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THE SPECTRAL THEORY OF SECOND ORDER TWO-POINT DIFFERENTIAL OPERATORS III. THE EIGENVALUES AND THEIR ASYMPTOTIC FORMULAS

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ABSTRACT. In the third part of a four-part series, the eigenvalues of a two-point differential operator L in $L^2[0,1]$ are calculated, along with the corresponding asymptotic formulas. L is determined by a formal differential operator $l = -D^2 + q$ and by independent boundary values B_1, B_2. The rates of convergence in the asymptotic formulas vary with the form of B_1, B_2 (Cases 1–4) and with the smoothness of q.

1. Introduction. In this paper, which is the third part in a four-part series, we continue our development of the spectral theory for a linear second order two-point differential operator L in the complex Hilbert space $L^2[0,1]$. In Part I [**14**] a priori estimates for the eigenvalues of L are derived, and the generalized eigenfunctions are shown to be complete. In Part II [**15**] the characteristic determinant of L is constructed utilizing operator theory methods. Using this representation of the characteristic determinant, here in Part III we calculate the actual eigenvalues of L, compute the corresponding algebraic multiplicities and ascents, and determine asymptotic formulas for the eigenvalues. We also establish the geometries and the growth rates for the characteristic determinant, which are the key results needed for Part IV where the L^2-expansion theory is developed.

Let L be the differential operator in $L^2[0,1]$ defined by

$$\mathcal{D}(L) = \{u \in H^2[0,1] \mid B_i(u) = 0, i = 1, 2\}, \qquad Lu = lu,$$

where

$$l = -\left(\frac{d}{dt}\right)^2 + q(t)\left(\frac{d}{dt}\right)^0$$

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is a second order formal differential operator on $[0,1]$ with $q \in C[0,1]$, B_1, B_2 are linearly independent boundary values given by

$$\begin{aligned} B_1(u) &= a_1 u'(0) + b_1 u'(1) + a_0 u(0) + b_0 u(1), \\ B_2(u) &= c_1 u'(0) + d_1 u'(1) + c_0 u(0) + d_0 u(1), \end{aligned}$$

and $H^2[0,1]$ is the Sobolev space consisting of all functions $u \in C^1[0,1]$ with u' absolutely continuous on $[0,1]$ and $u'' \in L^2[0,1]$. Let T and S be the differential and multiplication operators in $L^2[0,1]$ defined by

$$\mathcal{D}(T) = \{u \in H^2[0,1] \mid B_i(u) = 0, i = 1,2\}, \qquad Tu = -u'',$$

and $\mathcal{D}(S) = L^2[0,1]$, $Su = qu$, so

$$L = T + S. \tag{1.1}$$

Let

$$A = \begin{bmatrix} a_1 & b_1 & a_0 & b_0 \\ c_1 & d_1 & c_0 & d_0 \end{bmatrix}$$

be the boundary coefficient matrix, and let A_{ij}, $1 \le i < j \le 4$, denote the determinant of the 2×2 submatrix of A obtained by retaining the ith and jth columns.

One of the central topics of study in Part I is the characteristic determinant of T, which is given by equation (1.2) in Part I:

$$\begin{aligned} \tilde{\Delta}(\rho) = &-[A_{12}\rho^2 - i(A_{14} + A_{23})\rho + A_{34}]e^{i\rho} \\ &+ [A_{12}\rho^2 + i(A_{14} + A_{23})\rho + A_{34}]e^{-i\rho} + 2i[A_{13} + A_{24}]\rho. \end{aligned} \tag{1.2}$$

The nonzero eigenvalues of T are precisely the complex numbers $\lambda = \rho^2$ with the $\rho \neq 0$ being zeros of $\tilde{\Delta}$. By imposing various conditions on the six boundary parameters A_{ij}, the differential operators L and T are classified as belonging to one of five different cases, Cases 1–5. Each case determines the actual form of $\tilde{\Delta}$, and leads in a natural way to specific growth rates for $\tilde{\Delta}$ on various regions of the ρ-plane. These growth rates produce a priori estimates for the eigenvalues of L, viz., if $\lambda = \rho^2$ is any eigenvalue of L with $|\rho|$ sufficiently large, then ρ lies in the interior of a horizontal strip Ω in Cases 1–3 and in the interior of a logarithmic strip Ω in Case 4. See Section 8 of Part I for some comments on Case 5.

To actually calculate the eigenvalues of L, we replace $\tilde{\Delta}$ by the characteristic determinant of L, which is constructed in Part II:

$$\Delta(\rho) = \det \begin{bmatrix} B_1(u(\cdot;\rho)) & B_1(v(\cdot;\rho)) \\ B_2(u(\cdot;\rho)) & B_2(v(\cdot;\rho)) \end{bmatrix}, \tag{1.3}$$

where $u(\cdot;\rho)$ and $v(\cdot;\rho)$ are the solutions of the differential equation

$$\rho^2 u + u'' - qu = 0 \tag{1.4}$$

which satisfy the initial conditions $u(1;\rho) = e^{i\rho}$, $u'(1;\rho) = i\rho e^{i\rho}$ and $v(0;\rho) = 1$, $v'(0;\rho) = -i\rho$ for each $\rho \in \mathbf{C}$. From equation (4.7) in Part II the characteristic determinants Δ and $\tilde{\Delta}$ are related by the formula

$$\begin{aligned} \Delta(\rho) =& \tilde{\Delta}(\rho) \\ &+ A_{12} i\rho e^{-i\rho}\bigg[\int_0^1 q(s)\,ds + \frac{1}{2}\int_0^1 e^{2i\rho s} q(s)\,ds \\ &\qquad\qquad + \frac{1}{2}\int_0^1 e^{2i\rho(1-s)} q(s)\,ds + \Theta_1(\rho)\bigg] \\ &- A_{14} e^{-i\rho}\bigg[\int_0^1 q(s)\,ds + \frac{1}{2}\int_0^1 e^{2i\rho s} q(s)\,ds \\ &\qquad\qquad - \frac{1}{2}\int_0^1 e^{2i\rho(1-s)} q(s)\,ds + \Theta_2(\rho)\bigg] \\ &- A_{23} e^{-i\rho}\bigg[\int_0^1 q(s)\,ds - \frac{1}{2}\int_0^1 e^{2i\rho s} q(s)\,ds \\ &\qquad\qquad + \frac{1}{2}\int_0^1 e^{2i\rho(1-s)} q(s)\,ds + \Theta_3(\rho)\bigg] \\ &- A_{34}\frac{e^{-i\rho}}{i\rho}\bigg[\int_0^1 q(s)\,ds - \frac{1}{2}\int_0^1 e^{2i\rho s} q(s)\,ds \\ &\qquad\qquad - \frac{1}{2}\int_0^1 e^{2i\rho(1-s)} q(s)\,ds + \Theta_4(\rho)\bigg] \\ &- (A_{13} + A_{24})\bigg[\int_0^1 q(s)\,ds + \Theta_0(\rho)\bigg] \end{aligned} \tag{1.5}$$

for all $\rho \neq 0$ in $\mathbf{C}$, where the functions Θ_i, $i = 0, 1, \ldots, 4$, are analytic for $\rho \neq 0$ and of order $O(1/\rho)$ on a half plane $\operatorname{Im}\rho \geq -d$ for fixed $d > 0$.

We also have the following basic theorem from Part II (see Theorems 4.1 and 5.1).

Theorem 1.1. *Let $d > 0$ be any real number, and let $\lambda = \rho^2$ be any complex number with $\operatorname{Im} \rho \geq -d$ and $|\rho| > 2e^{2d}||q||_\infty$. Then*

(a) *The functions $u(\cdot;\rho)$, $v(\cdot;\rho)$ are linearly independent and form a basis for the solution space of the differential equation* (1.4).

(b) *$\lambda = \rho^2$ is an eigenvalue of L if and only if $\Delta(\rho) = 0$, in which case the algebraic multiplicity of λ is equal to the order of ρ as a zero of Δ.*

The general strategy for calculating the eigenvalues of L is to (i) form the "punctured strip" Ω_* from Ω by removing the points inside two sequences of circles Γ'_k, Γ''_k, (ii) use (1.5) and (1.2) to establish the growth rates of Δ and $\tilde{\Delta}$ on Ω_*, (iii) obtain the existence of the eigenvalues of L as an application of Rouché's theorem, (iv) discuss the algebraic multiplicities and ascents corresponding to the eigenvalues, and (v) derive asymptotic formulas for the eigenvalues. It is the growth rate of $\tilde{\Delta}$ on Ω_* which will initiate our study in Part IV of the projections associated with L and the subspace $S_\infty(L)$ consisting of the functions which have L^2-expansions in the generalized eigenfunctions of L.

The program outlined above is instituted in the following two sections, with Section 2 treating the closely related Cases 1, 2, 3 and Section 3 treating the difficult logarithmic Case 4. In each case we emphasize the geometry of the punctured strip Ω_*, which is needed for the theory of Part IV. Because of the structure of Δ, it is now necessary to subdivide Case 2 and Case 3 into Cases 2A, 2B and Cases 3A, 3B, respectively. The asymptotic formulas for the eigenvalues exhibit slower rates of convergence for Cases 2B, 3B and 4. This is caused in Case 2B by the multiple nature of the eigenvalues; in Case 3B it varies with the smoothness of the coefficient q; and in Case 4 it results from a combination of the inherent logarithmic geometry and the smoothness of the coefficient q.

