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CAUCHY TRANSFORMS OF MEASURES AND WEIGHTED SHIFT OPERATORS ON THE DISC ALGEBRA

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ABSTRACT. We consider families $\mathcal{F}_\alpha$, $\alpha > 0$, of analytic functions $F_\mu(z)$ on the unit disc that are obtained by integrating $(1 - e^{i\theta}z)^{-\alpha}$ with respect to complex measures μ on the unit circle. These families are Banach spaces which are isometrically isomorphic to the dual space of the disc algebra. The collection $\mathfrak{M}_\alpha$ of all multipliers of $\mathcal{F}_\alpha$ is shown to be the set of adjoints of the commutant of a certain weighted shift operator on the disc algebra.

It is known that if $0 < \alpha < \beta$, then $\mathfrak{M}_\alpha \subset \mathfrak{M}_\beta$. We show that this inclusion is proper in a number of cases. Also, for various α we find conditions on the sequence of Taylor coefficients of an analytic function that imply that the function is a multiplier of $\mathcal{F}_\alpha$.

1. Introduction. In this paper we consider families $\mathcal{F}_\alpha$ of Cauchy transforms of complex Borel measures on the unit circle $\mathbf{T}$ in the complex plane. For $\alpha > 0$ let $\mathcal{F}_\alpha$ consist of all functions f on the unit disc of the form

$$F_\mu(z) = \int_{\mathbf{T}} \frac{1}{(1 - e^{i\theta}z)^\alpha}\, d\mu(e^{i\theta}),$$

where μ is a complex Borel measure on $\mathbf{T}$. If $[\mu]$ is the equivalence class of all measures representing F_μ, then the norm $||F_\mu||_\alpha = ||[\mu]||$ makes $\mathcal{F}_\alpha$ into a Banach space.

The family $\mathcal{F}_1$ is of classical interest [**8, 9, 20**]. For example, it includes the Hardy spaces H^p for $p \geq 1$. The families $\mathcal{F}_\alpha$, $\alpha > 0$, were defined by T.H. MacGregor [**16**] in connection with geometric function theory. In particular, [**16**] includes the result that for $0 < \alpha < \beta$, $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$. We give a proof of the stronger result that $\mathcal{F}_\alpha \subset \mathcal{F}_{\beta a}$, the subset of $\mathcal{F}_\beta$ consisting of transforms of absolutely continuous measures,

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and we also prove that the inclusion map is a compact operator of norm one.

A second result in [**16**] is that for α and β positive, if f is in $\mathcal{F}_\alpha$ and g is in $\mathcal{F}_\beta$, then fg is in $\mathcal{F}_{\alpha+\beta}$. We give a new proof of this result and add the observation that $||fg||_{\alpha+\beta} \leq ||f||_\alpha ||g||_\beta$.

The main focus of this paper is on holomorphic functions ϕ on the unit disc with the property that if f is in $\mathcal{F}_\alpha$, then ϕf is in $\mathcal{F}_\alpha$. Such a function is called a multiplier of $\mathcal{F}_\alpha$, and the space of all such ϕ is denoted $\mathfrak{M}_\alpha$. It follows from the closed graph theorem that a multiplier ϕ induces a bounded linear operator M_ϕ on $\mathcal{F}_\alpha$ defined by $M_\phi f = \phi f$. These operators are all adjoints of operators on the disc algebra, which is the predual of $\mathcal{F}_\alpha$. They are closely related to certain weighted shift operators on the disc algebra.

The space $\mathcal{F}_1$ and its multiplier family $\mathfrak{M}_1$ have been considered extensively in the Soviet literature [**20, 19, 12**]. For example, in [**19**] S.A. Vinogradov proved that the function $\phi(z) = \sum_{n=0}^\infty a_n z^n$ is in $\mathfrak{M}_1$ if $\sum_{n=0}^\infty |a_n| \log(n+2) < \infty$. In addition, various necessary conditions for membership in $\mathfrak{M}_1$ are established in [**19**].

More recently, the first author and MacGregor studied properties of the families $\mathfrak{M}_\alpha$ in [**11**]. In particular, they used the inclusion of $\mathcal{F}_\alpha \mathcal{F}_\beta$ in $\mathcal{F}_{\alpha+\beta}$ to prove that if $0 < \alpha < \beta$, then $\mathfrak{M}_\alpha \subset \mathfrak{M}_\beta$. We include the proof here with the additional result that if $\alpha < \beta$ and $\phi \in \mathfrak{M}_\alpha$, then $||M_\phi||_\beta \leq ||M_\phi||_\alpha$.

In the case $0 < \alpha < 1$, it is known that $\mathfrak{M}_\alpha \neq \mathfrak{M}_1$ [**10**]. Some of the major results of this paper concern conditions on a function $\phi(z) = \sum_{n=0}^\infty a_n z^n$ that are sufficient to imply that ϕ is in $\mathfrak{M}_\alpha$. We show that for $0 < \alpha < 1$, if $\sum_{n=0}^\infty |a_n|(n+1)^{1-\alpha} \log(n+1) < \infty$, then $\phi \in \mathfrak{M}_\alpha$. For $0 < \alpha < \beta < 1$, we use this condition to show that there is a function in $\mathfrak{M}_\beta$ not in $\mathfrak{M}_\alpha$. We also prove that if $\sum_{n=0}^\infty |a_n| < \infty$, then $\phi \in \mathfrak{M}_2$. Using this, we show that there is a function in $\mathfrak{M}_2$ that is not in $\mathfrak{M}_1$. The general question of whether $\mathfrak{M}_\alpha$ is a proper subset of $\mathfrak{M}_\beta$ when $\alpha < \beta$ remains open.

