 1974.

8. J. Diestel and J.J. Uhl, Jr., *Vector measures*, Math. Survey Monographs **15**, Amer. Math. Soc., Providence, R.I. (1977).

9. M. Girardi, *Dentability, trees and Dunford-Pettis operators on L^1*, Pacific J. Math. (1991), 59–79.

10. ———, *Dunford–Pettis operators on L^1 and the complete continuity property*, Ph.D. thesis, University of Illinois, 1989.

11. Maria Girardi and J.J. Uhl, Jr., *Slices, RNP, strong regularity and martingales*, Bull. Austral. Math. Soc. **41** (1990), 411–415.

12. N. Ghoussoub and H. Rosenthal, *Martingales, G_δ-embeddings and quotients of L^1*, Math. Ann. **264** (1983), 321–332.

13. S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math **313** (1980), 72–104.

14. Thomas J. Jech, *Set theory*, Academic Press, New York, 1978.

15. Minos A. Petrakis, *A remark on the complete continuity property*, Proc. Analysis Conf., to appear.

16. A. Wessel, *Some results on Dunford-Pettis operators, strong regularity and the Radon Nikodym property*, Seminaire d' Analyse Fonctionnell, Publ. Math. Univ. **7**, Paris, 1985–6.

Technical University of Crete, Department of General Studies, University Campus, 73100 Chania-Crete, Greece

E-mail address: `minos@thalis.aml.tuc.gr`.