Our method for determining the eigenvalues of L has many similarities with those used by Birkhoff [**2**], Tamarkin [**21**], Stone [**20**], and Naimark [**17**]. It is distinguished by the highly refined representation of Δ given in (1.5), which is sensitive enough to permit the treatment of *all* cases as parts of the same general theory, thereby achieving a comprehensive and unified theory. It does not require the boundary values B_1, B_2 to be normalized, and it treats both regular boundary conditions (Cases 1, 2, 3A) and irregular boundary conditions (Cases 3B, 4).

The general forms for the eigenvalues in the regular Cases 1, 2, 3A are well-known (see Theorem 4.2 in [**17**, p. 64]). Because of our special delineation of the cases, we are able to explicitly evaluate the eigenvalue parameters ξ_0 and η_0, which are 1 or -1 in most cases, and hence, establish theorems which contain very explicit forms for the eigenvalues. The eigenvalue literature for the irregular Cases 3B, 4 has only partial results for the form of the eigenvalues (see Theorems III and IV in [**20**, pp. 29–30]). This paper contains the first complete treatment of these cases.

Remark 1.2. A very subtle feature of this four-part series is the smoothness required of the coefficient q. Nowhere is this more apparent than in this paper, where the asymptotic formulas for the eigenvalues vary dramatically with the smoothness of q (see Theorem 2.4, Remark 2.5 and Theorem 3.2). To keep the presentation as simple as possible and to minimize the technical difficulties, we have required $q \in C[0,1]$ in general, and have increased the smoothness to $q \in C^1[0,1]$ for Case 3B and Case 4 so as to achieve effective rates of convergence for the eigenvalues. It is possible to replace the smoothness classes $C[0,1]$ and $C^1[0,1]$ by the Sobolev spaces $H^0[0,1]$ and $H^1[0,1]$, respectively, and develop the spectral theory in this more general setting. Indeed, for $q \in H^0[0,1] = L^2[0,1]$ the major change needed occurs in Part I [**14**], where the operator S can be defined by $\mathcal{D}(S) = H^1[0,1]$, $Su = qu$, and then in Theorem 3.1 [**14**] one shows that $R_\lambda(L) = R_\lambda(T)[I - SR_\lambda(T)]^{-1}$ provided $||SR_\lambda(T)|| \le 1/2$. The operator $SR_\lambda(T)$ is a bounded integral operator defined on all of $L^2[0,1]$; by carefully estimating its norm it follows that the generalized eigenfunctions of L are still complete in $L^2[0,1]$. Parts II–IV can then be modified to this more general setting.

2. The eigenvalues of L for Cases 1, 2 and 3. Assume the differential operators L and T belong to Case 1, Case 2 or Case 3, where the specific case is identified by the following conditions on the parameters A_{ij}:

Case 1. $A_{12} \neq 0$

Case 2A. $A_{12} = 0$, $A_{14} + A_{23} \neq 0$, $A_{14} + A_{23} \neq \mp(A_{13} + A_{24})$

Case 2B. $A_{12} = 0$, $A_{14} + A_{23} \neq 0$, $A_{14} + A_{23} = \mp(A_{13} + A_{24})$

Case 3A. $A_{12} = 0$, $A_{14}+A_{23} = 0$, $A_{34} \neq 0$, $A_{13}+A_{24} = 0$, $A_{13} = A_{24}$

Case 3B. $A_{12} = 0$, $A_{14}+A_{23} = 0$, $A_{34} \neq 0$, $A_{13}+A_{24} = 0$, $A_{13} \neq A_{24}$.

The conditions in Case 3A are equivalent to the conditions $A_{12} = A_{13} = A_{14} = A_{23} = A_{24} = 0$, $A_{34} \neq 0$, which corresponds to Dirichlet boundary conditions (see Section 10 of [**11**]). In Case 3B we also have $A_{14} \neq A_{23}$ by Theorem 2.2 in [**10**], so $A_{23} = -A_{14} \neq 0$.

From (1.5) and (1.2) we can write Δ in the form

$$\Delta(\rho) = \rho^p e^{-i\rho}[f(\rho) + g(\rho)] \tag{2.1}$$

for all $\rho \neq 0$, where the integer p is equal to 2, 1, 0 for Cases 1, 2, 3, respectively, where the function f is an entire function of the form

$$f(\rho) = \alpha_0[e^{i\rho} - \xi_0][e^{i\rho} - \eta_0]$$

for $\rho \in \mathbf{C}$, and where the function g is analytic for $\rho \neq 0$ in $\mathbf{C}$. Specifically, for Case 1: $\alpha_0 = -A_{12}$ and $\xi_0 = 1$, $\eta_0 = -1$; for Case 2: $\alpha_0 = i(A_{14} + A_{23})$ with ξ_0 and η_0 being the roots of the quadratic polynomial

$$Q(z) = i(A_{14} + A_{23})z^2 + 2i(A_{13} + A_{24})z + i(A_{14} + A_{23}),$$

where $\xi_0 \neq \eta_0$ in Case 2A and $\xi_0 = \eta_0 = \pm 1$ in Case 2B; and for Case 3: $\alpha_0 = -A_{34}$ and $\xi_0 = 1$, $\eta_0 = -1$. In Case 2A we also have $\xi_0\eta_0 = 1$, so let us assume that $|\xi_0| \leq 1$ and $|\eta_0| \geq 1$, which implies that $-\ln|\xi_0| \geq 0$ and $-\ln|\eta_0| = \ln|\xi_0| \leq 0$.

Fix any real number d with $0 \leq -\ln|\xi_0| < d$, and form the horizontal strip

$$\Omega = \{\rho = a + ib \in \mathbf{C} \mid |b| \leq d\}.$$

For Case 3B we make the additional assumptions that $d \geq 1$ and

$$\frac{\beta_0}{d} \leq \frac{1}{2}||S||^{-1}, \tag{2.2}$$

where $\beta_0 := (2/|A_{34}|)\{4|A_{14}| + 8|A_{13}| + 6|A_{34}|\} > 0$. Using the a priori estimates of Part I (see Theorems 4.1, 5.1 and 6.1 in [**14**]), we can choose a constant $r_0 > 2e^{2d}||q||_\infty$ such that if $\lambda = \rho^2 \in \mathbf{C}$ is any eigenvalue of L with $|\rho| \geq r_0$, then ρ must lie in the interior of Ω. Combining this result with Theorem 1.1, we have

Theorem 2.1. *Let $\lambda = \rho^2 \in \mathbf{C}$ with $|\rho| \geq r_0$. Then $\lambda = \rho^2$ is an eigenvalue of L if and only if $-d < \operatorname{Im}\rho < d$ and $\Delta(\rho) = 0$.*

In view of the theorem we proceed to determine all $\rho \in \mathbf{C}$ which are zeros of Δ with $-d < \operatorname{Im}\rho < d$ and $|\rho| \geq r_0$. Observe that the function g appearing in (2.1) is of order $O(1/\rho)$ on the half plane $\operatorname{Im}\rho \geq -d$ for Cases 1, 2 and 3A, i.e., there exists a constant $\gamma_0 > 0$ such that

$$|g(\rho)| \leq \frac{\gamma_0}{|\rho|} \tag{2.3}$$

for all $\rho \in \mathbf{C}$ with $\operatorname{Im}\rho \geq -d$ and $|\rho| \geq r_0$. In Case 3B we have

$$g(\rho) = -A_{14}\int_0^1 e^{2i\rho s} q(s)\,ds + A_{14}\int_0^1 e^{2i\rho(1-s)} q(s)\,ds + h(\rho)$$

for $\rho \neq 0$ in $\mathbf{C}$, where the function h is analytic for $\rho \neq 0$ and of order $O(1/\rho)$ on the half plane $\operatorname{Im}\rho \geq -d$. For any function $\tilde{q} \in C^1[0,1]$ and for any complex number $\rho \neq 0$ with $\operatorname{Im}\rho \geq -d$, we have

$$\begin{aligned}\int_0^1 e^{2i\rho s} q(s)\,ds = &\int_0^1 e^{2i\rho s}[q(s) - \tilde{q}(s)]\,ds \\ &+ \frac{\tilde{q}(1)}{2i\rho}e^{2i\rho} - \frac{\tilde{q}(0)}{2i\rho} - \frac{1}{2i\rho}\int_0^1 e^{2i\rho s}\tilde{q}'(s)\,ds,\end{aligned}$$

so

$$\left|\int_0^1 e^{2i\rho s} q(s)\,ds\right| \leq e^{2d}||q - \tilde{q}||_\infty + \frac{e^{2d}}{|\rho|}\left[||\tilde{q}||_\infty + \frac{1}{2}||\tilde{q}'||_\infty\right]. \tag{2.4}$$