2. Spaces of Cauchy transforms. Letting $\mathbf{T}$ be the unit circle in the complex plane, we consider the spaces $C(\mathbf{T})$ and $M(\mathbf{T})$ of continuous complex-valued functions and complex Borel measures on $\mathbf{T}$, respectively. The space $C(\mathbf{T})$ is a Banach space under the norm

$||f|| = \sup\{|f(\zeta)| : \zeta \in \mathbf{T}\}$ for $f \in C(\mathbf{T})$, and $M(\mathbf{T})$ is its dual space with the duality given by

$$\langle f, \mu \rangle = \int_{\mathbf{T}} f \, d\mu$$

for $\mu \in M(\mathbf{T})$. The norm of a complex measure is its total variation. The disk algebra $\mathbf{A}$ is the subspace of $C(\mathbf{T})$ consisting of those functions having a continuous extension to the closed unit disc that is analytic in the interior $\mathbf{D}$. Equivalently, a function h in $C(\mathbf{T})$ is in $\mathbf{A}$, provided

$$\int_0^{2\pi} h(e^{i\theta}) e^{in\theta} \, d\theta = 0$$

for $n = 1, 2, \dots$. The measures μ that annihilate $\mathbf{A}$ satisfy in particular

$$\int_{\mathbf{T}} e^{in\theta} \, d\mu(e^{i\theta}) = 0$$

for $n = 0, 1, 2, \dots$. By the F. and M. Riesz theorem, such measures are absolutely continuous and their Radon-Nikodym derivatives constitute the space H_0^1. Thus, the dual space $\mathbf{A}^\sharp$ of $\mathbf{A}$ may be identified with the quotient space $M(\mathbf{T})/H_0^1$. The equivalence class of a complex measure μ will be written $[\mu]$ or $\mu + H_0^1$. We will next describe other representations of $\mathbf{A}^\sharp$ that play a central role in this paper.

Fix $\alpha > 0$. Let $k_z^\alpha(\zeta) = 1/(1 - \zeta z)^\alpha$ for $z \in \mathbf{D}$ and $\zeta \in \mathbf{T}$. Then $k_z^\alpha \in \mathbf{A}$, and for $\mu \in M(\mathbf{T})$, we define a Cauchy transform $F_\mu : \mathbf{D} \to \mathbf{C}$ of μ by

$$F_\mu(z) = \langle k_z^\alpha, [\mu] \rangle = \int_{\mathbf{T}} \frac{1}{(1 - e^{i\theta} z)^\alpha} \, d\mu(e^{i\theta}).$$

Clearly, each F_μ is analytic in $\mathbf{D}$, and we define $\mathcal{F}_\alpha$ to be the collection of all F_μ as μ varies over $M(\mathbf{T})$. From the binomial series expansion of k_z^α we see that

$$F_\mu(z) = \sum_{n=0}^{\infty} A_n(\alpha) \mu_n z^n,$$

where $A_n(\alpha)$ is the binomial coefficient

$$A_n(\alpha) = (-1)^n \binom{-\alpha}{n} = \binom{n + \alpha - 1}{n}$$

and

$$\mu_n = \int_{\mathbf{T}} e^{in\theta}\, d\mu(e^{i\theta}).$$

An application of Stirling's formula shows that a necessary condition for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ to belong to $\mathcal{F}_\alpha$ is $a_n = O(n^{\alpha-1})$. The mapping $\mu \mapsto F_\mu$ is linear, and its null space is the set of measures encountered above, viz. $d\mu = g\, d\theta$ for $g \in H_0^1$. Thus, $\mathcal{F}_\alpha$ may be identified with $\mathbf{A}^\sharp = M(\mathbf{T})/H_0^1$, and for $h \in \mathbf{A}$, we write

$$\langle h, F_\mu \rangle_\alpha = \langle h, [\mu] \rangle$$

and

$$||F_\mu||_\alpha = ||[\mu]||.$$

For example,

$$\langle z^n, z^m \rangle_\alpha = \frac{\delta_{mn}}{\binom{n+\alpha-1}{n}}$$

where $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ otherwise.

If μ is a complex measure that is absolutely continuous with respect to normalized Lebesgue measure m on $\mathbf{T}$ and the Radon-Nikodym derivative of μ is a trigonometric polynomial, then F_μ is a polynomial. Since the set of such measures is weak* dense in $M(\mathbf{T})$, it follows that the polynomials are weak* dense in $\mathcal{F}_\alpha$. In fact, the measures $d\mu = p\, dm$, where p is a trigonometric polynomial in the unit ball of $L^1(m)$, are weak* dense in the unit ball of $M(\mathbf{T})$. It follows that the polynomials in the unit ball of $\mathcal{F}_\alpha$ are weak* dense in the unit ball of $\mathcal{F}_\alpha$.

The norm closure in $M(\mathbf{T})$ of the measures $p\, dm$ is the set of complex measures that are absolutely continuous with respect to m and have Radon-Nikodym derivatives in $L^1(m)$. The set of F_μ corresponding to these measures is a norm closed subspace of $\mathcal{F}_\alpha$ which we denote $\mathcal{F}_{\alpha a}$. Thus $\mathcal{F}_{\alpha a}$ is the norm closure of the polynomials in $\mathcal{F}_\alpha$. (See [**3**].)

3. Diagonal operators on the disc algebra. We are interested in the multipliers of $\mathcal{F}_\alpha$, and these will be introduced in Section 4. In this preliminary section we gather together some known facts about convolution operators, or coefficient multipliers of Fourier series, and present a couple of observations on the spaces $\mathcal{F}_\alpha$. To emphasize the analogy between multipliers of $\mathcal{F}_\alpha$ and Toeplitz operators on the Hardy space H^2, we will refer to convolution operators as diagonal operators.

A bounded operator is called compact in case it maps the unit ball to a precompact set. Since $\mathbf{A}$ has the approximation property (in fact it has a Schauder basis [**2**]), every compact operator on $\mathbf{A}$ is a norm limit of finite rank operators (see [**15**]). A basic result, which appears here as Theorem 3.1, is due to Fekete [**5**] and characterizes the bounded convolution operators (see [**21**] and [**14**]). Also, the compact convolution operators have been characterized by Akeman [**1**], Gaudry [**6**] and Kitchen [**13**]. Their results pertain to $C(\mathbf{T})$ and more general spaces, but a version for $\mathbf{A}$ can be proved using similar methods. A bounded operator D on $\mathbf{A}$ is called diagonal in case every z^n is an eigenvector for D.

Theorem 3.1 (Fekete). *A sequence $\{a_n\}$ is the sequence of eigenvalues of a diagonal operator D with respect to the eigenvectors $\{z^n\}_{n=0}^{\infty}$ if and only if there exists a complex Borel measure ν on $\mathbf{T}$ such that, for all $n \geq 0$, $a_n = \langle z^n, \nu \rangle$. In this case $Dh(z) = \nu * h(z) = \int_{\mathbf{T}} h(zw)\, d\nu(w)$, and $||D|| = ||[\nu]||$.*

It is also of interest to characterize those diagonal operators that are compact (see [**1, 13**]). We sketch a proof for the sake of completeness.

Theorem 3.2. *The diagonal operator D induced by a complex measure ν is compact if and only if $\nu \ll m$. In this case, if $\phi = d\nu/dm$, then $||D|| = ||\phi + H_0^1||$.*

Proof. If ν is a complex measure that is absolutely continuous with respect to m, and if ϕ is its Radon-Nikodym derivative, then an $L^1(m)$ approximation of ϕ by trigonometric polynomials leads to a norm approximation of the diagonal operator induced by ν by finite rank operators. Conversely, suppose D is a compact diagonal operator and D is the norm limit of a sequence of finite rank operators C_n. Let M_θ be the isometry defined by $M_\theta h(z) = h(e^{i\theta} z)$ for h in $\mathbf{A}$. Then the integrals $D_n = (1/2\pi) \int_0^{2\pi} M_\theta C_n M_{-\theta}\, d\theta$ are finite rank diagonal operators that converge to D in norm. Each D_n is induced by a measure $p_n dm$, where p_n is a trigonometric polynomial, and, by Theorem 3.1,

D is induced by a complex measure ν. Since $|||[d\nu - p_n dm]||| \to 0$, it follows that ν is absolutely continuous. $\square$

As an example, suppose that the sequence $\{a_n\}$ is convex and monotonically decreasing to 0. Then the function $\psi(e^{i\theta}) = a_0 + 2\sum_{n=1}^{\infty} a_n \cos n\theta$ is nonnegative, belongs to $L^1(m)$ and has L^1-norm a_0 (see [**21**, Chapter 5]). Letting $d\nu = \psi dm$, we see that $a_n = \langle z^n, \nu\rangle$ and, by Fekete's theorem, there is a diagonal operator D having $\{a_n\}$ as its sequence of eigenvalues. Since $\nu \geq 0$, $||D|| = a_0$. In addition, Theorem 3.2 implies that D is compact.

Example 3.3. Fix $\alpha > 0$, and let $a_n = (n+1)/(n+\alpha) = A_n(\alpha)/A_{n+1}(\alpha)$ for $n \geq 0$. Since the sequence of differences $a_n - 1$ or of their negatives satisfies the above conditions, it follows that the function

$$\psi_\alpha(e^{i\theta}) = (a_0 - 1) + 2\sum_{n=1}^{\infty}(a_n - 1)\cos n\theta$$

is an $L^1(m)$ function of norm $|1-\alpha|/\alpha$. Hence, if $d\mu = \psi_\alpha dm + d\delta_1$, where δ_1 is the point mass at 1, then $a_n = \langle z^n, \mu\rangle$ for $n \geq 0$, and hence $\{a_n\}$ is the sequence of eigenvalues of a diagonal operator D_α. Also note that if E_α is the diagonal operator induced by the sequence $\{(1-\alpha)/(n+\alpha)\}$, then E_α is compact and $D_\alpha = 1 + E_\alpha$, where we write simply 1 for the identity operator.