The same argument shows that

$$\left|\int_0^1 e^{2i\rho(1-s)} q(s)\,ds\right| \leq e^{2d}||q - \tilde{q}||_\infty + \frac{e^{2d}}{|\rho|}\left[||\tilde{q}||_\infty + \frac{1}{2}||\tilde{q}'||_\infty\right]. \tag{2.5}$$

Thus, in Case 3B we have

$$|g(\rho)| \le 2|A_{14}|e^{2d}\left\{||q-\tilde{q}||_\infty + \frac{1}{|\rho|}\left[||\tilde{q}||_\infty + \frac{1}{2}||\tilde{q}'||_\infty\right]\right\} + \frac{\gamma_1}{|\rho|} \tag{2.6}$$

for all $\rho \in \mathbf{C}$ with $\operatorname{Im}\rho \ge -d$ and $|\rho| \ge r_0$. It should be emphasized that the function $\tilde{q} \in C^1[0,1]$ in (2.6) is completely arbitrary, and this method for estimating the integrals is one of the methods used to prove the Riemann-Lebesque theorem (see [**22**, p. 11–12]). In the special case $q \in C^1[0,1]$ we can choose $\tilde{q} = q$, and then in (2.6) we obtain the improved estimate

$$|g(\rho)| \le \frac{\gamma_0}{|\rho|} \tag{2.7}$$

for all $\rho \in \mathbf{C}$ with $\operatorname{Im}\rho \ge -d$ and $|\rho| \ge r_0$. To simplify the discussion, in the sequel we will assume that $q \in C^1[0,1]$ for Case 3B, and hence, we can use (2.3) throughout our study of Cases 1, 2 and 3 (see Remark 2.5 below).

Next, we set up the geometry for the various cases. Clearly the zeros of f are given by

$$\begin{aligned} \mu'_k &= (2k\pi + \operatorname{Arg}\xi_0) - i\ln|\xi_0|, \qquad k = 0, \pm 1, \pm 2, \dots, \\ \mu''_k &= (2k\pi + \operatorname{Arg}\eta_0) - i\ln|\eta_0|, \qquad k = 0, \pm 1, \pm 2, \dots, \end{aligned} \tag{2.8}$$

where the μ'_k, μ''_k all lie in the interior of the horizontal strip Ω. In case $\xi_0 \ne \eta_0$ (Cases 1, 2A and 3) we have $\mu'_k \ne \mu''_l$ for all k, l with each μ'_k and each μ''_k a zero of order 1 of f. In case $\xi_0 = \eta_0$ (Case 2B) we have

$$\mu'_k = \mu''_k := \mu_k \quad \text{for } k = 0, \pm 1, \pm 2, \dots,$$

and in this case each μ_k is a zero of order 2 of f. In all cases we will show that the zeros of Δ and $f+g$ in Ω appear as perturbations of the μ'_k, μ''_k.

Since $-\pi < \operatorname{Arg}\xi_0 \le \pi$ and $-\pi < \operatorname{Arg}\eta_0 \le \pi$, we can select a constant ω with $\pi \le \omega \le 3\pi/2$ such that $\omega - 2\pi < \operatorname{Arg}\xi_0 < \omega$ and $\omega - 2\pi < \operatorname{Arg}\eta_0 < \omega$. Then for $k = 1, 2, \dots$ we introduce the rectangles

$$R_k = \{\rho \in \Omega \mid \omega - 2k\pi \le \operatorname{Re}\rho \le \omega + 2(k-1)\pi\}.$$

Clearly the points μ_0', μ_0'' lie in the interior of the rectangle R_1. Choose a real number δ with $0 < \delta \leq \pi/4$ such that the two closed disks $|\rho - \mu_0'| \leq \delta$ and $|\rho - \mu_0''| \leq \delta$ both lie in the interior of R_1 and such that these two disks are disjoint in the cases where $\xi_0 \neq \eta_0$. For $k = 0, \pm 1, \pm 2, \ldots$ form the circles

$$\Gamma_k' = \{\rho \in \mathbf{C} \mid |\rho - \mu_k'| = \delta\}, \qquad \Gamma_k'' = \{\rho \in \mathbf{C} \mid |\rho - \mu_k''| = \delta\},$$

where in the case $\xi_0 = \eta_0$ we set

$$\Gamma_k' = \Gamma_k'' := \Gamma_k \quad \text{ for } k = 0, \pm 1, \pm 2, \ldots .$$

The following properties are obvious from these definitions: (i) the circles Γ_k', Γ_k'' lie in the interior of Ω, (ii) the Γ_k', Γ_l'' and the points inside them do not overlap each other in the cases where $\xi_0 \neq \eta_0$, (iii) the Γ_k and the points inside them do not overlap each other in the case $\xi_0 = \eta_0$, and (iv) for each positive integer k_0 the circles Γ_k', Γ_k'', $|k| < k_0$, lie in the interior of the rectangle R_{k_0}, while the circles Γ_k', Γ_k'', $|k| \geq k_0$, lie in the exterior of R_{k_0}. Let Ω_* be the region in the ρ-plane consisting of Ω with all the points inside the Γ_k', Γ_k'' removed. In the sequel we refer to Ω_* as a *punctured strip*.

Turning to the growth rate of Δ, it is clear that $f(\rho) \neq 0$ for all $\rho \in R_1$ which do not lie in the circles Γ_0', Γ_0''. Set

$$m_0 = \min\{|f(\rho)| \mid \rho \in R_1 \text{ with } \rho \text{ not lying in } \Gamma_0', \Gamma_0''\} > 0.$$

Since $f(\rho + 2\pi) = f(\rho)$ for all $\rho \in \mathbf{C}$, it follows that

$$|f(\rho)| \geq m_0 \tag{2.9}$$

for all $\rho \in \Omega_*$. Choose a positive integer k_0 such that $x_0 := \omega - 2k_0\pi \leq -r_0$, $y_0 := \omega + 2(k_0 - 1)\pi \geq r_0$, and such that in terms of (2.3)

$$\frac{\gamma_0}{|a|} \leq \frac{m_0}{2} \tag{2.10}$$

for all $a \in \mathbf{R}$ with $a \leq x_0$ or $a \geq y_0$. Combining all the above pieces, we have

$$|g(\rho)| \leq \frac{m_0}{2} \tag{2.11}$$

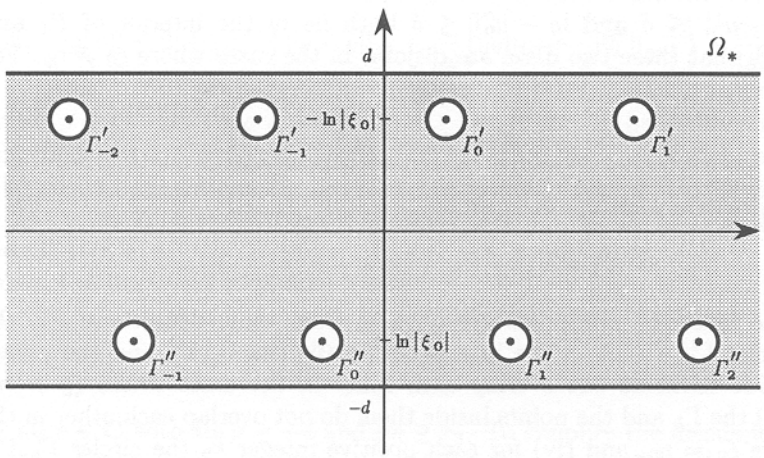

FIGURE 1. Punctured strip Ω_* in Cases 1, 2, 3.

for all $\rho \in \Omega$ with $\operatorname{Re}\rho \le x_0$ or $\operatorname{Re}\rho \ge y_0$, and hence,

$$|g(\rho)| \le \frac{m_0}{2} < m_0 \le |f(\rho)| \tag{2.12}$$

and

$$|\Delta(\rho)| = |\rho|^p |e^{-i\rho}||f(\rho) + g(\rho)| \ge \frac{m_0}{2} e^{-d}|\rho|^p \tag{2.13}$$

for all $\rho \in \Omega_*$ with $\operatorname{Re}\rho \le x_0$ or $\operatorname{Re}\rho \ge y_0$.

The estimate (2.13) is our principal result for the growth of Δ on the punctured strip Ω_*. It immediately implies that Δ and $f+g$ have no zeros in Ω_* when $\operatorname{Re}\rho \le x_0$ or $\operatorname{Re}\rho \ge y_0$. At this point we divide the discussion into the cases where $\xi_0 \ne \eta_0$ and $\xi_0 = \eta_0$.

Cases 1, 2A and 3. $\xi_0 \ne \eta_0$. Let us consider the circles Γ'_k, Γ''_k for $|k| \ge k_0$, which lie in the exterior of the rectangle $R_{k_0} = [x_0, y_0] \times [-d, d]$. From (2.12) we have $|g(\rho)| < |f(\rho)|$ for all points

ρ on Γ'_k, Γ''_k for $|k| \geq k_0$, and hence, by Rouché's theorem Δ and $f+g$ have precisely the same number of zeros as f inside Γ'_k and Γ''_k for all $|k| \geq k_0$. But f has only the single zero μ'_k of order 1 inside Γ'_k and only the single zero μ''_k of order 1 inside Γ''_k, implying that Δ has exactly one zero ρ'_k inside Γ'_k with ρ'_k having order 1 for $|k| \geq k_0$, and Δ has exactly one zero ρ''_k inside Γ''_k with ρ''_k having order 1 for $|k| \geq k_0$.

Setting

$$\lambda'_k = (\rho'_k)^2, \qquad k = k_0, k_0+1, \dots ,$$
$$\lambda''_k = (\rho''_k)^2, \qquad k = k_0, k_0+1, \dots ,$$

it follows from Theorem 2.1 that the λ'_k, λ''_k, $k = k_0, k_0+1, \dots$, are eigenvalues of L, and the corresponding algebraic multiplicities and ascents are (see Section 5 in Part II)

$$\begin{aligned} \nu(\lambda'_k) &= m(\lambda'_k) = 1, \qquad k = k_0, k_0+1, \dots , \\ \nu(\lambda''_k) &= m(\lambda''_k) = 1, \qquad k = k_0, k_0+1, \dots . \end{aligned} \tag{2.14}$$

Now suppose that $\lambda_0 = (\rho_0)^2$ is any eigenvalue of L which is distinct from λ'_k, λ''_k, $k = k_0, k_0+1, \dots$. By replacing ρ_0 by $-\rho_0$ if necessary, we can assume that $\operatorname{Re}\rho_0 \geq 0$. There are two possibilities for ρ_0: ρ_0 can lie in the disk $|\rho| < r_0$ or ρ_0 can belong to the rectangle R_{k_0}. Clearly only a finite number of such ρ_0 are possible since they are zeros of the entire function Δ. Thus, we conclude that λ'_k, λ''_k, $k = k_0, k_0+1, \dots$, account for all but a finite number of the eigenvalues of L.

Finally, let us derive asymptotic formulas for the zeros ρ'_k, ρ''_k, $k = k_0, k_0+1, \dots$, of Δ. Indeed, we first introduce the entire function $G(\rho) = \alpha_0[e^{i\rho} - \eta_0]$ and set

$$M_0 = \min\{|G(\rho)| \mid \rho \in \mathbf{C} \text{ with } |\rho - \mu'_0| \leq \delta\} > 0.$$

Because G has period 2π, it follows that $|G(\rho'_k)| \geq M_0$ for $k = k_0, k_0+1, \dots$. If we set $\zeta'_k = -g(\rho'_k)/\xi_0 G(\rho'_k)$, $k = k_0, k_0+1, \dots$, then we can rewrite the equation $f(\rho'_k) + g(\rho'_k) = 0$ as $e^{i\rho'_k} = \xi_0 + \xi_0\zeta'_k$, and upon dividing by $e^{i\mu'_k} = \xi_0$, it becomes $e^{i(\rho'_k - \mu'_k)} = 1 + \zeta'_k$. But $|\operatorname{Re}(\rho'_k - \mu'_k)| \leq |\rho'_k - \mu'_k| < \pi/4$, so

$$\rho'_k - \mu'_k = -i \operatorname{Log}[1 + \zeta'_k], \qquad k = k_0, k_0+1, \dots . \tag{2.15}$$

Now fix any integer $k \geq k_0$, and consider ρ'_k. Clearly

$$|\rho'_k| \geq |\mu'_k| - |\rho'_k - \mu'_k| \geq 2k\pi + \operatorname{Arg}\xi_0 - \delta \geq 6k - 5 \geq k$$

and

$$|\zeta_k'| = \frac{|g(\rho_k')|}{|\xi_0||G(\rho_k')|} \le \frac{\gamma_0}{|\xi_0|M_0|\rho_k'|} \le \frac{\gamma}{k}. \tag{2.16}$$

Since

$$-i\,\mathrm{Log}\,[1+z] = zH(z) \quad \text{for } |z|<1,$$

with H analytic on the disk $|z|<1$, from (2.15) and (2.16) we obtain the estimate

$$|\rho_k' - \mu_k'| \le \frac{\gamma}{k}, \qquad k = k_0, k_0+1, \dots\,, \tag{2.17}$$

for an appropriate constant $\gamma > 0$. A similar argument shows that

$$|\rho_k'' - \mu_k''| \le \frac{\gamma}{k}, \qquad k = k_0, k_0+1, \dots\,. \tag{2.18}$$

(2.17) and (2.18) are the desired asymptotic formulas.

Let us summarize these results for the eigenvalues in the following theorems, where in each case we insert the appropriate values for the constants ξ_0 and η_0.

Theorem 2.2. *Let the differential operator L belong to Case* 1. *Then the elements of the spectrum $\sigma(L)$ can be listed as two sequences $\lambda_k' = (\rho_k')^2$, $k = k_0, k_0+1, \dots$, and $\lambda_k'' = (\rho_k'')^2$, $k = k_0, k_0+1, \dots$, plus a finite number of additional points, where*

$$\rho_k' = 2k\pi + \varepsilon_k' \quad \textit{with } |\varepsilon_k'| \le \frac{\gamma}{k}, \quad k = k_0, k_0+1, \dots\,,$$
$$\rho_k'' = (2k+1)\pi + \varepsilon_k'' \quad \textit{with } |\varepsilon_k''| \le \frac{\gamma}{k}, \quad k = k_0, k_0+1, \dots\,,$$

and where the corresponding algebraic multiplicities and ascents are $\nu(\lambda_k') = m(\lambda_k') = 1$, $k = k_0, k_0+1, \dots$, and $\nu(\lambda_k'') = m(\lambda_k'') = 1$, $k = k_0, k_0+1, \dots$, respectively.

Theorem 2.3. *Let the differential operator L belong to Case* 2A, *and let $\xi_0, \eta_0 = 1/\xi_0$ be the distinct roots of the quadratic polynomial Q with $|\xi_0| \le 1$. Then the elements of the spectrum $\sigma(L)$ can be listed* *as two sequences* $\lambda_k' = (\rho_k')^2$, $k = k_0, k_0+1, \dots$, *and* $\lambda_k'' = (\rho_k'')^2$, $k = k_0, k_0+1, \dots$, *plus a finite number of additional points, where*

$$\rho_k' = (2k\pi + \operatorname{Arg}\xi_0) - i\ln|\xi_0| + \varepsilon_k' \quad \textit{with } |\varepsilon_k'| \le \frac{\gamma}{k}, \quad k = k_0, k_0+1, \dots,$$
$$\rho_k'' = (2k\pi + \operatorname{Arg}\eta_0) - i\ln|\eta_0| + \varepsilon_k'' \quad \textit{with } |\varepsilon_k''| \le \frac{\gamma}{k}, \quad k = k_0, k_0+1, \dots,$$

and where the corresponding algebraic multiplicities and ascents are $\nu(\lambda_k') = m(\lambda_k') = 1$, $k = k_0, k_0+1, \dots$, *and* $\nu(\lambda_k'') = m(\lambda_k'') = 1$, $k = k_0, k_0+1$, *respectively.*

Theorem 2.4. *Let the differential operator* L *belong to Case* 3, *with* $q \in C^1[0,1]$ *for Case* 3B. *Then the elements of the spectrum* $\sigma(L)$ *can be listed as two sequences* $\lambda_k' = (\rho_k')^2$, $k = k_0, k_0+1, \dots$, *and* $\lambda_k'' = (\rho_k'')^2$, $k = k_0, k_0+1, \dots$, *plus a finite number of additional points, where*

$$\rho_k' = 2k\pi + \varepsilon_k' \quad \textit{with } |\varepsilon_k'| \le \frac{\gamma}{k}, \quad k = k_0, k_0+1, \dots,$$
$$\rho_k'' = (2k+1)\pi + \varepsilon_k'' \quad \textit{with } |\varepsilon_k''| \le \frac{\gamma}{k}, \quad k = k_0, k_0+1, \dots,$$

and where the corresponding algebraic multiplicities and ascents are $\nu(\lambda_k') = m(\lambda_k') = 1$, $k = k_0, k_0+1, \dots$, *and* $\nu(\lambda_k'') = m(\lambda_k'') = 1$, $k = k_0, k_0+1, \dots$, *respectively.*