It was shown in [**16**] that $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ for $\alpha < \beta$. We illustrate the use of diagonal operators by establishing the following stronger inclusion relation and also show that the inclusion map is compact.

Theorem 3.4. *If $\alpha < \beta$, then $\mathcal{F}_\alpha \subset \mathcal{F}_{\beta a}$, and the inclusion map is a compact operator of norm one.*

Proof. If $f \in \mathcal{F}_\alpha$, then there exists a measure μ such that

$$f(z) = \int_{\mathbf{T}} \frac{1}{(1-\zeta z)^\alpha}\, d\mu(\zeta) = \sum_{n=0}^{\infty} A_n(\alpha)\mu_n z^n.$$

Consider the quotients $q_n = A_n(\alpha)/A_n(\beta)$. We have

$$q_n = \frac{\Gamma(\beta)\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+\beta)},$$

and a computation shows that

$$\begin{aligned} q_{n+2} - 2q_{n+1} + q_n \\ &= \frac{\Gamma(\beta)\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+\beta)}\left(\frac{(n+\alpha+1)(n+\alpha)}{(n+\beta+1)(n+\beta)} - 2\frac{n+\alpha}{n+\beta} + 1\right) \\ &\geq 0. \end{aligned}$$

Thus, the sequence $\{q_n\}_0^\infty$ is a convex sequence of positive numbers converging monotonically to 0, and consequently the Fourier series

$$q(e^{i\theta}) \sim q_0 + 2\sum_{n=1}^{\infty} q_n \cos n\theta$$

is that of an L^1 function of norm $q_0 = 1$.

Define ν to be the measure obtained as the convolution of μ with the function q. Then ν is absolutely continuous and $\nu_n = q_n\mu_n$, or equivalently,

$$A_n(\alpha)\mu_n = A_n(\beta)\nu_n.$$

Consequently,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} A_n(\beta)\nu_n z^n \\ &= \int_{\mathbf{T}} \frac{1}{(1-\zeta z)^\beta}\, d\nu(\zeta), \end{aligned}$$

which establishes the result. $\square$

The inclusion of the following theorem was first obtained in [**16**] with different methods. The norm inequality is new.

Theorem 3.5. *For all* $\alpha, \beta > 0$, $\mathcal{F}_\alpha \mathcal{F}_\beta \subset \mathcal{F}_{\alpha+\beta}$, *and for* $f \in \mathcal{F}_\alpha$ *and* $g \in \mathcal{F}_\beta$, $||fg||_{\alpha+\beta} \leq ||f||_\alpha ||g||_\beta$.

Proof. For each function h in $C(\mathbf{T})$ we consider its extension to the closed unit disc obtained by taking its Poisson integral on the interior and, for each probability measure ρ on $[0,1]$, define H_ρ on $\mathbf{T}\times\mathbf{T}$ by

$$H_\rho(z,w) = \int_0^1 h(tz+(1-t)w)\,d\rho(t).$$

Then H_ρ is continuous and $||H_\rho||_\infty = ||h||$. If μ and ν are complex Borel measures on $\mathbf{T}$, then we obtain a bounded linear functional ϕ_ρ on $C(\mathbf{T})$ as follows:

$$\phi_\rho(h) = \int_{\mathbf{T}\times\mathbf{T}} H_\rho\,d\mu\times\nu.$$

Since H_ρ has the same norm as h, it follows that $||\phi_\rho|| \le ||\mu||\,||\nu||$. By the Riesz representation theorem, there exists a complex measure λ on $\mathbf{T}$ such that

$$\int_{\mathbf{T}} h\,d\lambda = \int_{\mathbf{T}\times\mathbf{T}} H_\rho\,d\mu\times\nu$$

and $||\lambda|| \le ||\mu||\,||\nu||$.

The special case with $d\rho(t) = (1/B(\alpha,\beta))t^{\alpha-1}(1-t)^{\beta-1}\,dt$, where α and β are positive numbers and $B(\alpha,\beta)$ is the beta function, is of interest to us. In this case we obtain the negatively indexed Fourier coefficients $\lambda_n = \hat{\lambda}(-n)$ of λ by taking $h(z)=z^n$ and evaluating $\phi_\rho(h)$. Thus,

$$\lambda_n = \int_{\mathbf{T}\times\mathbf{T}}\int_0^1 [tz+(1-t)w]^n\,d\rho(t)\,d\mu\times\nu(z,w).$$

By virtue of the following calculation,

$$\begin{aligned}\binom{n}{k}\int_0^1 t^k(1-t)^{n-k}\,d\rho(t) &= \binom{n}{k}\frac{B(k+\alpha,n-k+\beta)}{B(\alpha,\beta)}\\ &= \frac{A_k(\alpha)A_{n-k}(\beta)}{A_n(\alpha+\beta)},\end{aligned}$$

we obtain

$$\begin{aligned}\lambda_n &= \sum_{k=0}^n \binom{n}{k}\int_0^1 t^k(1-t)^{n-k}\,d\rho(t)\int_{\mathbf{T}\times\mathbf{T}} z^k w^{n-k}\,d\mu\times\nu(z,w)\\ &= \sum_{k=0}^n \frac{A_k(\alpha)A_{n-k}(\beta)}{A_n(\alpha+\beta)}\mu_k\nu_{n-k}.\end{aligned}$$