Remark 2.5. In Case 3B with $q \in C[0,1]$, we can proceed as follows. In terms of (2.6) select a function $\hat{q} \in C^1[0,1]$ such that

$$2|A_{14}|e^{2d}||q - \hat{q}||_\infty \le \frac{m_0}{4}, \tag{2.19}$$

and then for this fixed $\hat{q}$ choose a positive integer k_0 such that $x_0 := \omega - 2k_0\pi \le -r_0$, $y_0 := \omega + 2(k_0-1)\pi \ge r_0$, and

$$\frac{2|A_{14}|e^{2d}}{|a|}\left[||\hat{q}||_\infty + \frac{1}{2}||\hat{q}'||_\infty\right] + \frac{\gamma_1}{|a|} \le \frac{m_0}{4} \tag{2.20}$$

for all $a \in \mathbf{R}$ with $a \le x_0$ or $a \ge y_0$. These results lead immediately to the estimates

$$|g(\rho)| \le \frac{m_0}{2} < m_0 \le |f(\rho)| \tag{2.21}$$

and

$$|\Delta(\rho)| \geq \frac{m_0}{2} e^{-d} \tag{2.22}$$

for all $\rho \in \Omega_*$ with $\operatorname{Re} \rho \leq x_0$ or $\operatorname{Re} \rho \geq y_0$. From this point on, we can follow our earlier treatment of Case 3B verbatim, except for the asymptotic formulas we obtain the weaker results

$$|\rho_k' - \mu_k'| \leq \gamma \left\{ ||q - \tilde{q}||_\infty + \frac{1}{k} \left[||\tilde{q}||_\infty + \frac{1}{2} ||\tilde{q}'||_\infty + 1 \right] \right\} \tag{2.23}$$

and

$$|\rho_k'' - \mu_k''| \leq \gamma \left\{ ||q - \tilde{q}||_\infty + \frac{1}{k} \left[||\tilde{q}||_\infty + \frac{1}{2} ||\tilde{q}'||_\infty + 1 \right] \right\} \tag{2.24}$$

for $k = k_0, k_0 + 1, \ldots$, where the function $\tilde{q} \in C^1[0, 1]$ is arbitrary. This shows that

$$\lim_{k \to \infty} (\rho_k' - \mu_k') = \lim_{k \to \infty} (\rho_k'' - \mu_k'') = 0, \tag{2.25}$$

but it does not give us an effective rate of convergence for Case 3B.

Case 2B. $\xi_0 = \eta_0 = \pm 1$. Let us consider the circles Γ_k, $|k| \geq k_0$, which lie in the exterior of the rectangle $R_{k_0} = [x_0, y_0,] \times [-d, d]$. From (2.12) we have $|g(\rho)| < |f(\rho)|$ for all points ρ on Γ_k for $|k| \geq k_0$, and hence, by Rouché's theorem Δ and $f + g$ have precisely the same number of zeros as f inside Γ_k for all $|k| \geq k_0$. But f has only the single zero μ_k of order 2 inside Γ_k, implying that Δ has two zeros ρ_k' and ρ_k'' inside Γ_k for $|k| \geq k_0$, where either $\rho_k' \neq \rho_k''$ with ρ_k' and ρ_k'' both being zeros of order 1 or $\rho_k' = \rho_k''$ with ρ_k' being a zero of order 2.

Setting

$$\begin{aligned} \lambda_k' &= (\rho_k')^2, \qquad k = k_0, k_0 + 1, \ldots, \\ \lambda_k'' &= (\rho_k'')^2, \qquad k = k_0, k_0 + 1, \ldots, \end{aligned}$$

it follows that the λ_k', λ_k'', $k = k_0, k_0 + 1, \ldots$, are eigenvalues of L. In addition, if $\rho_k' \neq \rho_k''$, then $\lambda_k' \neq \lambda_k''$ and the algebraic multiplicities and ascents are

$$\nu(\lambda_k') = m(\lambda_k') = 1, \qquad \nu(\lambda_k'') = m(\lambda_k'') = 1; \tag{2.26}$$

if $\rho'_k = \rho''_k$, then $\lambda'_k = \lambda''_k$ and the algebraic multiplicities and ascents are

$$\nu(\lambda'_k) = 2, \qquad m(\lambda'_k) = 1 \quad \text{or} \quad m(\lambda'_k) = 2. \tag{2.27}$$

Sections 7 and 8 of [**11**] have models where the eigenvalues satisfy (2.26), while Sections 5 and 6 of [**11**] contain models satisfying (2.27). Also, as in the previous cases L can have only a finite number of eigenvalues in addition to the λ'_k, λ''_k, $k = k_0, k_0 + 1, \dots$.

Finally, we derive asymptotic formulas for the zeros ρ'_k, ρ''_k, $k = k_0, k_0 + 1, \dots$, of Δ. Fix any index $k \geq k_0$. We know that $f(\rho'_k) + g(\rho'_k) = 0$, so for the appropriate square root

$$e^{i\rho'_k} = \xi_0 + \sqrt{-\frac{1}{\alpha_0} g(\rho'_k)}$$

or

$$e^{i(\rho'_k - \mu_k)} = 1 + \underbrace{\frac{1}{\xi_0}\sqrt{-\frac{1}{\alpha_0} g(\rho'_k)}}_{\zeta'_k}.$$

Proceeding as in the previous cases, we get

$$\rho'_k - \mu_k = -i \operatorname{Log}[1 + \zeta'_k]$$

with $|\rho'_k| \geq k$ and

$$|\zeta'_k| = \frac{1}{|\xi_0|}\sqrt{\frac{1}{|\alpha_0|}|g(\rho'_k)|} \leq \frac{1}{|\xi_0|}\sqrt{\frac{\gamma_0}{|\alpha_0||\rho'_k|}} \leq \frac{\gamma}{\sqrt{k}},$$

and hence,

$$|\rho'_k - \mu_k| \leq \frac{\gamma}{\sqrt{k}}, \qquad k = k_0, k_0 + 1, \dots . \tag{2.28}$$

The same argument shows that

$$|\rho''_k - \mu_k| \leq \frac{\gamma}{\sqrt{k}}, \qquad k = k_0, k_0 + 1, \dots . \tag{2.29}$$

We summarize the above results as a theorem, incorporating the fact that $\xi_0 = \eta_0 = \pm 1$.

Theorem 2.6. *Let the differential operator L belong to Case* 2B*, and let $\xi_0 = \eta_0 = \pm 1$ be the double root of the quadratic polynomial Q. Then the elements of the spectrum $\sigma(L)$ can be listed as two sequences $\lambda'_k = (\rho'_k)^2$, $k = k_0, k_0+1, \dots,$ and $\lambda''_k = (\rho''_k)^2$, $k = k_0, k_0+1, \dots,$ plus a finite number of additional points, where*

$$\rho'_k = 2k\pi + \operatorname{Arg}\xi_0 + \varepsilon'_k \quad \text{with } |\varepsilon'_k| \le \frac{\gamma}{\sqrt{k}}, \quad k = k_0, k_0+1, \dots,$$
$$\rho''_k = 2k\pi + \operatorname{Arg}\xi_0 + \varepsilon''_k \quad \text{with } |\varepsilon''_k| \le \frac{\gamma}{\sqrt{k}}, \quad k = k_0, k_0+1, \dots.$$

For each $k \ge k_0$ if $\rho'_k \ne \rho''_k$, then $\lambda'_k \ne \lambda''_k$ and the corresponding algebraic multiplicities and ascents are $\nu(\lambda'_k) = m(\lambda'_k) = 1$ and $\nu(\lambda''_k) = m(\lambda''_k) = 1$, while if $\rho'_k = \rho''_k$, then $\lambda'_k = \lambda''_k$ and the corresponding algebraic multiplicity is $\nu(\lambda'_k) = 2$ and the corresponding ascent is $m(\lambda'_k) = 1$ or $m(\lambda'_k) = 2$.

Remark 2.7. The results of this section are directly applicable to the differential operator T and its characteristic determinant $\tilde{\Delta}$, with some obvious simplifications. Note that L and T share the same integer p, the same function f, and the same parameters ξ_0 and η_0 (see equation (2.1)). Thus, in setting up the geometry for T, we can use the same punctured strip Ω_* and the same circles $\Gamma'_k, \Gamma''_k, \Gamma_k$ as were used for L. From the above we have the following results:

$$|\tilde{\Delta}(\rho)| \ge \frac{m_0}{2} e^{-d} |\rho|^p \tag{2.30}$$

for all $\rho \in \Omega_*$ with $\operatorname{Re}\rho \le x_0$ or $\operatorname{Re}\rho \ge y_0$; $\tilde{\Delta}$ has zeros $\tilde{\rho}'_k, \tilde{\rho}''_k$ inside Γ'_k, Γ''_k for $|k| \ge k_0$; and the two sequences $\tilde{\lambda}'_k = (\tilde{\rho}'_k)^2$, $k = k_0, k_0+1, \dots,$ and $\tilde{\lambda}''_k = (\tilde{\rho}''_k)^2$, $k = k_0, k_0+1, \dots,$ are eigenvalues for T, accounting for all but a finite number of the eigenvalues of T. The corresponding algebraic multiplicities, ascents and asymptotic formulas are specified in the above theorems on a case by case basis. Many of these results are included in Theorems 1.1 and 2.1 in [**11**], and they will play a key role in Part IV.

3. The eigenvalues of L for Case 4. We conclude this paper by calculating the eigenvalues for the logarithmic case, which is technically the most difficult case. Let the differential operators L and T belong to Case 4:

Case 4. $A_{12} = 0$, $A_{14} + A_{23} = 0$, $A_{34} \neq 0$, $A_{13} + A_{24} \neq 0$.