Hence

$$A_n(\alpha+\beta)\lambda_n = \sum_{k=0}^{n} A_k(\alpha)\mu_k A_{n-k}(\beta)\nu_{n-k}.$$

Thus, if $f \in \mathcal{F}_\alpha$ and $g \in \mathcal{F}_\beta$, then there are measures μ and ν such that $\hat{f}(n) = A_n(\alpha)\mu_n$ and $\hat{g}(n) = A_n(\beta)\nu_n$. Defining λ as above, we see that if $h = fg$, then $\hat{h}(n) = A_n(\alpha+\beta)\lambda_n$, and thus $h \in \mathcal{F}_{\alpha+\beta}$. Since $||\lambda|| \le ||\mu||\,||\nu||$, it follows that $||h||_{\alpha+\beta} \le ||f||_\alpha ||g||_\beta$. □

We remark that the inclusion of $\mathcal{F}_\alpha$ in $\mathcal{F}_\beta$ for $\alpha < \beta$ in Theorem 3.4 is a corollary of Theorem 3.5. Note that Theorem 3.5 cannot be improved to get $\mathcal{F}_\alpha\mathcal{F}_\beta$ included in $\mathcal{F}_{\alpha+\beta,a}$, since, for example, $(1-z)^{-\alpha}(1-z)^{-\beta} = (1-z)^{-(\alpha+\beta)}$, which is not in $\mathcal{F}_{\alpha+\beta,a}$.

4. Weighted shifts on the disc algebra.

Definition 4.1. A *weighted shift* on $\mathbf{A}$ is an operator of the form $W = DS$, where D is a diagonal operator and $Sh(z) = (h(z)-h(0))/z$ for $h \in \mathbf{A}$.

The operator S is the "backwards shift operator" on $\mathbf{A}$ since

$$Sz^n = \begin{cases} 0 & \text{if } n = 0, \\ z^{n-1} & \text{if } n > 0. \end{cases}$$

Define weighted shift operators S_α on $\mathbf{A}$ by $S_\alpha = D_\alpha S$. Thus,

$$\begin{aligned} S_\alpha z^n &= \begin{cases} 0 & \text{if } n = 0, \\ (A_{n-1}(\alpha)/A_n(\alpha))z^{n-1} & \text{if } n > 0 \end{cases} \\ &= \begin{cases} 0 & \text{if } n = 0, \\ (n/(n-1+\alpha))z^{n-1} & \text{if } n > 0. \end{cases} \end{aligned}$$

By Example 3.3, $S_\alpha = S + E_\alpha S$, and thus S_α is a compact perturbation of S. Also, for $n \ge 1$,

$$\begin{aligned} \langle S_\alpha z^n, z^m\rangle_\alpha &= \frac{n}{n-1+\alpha}\langle z^{n-1}, z^m\rangle_\alpha \\ &= \frac{\delta_{n-1,m}}{\binom{n+\alpha-1}{n}} \\ &= \langle z^n, z^{m+1}\rangle_\alpha, \end{aligned}$$

and hence for $k \geq 0$,

$$\langle S_\alpha^k z^n, z^m \rangle_\alpha = \langle z^n, z^{m+k} \rangle_\alpha.$$

Also, it follows that

$$S_\alpha^k z^n = \begin{cases} 0 & \text{if } n < k \\ (A_{n-k}(\alpha)/A_n(\alpha)) z^{n-k} & \text{if } n \geq k. \end{cases}$$

For an analytic function $\phi(z) = \sum_{k=0}^\infty \hat{\phi}(k) z^k$, we define a linear transformation $\phi(S_\alpha)$ on polynomials p by

$$\phi(S_\alpha) = \sum_{k=0}^\infty \hat{\phi}(k) S_\alpha^k.$$

Note that the sum is finite since $S_\alpha^n p = 0$ whenever n exceeds the degree of p. Also, for $\phi \in \mathcal{F}_\alpha$ we define M_ϕ by $M_\phi q = \phi q$ for polynomials q. Then for $\phi \in \mathcal{F}_\alpha$, we have

$$\begin{aligned} \langle \phi(S_\alpha) z^m, z^n \rangle_\alpha &= \left\langle \sum_{k=0}^\infty \hat{\phi}(k) S_\alpha^k z^m, z^n \right\rangle_\alpha \\ &= \sum_{k=0}^\infty \hat{\phi}(k) \langle z^m, M_z^k z^n \rangle_\alpha \\ &= \langle z^m, M_\phi z^n \rangle_\alpha. \end{aligned}$$

Thus, for polynomials p and q we have the duality relation

$$\langle \phi(S_\alpha) p, q \rangle_\alpha = \langle p, M_\phi q \rangle_\alpha.$$

An operator on a Banach space $\mathbf{X}$ is called closed in case its graph is a closed subspace of $\mathbf{X} \times \mathbf{X}$. It is called closable in case it has a closed extension. An operator is closable provided $(0, F)$ is in the closure of its graph only if $F = 0$.