Proceeding as in Section 7 of Part I [**14**], let $\mu = -2i(A_{13}+A_{24})/A_{34}$; let α and β be real numbers satisfying $0 < \alpha \leq 1/2$, $\beta \geq 2$, and

$$\frac{6\beta_0}{|A_{34}|\beta} \leq \frac{1}{2}||S||^{-1},$$

where $\beta_0 := 2|A_{14}| + 2|A_{13}| + 2|A_{24}| + 2|A_{34}| > 0$; and then form the logarithmic strip

$$\begin{aligned}\Omega &= \left\{\rho = a + ib \in \mathbf{C} \,\middle|\, \frac{\alpha}{|\mu|}e^{|b|} \leq |a| \leq \frac{\beta}{|\mu|}e^{|b|}\right\}\\ &= \left\{\rho = a + ib \in \mathbf{C} \,\middle|\, |a| \geq \frac{\alpha}{|\mu|} \text{ and } \ln\frac{|\mu||a|}{\beta} \leq |b| \leq \ln\frac{|\mu||a|}{\alpha}\right\},\end{aligned}$$

where the curve $|b| = \ln|\mu||a|$ runs down the "middle" of Ω. Setting $\xi = [1 + (|\mu|^2/\alpha^2)]^{1/2}$ and $\eta = [(\beta^2/|\mu|^2) + 1]^{1/2}$, for $\rho = a + ib \in \Omega$ we have the additional inequalities

$$|\rho| = [a^2 + b^2]^{1/2} \leq [a^2 + (|\mu|^2 a^2/\alpha^2)]^{1/2} = \xi|a|, \tag{3.1}$$

$$|\rho| = [a^2 + b^2]^{1/2} \leq [(\beta^2/|\mu|^2)e^{2|b|} + e^{2|b|}]^{1/2} = \eta e^{|b|}, \tag{3.2}$$

and

$$\ln\frac{|\rho|}{\eta} \leq |b|. \tag{3.3}$$

Fix any real number $d > 0$. Applying the a priori estimates of Part I (see Theorem 7.1 in [**14**]), there exists a constant r_0 with

$$r_0 > 2e^{2d}||q||_\infty, \qquad r_0 > \eta e^d > \frac{\beta}{|\mu|}, \qquad r_0 \geq \eta^2 \tag{3.4}$$

such that: if $\lambda = \rho^2 \in \mathbf{C}$ is any eigenvalue of L with $|\rho| \geq r_0$, then ρ must lie in the interior of Ω. Note that for any point $\rho = a + ib \in \Omega$ with $|\rho| \geq r_0$, we have $|b| > d$ by (3.3) and (3.4), $(1/2)\ln|\rho| \geq \ln \eta$, and from (3.3)

$$\frac{1}{2}\ln|\rho| \leq |b|. \tag{3.5}$$

Combining these results with Theorem 1.1, we obtain the following fundamental theorem for calculating the eigenvalues.

Theorem 3.1. *Let $\lambda = \rho^2 \in \mathbf{C}$ with $\rho = a + ib$, and assume that $|a| \geq r_0$ and $b \geq -d$. Then $\lambda = \rho^2$ is an eigenvalue of L if and only if $\ln(|\mu||a|/\beta) < b < \ln(|\mu||a|/\alpha)$ and $\Delta(\rho) = 0$.*

For the characteristic determinant in Case 4, we employ (1.5) and (1.2) to write Δ in the form

$$\Delta(\rho) = A_{34}e^{-i\rho}\{1 + g(\rho) + \phi(\rho)\} - A_{34}\mu\rho\{1 + \psi(\rho)\} \tag{3.6}$$

for all $\rho \neq 0$, where

$$g(\rho) = -\frac{A_{14}}{A_{34}}\int_0^1 e^{2i\rho s}q(s)\,ds + \frac{A_{14}}{A_{34}}\int_0^1 e^{2i\rho(1-s)}q(s)\,ds \quad \text{for } \rho \in \mathbf{C}$$

and the functions ϕ and ψ are analytic for $\rho \neq 0$ and of order $O(1/\rho)$ on the half plane $\operatorname{Im}\rho \geq -d$:

$$|\phi(\rho)| \leq \frac{\gamma_0}{|\rho|} \quad \text{and} \quad |\psi(\rho)| \leq \frac{\gamma_0}{|\rho|} \tag{3.7}$$

for all $\rho \in \mathbf{C}$ with $\operatorname{Im}\rho \geq -d$ and $|\rho| \geq r_0$. While an estimate like (2.6) is valid for the function g on the half plane $\operatorname{Im}\rho \geq -d$, we can get a better result by restricting ρ to Ω. Indeed, take any point $\rho = a + ib \in \Omega$ with $b \geq -d$ and $|\rho| \geq r_0$. From the above we have $b > d$ and $b = |b| \geq (1/2)\ln|\rho|$, and hence,

$$\begin{aligned} \left|\int_0^1 e^{2i\rho s}q(s)\,ds\right| &\leq ||q||_\infty \int_0^1 e^{-2bs}\,ds \\ &= \frac{1}{2b}||q||_\infty[1 - e^{-2b}] \leq \frac{||q||_\infty}{\ln|\rho|}. \end{aligned} \tag{3.8}$$

The same estimate is obtained for the integral $\int_0^1 e^{2i\rho(1-s)}q(s)\,ds$, so we conclude that

$$|g(\rho)| \le \frac{2|A_{14}|\,||q||_\infty}{|A_{34}|\ln|\rho|} \tag{3.9}$$

for all $\rho \in \Omega$ with $\operatorname{Im}\rho \ge -d$ and $|\rho| \ge r_0$. Of course, if $q \in C^1[0,1]$, then as in the last section we obtain the improved estimate

$$|g(\rho)| \le \frac{2|A_{14}|e^{2d}}{|A_{34}|\,|\rho|}[||q||_\infty + ||q'||_\infty] \tag{3.10}$$

for all $\rho \in \mathbf{C}$ with $\operatorname{Im}\rho \ge -d$ and $|\rho| \ge r_0$.

In view of Theorem 3.1, we proceed to determine the zeros $\rho = a+ib$ of Δ satisfying $|a| \ge r_0$ and $\ln(|\mu||a|/\beta) < b < \ln(|\mu||a|/\alpha)$, which obviously lie in Quadrants I and II. Let

$$\Omega' = \left\{\rho = a+ib \in \mathbf{C} \,\middle|\, a \ge r_0, \ln\frac{|\mu|a}{\beta} \le b \le \ln\frac{|\mu|a}{\alpha}\right\},$$
$$\Omega'' = \left\{\rho = a+ib \in \mathbf{C} \,\middle|\, a \le -r_0, \ln\frac{|\mu|(-a)}{\beta} \le b \le \ln\frac{|\mu|(-a)}{\alpha}\right\}.$$

We will give a detailed analysis of the zeros of Δ in Ω', and then simply state the analogous results for the zeros in Ω''.

Fix any real number δ with $0 < \delta \le \pi/4$ and $0 < \delta < (\ln 2)/(|\mu|+1)$, and then for $k = 1, 2, \ldots$ define

$$\begin{cases} \alpha_k' = 2k\pi - \operatorname{Arg}\mu, \qquad \beta_k' = \ln|\mu|\alpha_k', \\ \mu_k' = \alpha_k' + i\beta_k', \end{cases}$$

and introduce the circles

$$\Gamma_k' = \{\rho \in \mathbf{C} \mid |\rho - \mu_k'| = \delta\}.$$

Choose a positive integer $k_1 \ge 2$ such that $y_1 := \alpha_{k_1}' - \pi \ge r_0$. Note that

$$\alpha_k' - \pi \ge r_0 > \frac{\beta}{|\mu|} \quad \text{and} \quad \alpha_k' \ge 3\pi \tag{3.11}$$

for $k = k_1, k_1 + 1, \ldots$. Also, let us introduce the *logarithmic rectangles*

$$R'_k = \left\{ \rho = a + ib \in \mathbf{C} \,\middle|\, \alpha'_k - \pi \le a \le \alpha'_k + \pi, \ln \frac{|\mu|a}{\beta} \le b \le \ln \frac{|\mu|a}{\alpha} \right\}$$

for $k = k_1, k_1 + 1, \ldots$. Clearly the R'_k are all contained in Ω', and μ'_k lies in the interior of R'_k.

Next, fix any index $k \ge k_1$ and any point $\rho = a + ib \in \mathbf{C}$ with $|\rho - \mu'_k| \le \delta$. We assert that ρ lies in the interior of R'_k. Indeed, we clearly have $|a - \alpha'_k| \le \delta < \pi$ and $|b - \ln|\mu|\alpha'_k| \le \delta$, so

$$\begin{aligned} |b - \ln|\mu|a| &\le |b - \ln|\mu|\alpha'_k| + |\ln|\mu|a - \ln|\mu|\alpha'_k| \\ &\le \delta + |\mu|\,|a - \alpha'_k| \le \delta(|\mu| + 1) < \ln 2. \end{aligned}$$

It follows that $\ln(|\mu|a/2) < b < \ln(2|\mu|a)$ and $\ln(|\mu|a/\beta) < b < \ln(|\mu|a/\alpha)$. This establishes the assertion, and it is immediate that the circle Γ'_k lies in the interior of the logarithmic rectangle R'_k for $k = k_1, k_1 + 1, \ldots$. To complete the setup of the geometry, for $k = k_1, k_1 + 1, \ldots$ let Ω'_k be the *punctured logarithmic rectangle* formed from R'_k by removing all the points inside Γ'_k.