Lemma 4.2. *Both $\phi(S_\alpha)$ and M_ϕ are closable whenever $\phi \in \mathcal{F}_\alpha$.*

Proof. This is an elementary computation based on the preceding duality relation. □

Theorem 4.3. *For $\phi \in \mathcal{F}_\alpha$, $\phi(S_\alpha)$ is bounded if and only if M_ϕ is bounded. In this case $\phi(S_\alpha)$ has a natural extension to* $\mathbf{A}$*, and M_ϕ has a natural extension to $\mathcal{F}_\alpha$ given by $M_\phi F = \phi(S_\alpha)^\sharp F = \phi F$ for all F in $\mathcal{F}_\alpha$. Moreover, $\phi(\mathbf{D})$ is included in the point spectrum of $\phi(S_\alpha)$.*

Proof. Suppose $\phi \in \mathcal{F}_\alpha$ and M_ϕ is bounded. Its extension to $\mathcal{F}_\alpha$ will also be denoted by M_ϕ. We write $\|M_\phi\|_\alpha$ for the operator norm of M_ϕ acting on $\mathcal{F}_\alpha$. Then for all polynomials p and q,

$$\begin{aligned}|\langle \phi(S_\alpha)p, q\rangle_\alpha| &= |\langle p, M_\phi q\rangle_\alpha| \\ &\le ||M_\phi||_\alpha ||p|| \, ||q||_\alpha.\end{aligned}$$

By the remark in the next to the last paragraph of Section 2, the polynomials in the unit ball of $\mathcal{F}_\alpha$ are weak* dense in the unit ball of $\mathcal{F}_\alpha$, and thus

$$||\phi(S_\alpha)p|| \le ||M_\phi||_\alpha ||p||.$$

Therefore, $\phi(S_\alpha)$ is bounded, and it extends by continuity to the closure of the polynomials in $\mathbf{A}$, i.e., to all of $\mathbf{A}$. We will denote the extension to $\mathbf{A}$ by $\phi(S_\alpha)$ also.

Suppose $\phi \in \mathcal{F}_\alpha$ and $\phi(S_\alpha)$ is bounded. Then, for all polynomials p and q,

$$\begin{aligned}\langle p, M_\phi q\rangle_\alpha &= \langle \phi(S_\alpha)p, q\rangle_\alpha \\ &= \langle p, \phi(S_\alpha)^\sharp q\rangle_\alpha.\end{aligned}$$

Thus, $M_\phi q = \phi(S_\alpha)^\sharp q$ for all polynomials q. Hence, M_ϕ is bounded and $\phi(S_\alpha)^\sharp$ is a bounded extension of M_ϕ to $\mathcal{F}_\alpha$. It will be shown that $\phi(S_\alpha)^\sharp$ is multiplication by ϕ.

Suppose that $\phi(S_\alpha)$, or equivalently M_ϕ, is bounded. For each w in $\mathbf{D}$ and each polynomial q,

$$\begin{aligned}\langle \phi(S_\alpha)k_w^\alpha, q\rangle_\alpha &= \langle k_w^\alpha, M_\phi q\rangle_\alpha = \langle k_w^\alpha, \phi q\rangle_\alpha \\ &= \phi(w)q(w) = \langle \phi(w)k_w^\alpha, q\rangle_\alpha.\end{aligned}$$

Thus, $\phi(S_\alpha)k_w^\alpha = \phi(w)k_w^\alpha$, and hence $\phi(w)$ is an eigenvalue of $\phi(S_\alpha)$. For $F \in \mathcal{F}_\alpha$,

$$\begin{aligned}(\phi(S_\alpha)^\sharp F)(w) &= \langle k_w^\alpha, \phi(S_\alpha)^\sharp F\rangle_\alpha = \langle \phi(S_\alpha)k_w^\alpha, F\rangle_\alpha \\ &= \phi(w)\langle k_w^\alpha, F\rangle_\alpha = \phi(w)F(w),\end{aligned}$$

and thus the natural extension of M_ϕ to $\mathcal{F}_\alpha$ is just multiplication by ϕ. $\square$

Note that Theorem 4.3 implies that $\phi(\mathbf{D})$ is included in the spectrum of M_ϕ, which is a compact subset of the complex plane. This gives another proof of the result that if $\phi \in \mathcal{F}_\alpha$ and if M_ϕ is bounded, then $\phi \in H^\infty$.

In particular, it follows that the spectrum of S_α includes the closed unit disc. We will obtain the opposite inclusion by showing that the spectral radius of S_α is one. Recall that the spectral radius is given by the formula $\rho(S_\alpha) = \lim_m ||S_\alpha^m||^{1/m}$, so we must estimate $||S_\alpha^m||$. Let $D_{\alpha,k}$ be the diagonal operator induced by the weight sequence $\{(n+k)/(n+\alpha+k-1)\}_{n=0}^\infty$. Then $D_{\alpha,1} = D_\alpha$, and $D_{\alpha,k+1}S = SD_{\alpha,k}$. Hence $S_\alpha^m = (D_\alpha S)^m = P_m S^m$, where $P_m = \prod_{k=1}^m D_{\alpha,k}$. Note that P_m is the diagonal operator induced by the weight sequence $\{A_n(\alpha)/A_{n+m}(\alpha)\}$.

Lemma 4.4. *There exists a constant c such that, for $m \geq 1$,*

$$||S_\alpha^m|| \leq c(m+1)^{1-\alpha}\log(m+1)$$

whenever $0 < \alpha \leq 1$, and

$$||S_\alpha^m|| \leq c\log(m+1)$$

whenever $\alpha > 1$.

Proof. The norms of P_m and S^m will be estimated separately. The operator P_m is diagonal with weight sequence $(\prod_{k=1}^m (n+k)/(n+\alpha+k-1))_{n=0}^\infty$. Suppose that $0 < \alpha \leq 1$. Thus, the weight sequence of $P_m - 1$ is nonnegative, convex and monotonically decreasing to zero. Hence, by Theorem 3.1 and the discussion following Theorem 3.2, the norm of $P_m - 1$ is no greater than $\prod_{k=1}^m k/(\alpha+k-1) - 1 \leq 1/\binom{m+\alpha-1}{m} - 1$. By Stirling's formula, $1/\binom{m+\alpha-1}{m}$ is dominated by $\kappa(m+1)^{1-\alpha}$ for some constant κ. Thus,

$$||P_m|| \leq \kappa(m+1)^{1-\alpha}.$$

On the other hand, if $\alpha > 1$, then the weight sequence of $1 - P_m$ is nonnegative, convex and monotonically decreasing to zero. Hence, the norm of $1 - P_m$ is no greater than $1 - \prod_{k=1}^{m} k/(\alpha + k - 1) \leq 1$, and thus $||P_m|| \leq 2$.

Let T be multiplication by z on $\mathbf{A}$, $Th(z) = zh(z)$. Thus, T is an isometry on $\mathbf{A}$, and $T^m S^m = 1 - s_{m-1}$, where $s_m h$ is the mth partial sum of the Fourier series of h. Since $||s_m|| \leq \lambda \log(m+2)$ for some constant λ (see [**21**, volume 1, p. 67]), we have

$$\begin{aligned} ||S^m|| &= ||T^m S^m|| \\ &\leq \lambda \log(m+1) + 1 \\ &\leq \lambda' \log(m+1). \end{aligned}$$

Consequently, for $0 < \alpha \leq 1$,

$$\begin{aligned} ||S_\alpha^m|| &\leq ||P_m||\,||S^m|| \\ &\leq \kappa (m+1)^{1-\alpha} \lambda' \log(m+1), \end{aligned}$$

and putting $c = \kappa\lambda'$ completes the proof of this case. In case $\alpha > 1$,

$$||S_\alpha^m|| \leq ||P_m||\,||S^m|| \leq 2\lambda' \log(m+1),$$

which establishes the lemma. □

The above estimates suffice to establish the following theorem. A better estimate for $||S_2^m||$ will be obtained in Theorem 8.3.

Theorem 4.5. *The spectrum of S_α, and consequently also of M_z, is the closed unit disc.*

Proof. The spectrum includes $\mathbf{D}$ by Theorem 4.3. To see that it lies in $\overline{\mathbf{D}}$, apply Lemma 4.4 and the spectral radius formula. □

It would be of interest to know if, more generally, the spectrum of $\phi(S_\alpha)$ is $\overline{\phi(\mathbf{D})}$ whenever $\phi(S_\alpha)$ is bounded.

5. Multipliers of $\mathcal{F}_\alpha$.

Definition 5.1. A *multiplier* of $\mathcal{F}_\alpha$ is an *analytic function* ϕ such that $\phi F \in \mathcal{F}_\alpha$ for all $F \in \mathcal{F}_\alpha$.

If ϕ is a multiplier of $\mathcal{F}_\alpha$, then clearly $\phi = \phi 1 \in \mathcal{F}_\alpha$. Further, the closed graph theorem shows that M_ϕ is a bounded operator on $\mathcal{F}_\alpha$. Thus, if $\mathfrak{M}_\alpha$ is the set of all multipliers of $\mathcal{F}_\alpha$, then $\mathfrak{M}_\alpha = \{\phi \in \mathcal{F}_\alpha : M_\phi \text{ is a bounded operator on } \mathcal{F}_\alpha\}$. Hence, the following theorem is a consequence of Theorem 4.3.

Theorem 5.2. *The following are equivalent*:

i) *ϕ is a multiplier of $\mathcal{F}_\alpha$,*

ii) *M_ϕ is a bounded operator on $\mathcal{F}_\alpha$,*

iii) *$\phi(S_\alpha)$ is a bounded operator on* $\mathbf{A}$.

If the above hold, then $M_\phi = \phi(S_\alpha)^\sharp$ and $||M_\phi||_\alpha = ||\phi(S_\alpha)||$.

The case of $\alpha = 1$ in both the above theorem and Theorem 5.4 below were established by Vinogradov [**19**] with different methods.

Corollary 5.3. *If ϕ is a multiplier of $\mathcal{F}_\alpha$, then M_ϕ is weak* continuous.*

For each function $\phi(z) = \sum_{n=0}^{\infty} \hat{\phi}(n) z^n$ and each $w \in \mathbf{D}$, we define ϕ_w by $\phi_w(z) = \sum_{n=0}^{\infty} \hat{\phi}(n) w^n z^n$.