The next step is to establish the growth rate of Δ on each of the regions Ω'_k. For the constants δ and $\delta_0 := \max\{2/\alpha, 2\beta\}$ with $\delta < \ln 2 < \ln \delta_0$, we form the punctured rectangle

$$R_* = \{\rho = a + ib \in \mathbf{C} \mid -\pi \le a \le \pi, -\ln\delta_0 \le b \le \ln\delta_0, \text{ and } |\rho| \ge \delta\},$$

and set

$$m_0 = \min\{|e^{-i\rho} - 1| \mid \rho \in R_*\} > 0.$$

Let h be the function defined by

$$h(\rho) = -\frac{e^{-i\rho}}{\mu\rho} g(\rho) - \frac{e^{-i\rho}}{\mu\rho} \phi(\rho) + \psi(\rho) \quad \text{for } \rho \ne 0.$$

Clearly the function h is analytic for $\rho \ne 0$, and Δ can be written as

$$\Delta(\rho) = A_{34}\{e^{-i\rho} - \mu\rho[1 + h(\rho)]\} \tag{3.12}$$

for all $\rho \ne 0$.

To estimate the function h, take any point $\rho = a+ib \in \Omega$ with $b \geq -d$ and $|\rho| \geq r_0$. Then from the inequalities defining Ω,

$$\left|\frac{e^{-i\rho}}{\mu\rho}\right| = \frac{e^b}{|\mu||\rho|} \leq \frac{1}{|\mu||\rho|} \cdot \frac{|\mu||a|}{\alpha} \leq \frac{1}{\alpha},$$

and by (3.7), (3.9) and (3.10), we obtain

$$|h(\rho)| \leq \frac{\gamma_1}{\ln|\rho|} \tag{3.13}$$

in the general case $q \in C[0,1]$, and

$$|h(\rho)| \leq \frac{\gamma_1}{|\rho|} \tag{3.14}$$

in the special case $q \in C^1[0,1]$.

To complete the setup of Δ, observe that

$$e^{-i\mu'_k} = \mu\alpha'_k \tag{3.15}$$

for $k = k_1, k_1+1, \ldots$, and hence, introducing the sequences of functions f_k, g_k, $k = k_1, k_1+1, \ldots$, defined by

$$\begin{aligned} f_k(\rho) &= e^{-i(\rho-\mu'_k)} - 1 \quad \text{for } \rho \in \mathbf{C}, \\ g_k(\rho) &= -\left[\frac{i\beta'_k}{\alpha'_k} + \frac{1}{\alpha'_k}(\rho - \mu'_k)\right][1+h(\rho)] - h(\rho) \quad \text{for } \rho \neq 0, \end{aligned}$$

we can put Δ in its final form

$$\Delta(\rho) = A_{34}\mu\alpha'_k[f_k(\rho) + g_k(\rho)] \tag{3.16}$$

for all $\rho \neq 0$ and for $k = k_1, k_1+1, \ldots$. Here we have a family of representations for Δ depending on the integer k. We will use the kth representation to determine the growth rate of Δ on the kth region Ω'_k.

Fix any index $k \geq k_1$, and take any point $\rho = a+ib \in \Omega'_k$. Clearly $-\pi \leq a - \alpha'_k \leq \pi$ and

$$\begin{aligned} \ln\frac{a}{\beta\alpha'_k} &= \ln\frac{|\mu|a}{\beta} - \ln|\mu|\alpha'_k \leq b - \beta'_k \\ &\leq \ln\frac{|\mu|a}{\alpha} - \ln|\mu|\alpha'_k = \ln\frac{a}{\alpha\alpha'_k}. \end{aligned}$$

But $\alpha'_k \geq 2\pi$ or $\pi/\alpha'_k \leq 1/2$, and

$$\frac{a}{\alpha'_k} \geq \frac{\alpha'_k - \pi}{\alpha'_k} \geq \frac{1}{2}, \qquad \frac{a}{\alpha'_k} \leq \frac{\alpha'_k + \pi}{\alpha'_k} \leq 2,$$

so $-\ln(2\beta) \leq b - \beta'_k \leq \ln(2/\alpha)$. Thus, the point $\rho - \mu'_k$ belongs to the punctured rectangle R_* with $|\rho - \mu'_k| \leq \pi + \ln \delta_0$, and

$$|f_k(\rho)| \geq m_0. \tag{3.17}$$

Since $\lim_{k\to\infty} 1/\alpha'_k = \lim_{k\to\infty} \beta'_k/\alpha'_k = 0$, in terms of (3.13) we can choose a positive integer $k_0 \geq k_1$ such that

$$\left[\frac{\beta'_k}{\alpha'_k} + \frac{\pi + \ln \delta_0}{\alpha'_k}\right]\left[1 + \frac{\gamma_1}{\ln r_0}\right] \leq \frac{m_0}{4} \tag{3.18}$$

for all $k \geq k_0$ and such that

$$\frac{\gamma_1}{\ln |a|} \leq \frac{m_0}{4} \tag{3.19}$$

for all $a \in \mathbf{R}$ with $a \geq y_0 := \alpha'_{k_0} - \pi \geq r_0$. Then for each index $k \geq k_0$ and for each point $\rho = a + ib \in \Omega'_k$, we have

$$|\rho| \geq a \geq \alpha'_k - \pi \geq y_0 \geq r_0,$$

and hence, by (3.13), (3.18) and (3.19),

$$|g_k(\rho)| \leq \frac{m_0}{2} < m_0 \leq |f_k(\rho)|. \tag{3.20}$$

Also, since $\alpha'_k \geq 3\pi$, we have $a \geq \alpha'_k - \pi \geq 2\pi$ or $a/2 \geq \pi$, and by (3.1)

$$\alpha'_k \geq a - \pi \geq \frac{a}{2} \geq \frac{|\rho|}{2\xi}.$$

Therefore,

$$|\Delta(\rho)| \geq |A_{34}|\, |\mu|\alpha'_k \cdot \frac{m_0}{2} \geq \frac{m_0}{4\xi}|A_{34}|\, |\mu|\, |\rho|. \tag{3.21}$$

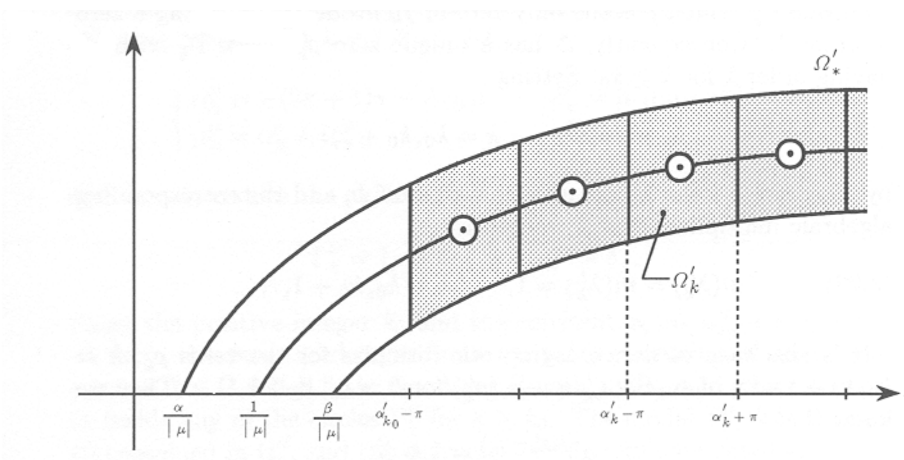

FIGURE 2. Punctured logarithmic strip Ω'_* in Case 4.

The estimate (3.20) is local in character in that it depends on k—it is valid only on the region Ω'_k. In contrast, the estimate (3.21) is global being independent of k. If we introduce the *punctured logarithmic strip*

$$\Omega'_* = \bigcup_{k=k_0}^{\infty} \Omega'_k,$$

then we see that Ω'_* consists of all points $\rho = a + ib \in \Omega$ which lie in Quadrant I with $a \geq y_0$ and which do not lie inside any of the circles Γ'_k for $k \geq k_0$, and from the above

$$|\Delta(\rho)| \geq \frac{m_0}{4\xi}|A_{34}|\,|\mu|\,|\rho| \geq \frac{\alpha m_0}{4\xi}|A_{34}|e^{|b|} \tag{3.22}$$

for all $\rho = a + ib \in \Omega'_*$.

With the basic estimates (3.20) and (3.22) established, consider one of the circles Γ'_k for $k \geq k_0$. Since (3.20) is valid for each point ρ on

Γ'_k, it follows that Δ and $f_k + g_k$ have the same number of zeros as f_k inside Γ'_k. But μ'_k is the only zero of f_k inside Γ'_k, μ'_k being a zero of order 1. Consequently, Δ has a unique zero ρ'_k inside Γ'_k with ρ'_k having order 1 for $k \geq k_0$. Setting

$$\lambda'_k = (\rho'_k)^2, \qquad k = k_0, k_0+1, \dots ,$$

by Theorem 3.1 the λ'_k are all eigenvalues of L, and the corresponding algebraic multiplicities and ascents are

$$\nu(\lambda'_k) = m(\lambda'_k) = 1, \qquad k = k_0, k_0+1, \dots . \tag{3.23}$$

It is also easy to derive asymptotic formulas for the zeros ρ'_k, $k = k_0, k_0+1, \dots$, of Δ. Set $\zeta'_k = -g_k(\rho'_k)$ for $k = k_0, k_0+1, \dots$. Then we know that

$$e^{-i(\rho'_k - \mu'_k)} = 1 + \zeta'_k$$

and

$$\rho'_k - \mu'_k = i \operatorname{Log}[1 + \zeta'_k] \tag{3.24}$$

for $k = k_0, k_0+1, \dots$, where

$$|\zeta'_k| \leq \left[\frac{\beta'_k}{\alpha'_k} + \frac{\delta}{\alpha'_k}\right]\left[1 + \frac{\gamma_1}{\ln r_0}\right] + \frac{\gamma_1}{\ln|\rho'_k|}.$$

Now for each $k \geq k_0$,

$$\begin{aligned} \alpha'_k &\geq 2k\pi - \pi \geq k, \\ \beta'_k &\leq \ln|\mu|(2k\pi + \pi) \leq \gamma_2 \ln k, \\ |\rho'_k| &\geq |\mu'_k| - |\rho'_k - \mu'_k| \geq \alpha'_k - \delta \geq 6k - 5 \geq k, \end{aligned}$$

which yields $|\zeta'_k| \leq \gamma_3/\ln k$ and

$$|\rho'_k - \mu'_k| \leq \frac{\gamma}{\ln k}, \qquad k = k_0, k_0+1, \dots . \tag{3.25}$$

In the special case $q \in C^1[0,1]$ we can use (3.14) to obtain the improved estimate

$$|\rho'_k - \mu'_k| \leq \frac{\gamma \ln k}{k}, \qquad k = k_0, k_0+1, \dots . \tag{3.26}$$

Up to this point we have concentrated on the zeros of Δ in the region Ω'. The same analysis is applicable to the region Ω'', leading to the following results. For $k = 1, 2, \dots$ let

$$\begin{cases} \alpha''_k = -(2k+1)\pi - \operatorname{Arg}\mu, & \beta''_k = \ln|\mu|(-\alpha''_k), \\ \mu''_k = \alpha''_k + i\beta''_k, \end{cases}$$

and then introduce the circles

$$\Gamma''_k = \{\rho \in \mathbf{C} \mid |\rho - \mu''_k| = \delta\}.$$

Using the positive integer k_0 and the constant $x_0 := \alpha''_{k_0} + \pi \le -r_0$, we form the punctured logarithmic strip Ω''_* consisting of all points $\rho = a + ib \in \Omega$ which lie in Quadrant II with $a \le x_0$ and which do not lie inside any of the circles Γ''_k for $k \ge k_0$. The circles Γ''_k, $k \ge k_0$, are all contained in Ω''_*, and the characteristic determinant satisfies

$$|\Delta(\rho)| \ge \frac{m_0}{4\xi}|A_{34}|\,|\mu|\,|\rho| \ge \frac{\alpha m_0}{4\xi}|A_{34}|e^{|b|} \tag{3.27}$$

for all $\rho = a + ib \in \Omega''_*$. Also, Δ has precisely one zero ρ''_k inside Γ''_k with ρ''_k having order 1 for $k \ge k_0$, and the sequence

$$\lambda''_k = (\rho''_k)^2, \qquad k = k_0, k_0+1, \dots,$$

consists entirely of eigenvalues of L. The algebraic multiplicities, ascents, and asymptotic formulas for the λ''_k, ρ''_k are identical to those for the λ'_k, ρ'_k:

$$\nu(\lambda''_k) = m(\lambda''_k) = 1, \qquad k = k_0, k_0+1, \dots, \tag{3.28}$$

$$|\rho''_k - \mu''_k| \le \frac{\gamma}{\ln k}, \qquad k = k_0, k_0+1, \dots, \tag{3.29}$$

and

$$|\rho''_k - \mu''_k| \le \frac{\gamma \ln k}{k}, \qquad k = k_0, k_0+1, \dots, \tag{3.30}$$

in the special case $q \in C^1[0,1]$.

Finally, we claim that the λ_k', λ_k'', $k = k_0, k_0+1, \ldots$, account for all but a finite number of the eigenvalues of L. Indeed, suppose $\lambda_0 = (\rho_0)^2$ is any eigenvalue of L distinct from the λ_k', λ_k''. Replacing ρ_0 by $-\rho_0$ if necessary, we can assume that $\operatorname{Im}\rho_0 \geq -d$. Now there are only two possible locations for ρ_0: either ρ_0 lies in the disk $|\rho| < r_0$ or $\rho_0 \in \Omega$ with $|\rho_0| \geq r_0$ and $x_0 < \operatorname{Re}\rho_0 < y_0$. Only a finite number of such ρ_0 are possible because they correspond to zeros of Δ from a bounded region of the ρ-plane. This establishes the claim.

We summarize the results for Case 4 in the next theorem, phrasing them in terms of the negatives of the μ_k'', ρ_k'' so that they are in the same format as Theorem 9.1 of [**11**].

Theorem 3.2. *Let the differential operator L belong to Case 4, and let $\mu = -2i(A_{13} + A_{24})/A_{34}$. Then the elements of the spectrum $\sigma(L)$ can be listed as two sequences $\lambda_k' = (\rho_k')^2$, $k = k_0, k_0+1, \ldots$, and $\lambda_k'' = (\rho_k'')^2$, $k = k_0, k_0+1, \ldots$, plus a finite number of additional points, where*

$$\rho_k' = (2k\pi - \operatorname{Arg}\mu) + i\ln|\mu|(2k\pi - \operatorname{Arg}\mu) + \varepsilon_k', \qquad k = k_0, k_0+1, \ldots ,$$

$$\rho_k'' = [(2k+1)\pi + \operatorname{Arg}\mu] - i\ln|\mu|[(2k+1)\pi + \operatorname{Arg}\mu] + \varepsilon_k'', \quad k = k_0, k_0+1, \ldots ,$$

with

$$|\varepsilon_k'| \leq \frac{\gamma}{\ln k}, \qquad |\varepsilon_k''| \leq \frac{\gamma}{\ln k}, \qquad k = k_0, k_0+1, \ldots ,$$

and in particular, with

$$|\varepsilon_k'| \leq \frac{\gamma \ln k}{k}, \qquad |\varepsilon_k''| \leq \frac{\gamma \ln k}{k}, \qquad k = k_0, k_0+1, \ldots ,$$

in the special case $q \in C^1[0,1]$. Moreover, the corresponding algebraic multiplicities and ascents are $\nu(\lambda_k') = m(\lambda_k') = 1$, $k = k_0, k_0+1, \ldots$, and $\nu(\lambda_k'') = m(\lambda_k'') = 1$, $k = k_0, k_0+1, \ldots$, respectively.

Remark 3.3. From the above the differential operator T in Case 4 satisfies:

$$|\tilde{\Delta}(\rho)| \geq \frac{m_0}{4\xi}|A_{34}|\,|\mu|\,|\rho| \geq \frac{\alpha m_0}{4\xi}|A_{34}|e^{|b|} \tag{3.31}$$

for all $\rho = a + ib \in \Omega'_* \cup \Omega''_*$; $\tilde{\Delta}$ has a unique zero $\tilde{\rho}'_k$ inside Γ'_k and a unique zero $\tilde{\rho}''_k$ inside Γ''_k for $k \geq k_0$, with $\tilde{\rho}'_k$ and $\tilde{\rho}''_k$ both being zeros of order 1; the two sequences $\tilde{\lambda}'_k = (\tilde{\rho}'_k)^2$, $k = k_0, k_0 + 1, \dots$, and $\tilde{\lambda}''_k = (\tilde{\rho}''_k)^2$, $k = k_0, k_0 + 1, \dots$, consist of eigenvalues of T, accounting for all but a finite number of the points in $\sigma(T)$; and the corresponding algebraic multiplicities and ascents are $\nu(\tilde{\lambda}'_k) = m(\tilde{\lambda}'_k) = 1$, $k = k_0, k_0 + 1, \dots$, and $\nu(\tilde{\lambda}''_k) = m(\tilde{\lambda}''_k) = 1$, $k = k_0, k_0 + 1, \dots$, respectively. Many of these results for T are contained in [**11**, Theorem 9.1], and they will be needed in Part IV.

Remark 3.4. Suppose the coefficient q is an even function about the point $t = 1/2$:

$$q(t) = q(1 - t) \quad \text{for all } t \in [0, 1].$$

Then for the integrals which appear in the function g, we have $\int_0^1 e^{2i\rho(1-s)} q(s)\, ds = \int_0^1 e^{2i\rho s} q(s)\, ds$, and in Case 3B and Case 4 the zeros ρ'_k, ρ''_k, $k = k_0, k_0 + 1, \dots$, satisfy the improved asymptotic formulas (2.17), (2.18) and (3.26), (3.30), respectively.

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