Theorem 5.4. *Suppose ϕ is holomorphic in* $\mathbf{D}$. *The following are equivalent*:

i) *ϕ is a multiplier of $\mathcal{F}_\alpha$,*

ii) *$\{\phi(z)/(1-\zeta z)^\alpha : \zeta \in \mathbf{T}\}$ is a bounded subset of $\mathcal{F}_\alpha$,*

iii) *$\{M_{\phi_r} : 0 < r < 1\}$ is norm bounded.*

If ϕ is a multiplier of $\mathcal{F}_\alpha$, then

$$\begin{aligned} ||M_\phi||_\alpha &= \sup\left\{ \left\| \frac{\phi(z)}{(1-\zeta z)^\alpha} \right\|_\alpha : \zeta \in \mathbf{T} \right\} \\ &= \sup\{||M_{\phi_r}||_\alpha : 0 < r < 1\}. \end{aligned}$$

Proof. Suppose that ϕ is a multiplier of $\mathcal{F}_\alpha$. Since $1/(1-\zeta z)^\alpha = \langle k_z^\alpha, \delta_\zeta \rangle$, where δ_ζ is the point mass at ζ, the condition in ii) is necessary, and the set in ii) lies in the ball of radius $||M_\phi||_\alpha$ in $\mathcal{F}_\alpha$. Conversely, suppose the condition is satisfied and the set in ii) lies in the ball of radius s. Let μ be a complex measure of norm one. Since the measures δ_ζ are the extreme points in the unit ball of $M(\mathbf{T})$ and $C(\mathbf{T})$ is separable, there exists a sequence of complex convex combinations μ_n of point masses that converges weak* to μ. Put $F_n(w) = \langle k_w^\alpha, \mu_n \rangle$, so (F_n) converges weak* to F_μ in $\mathcal{F}_\alpha$. For a fixed n, suppose $\mu_n = \sum_{j=1}^k c_j \delta_j$, where δ_j is a point mass at ζ_j and the c_j are complex numbers such that $\sum_{j=1}^k |c_j| = 1$. Then

$$\phi(w)F_n(w) = \sum_{j=1}^{k} \frac{c_j \phi(w)}{(1-\zeta_j w)^\alpha},$$

and hence ϕF_n lies in the ball of radius s in $\mathcal{F}_\alpha$. By Alaoglu's theorem, there is a weak* convergent subnet, say $(\phi F_{n'})$, converging to a G in the ball of radius s in $\mathcal{F}_\alpha$. Thus, for w in $\mathbf{D}$,

$$\begin{aligned} G(w) = \langle k_w^\alpha, G\rangle_\alpha &= \lim_{n'} \langle k_w^\alpha, \phi F_{n'}\rangle \\ &= \lim_{n'} \phi(w) F_{n'}(w) = \phi(w)F_\mu(w). \end{aligned}$$

Hence ϕ is a multiplier of $\mathcal{F}_\alpha$, and $||M_\phi F_\mu||_\alpha \le s$. Thus, $||M_\phi||_\alpha \le s$. It has been shown that i) and ii) are equivalent.

Suppose ϕ is a multiplier of $\mathcal{F}_\alpha$ and $0 < r < 1$. Fix polynomials p in $\mathbf{A}$ and q in $\mathcal{F}_\alpha$ each of norm one. Suppose the degree of each is less than or equal to N. Consider the polynomial P defined by

$$P(w) = \langle p, \phi_w q\rangle_\alpha.$$

Note that the degree of P is no greater than N. The modulus of this polynomial assumes its maximum at a point ζ of $\mathbf{T}$. Thus,

$$|\langle p, \phi_r q\rangle_\alpha| \le |\langle p, \phi_\zeta q\rangle_\alpha|.$$

But a calculation shows that $\langle p, \phi_\zeta q\rangle_\alpha = \langle p_\zeta, \phi q_{\bar\zeta}\rangle_\alpha$, and since each of p_ζ and $q_{\bar\zeta}$ have norm one, it follows that

$$|\langle p, \phi_r q\rangle_\alpha| \le ||M_\phi||_\alpha.$$

Therefore $||M_{\phi_r}||_\alpha \leq ||M_\phi||_\alpha$.

If condition iii) is satisfied and $s = \sup\{||M_{\phi_r}||_\alpha : 0 < r < 1\}$, then for any polynomial p in $\mathbf{A}$ and any F in $\mathcal{F}_\alpha$,

$$\langle p, M_\phi F\rangle_\alpha = \lim_{r\to 1} \langle p, M_{\phi_r} F\rangle_\alpha.$$

Consequently, $|\langle p, M_\phi F\rangle_\alpha| \leq s||p||\,||F||_\alpha$. Thus, M_ϕ is bounded, and its norm is no greater than s. This completes the proof. □

A final characterization of multipliers, or rather multiplication operators on $\mathcal{F}_\alpha$, is that they constitute the commutant of M_z. To prove this, we need to recall that an operator on a Banach space is a Fredholm operator in case its kernel is finite dimensional and its range is closed and has finite codimension. The index $i(A)$ of a Fredholm operator A is the difference of the dimension of its kernel and the codimension of its range. The following lemma is standard. See, for example, [**18**].

Lemma 5.5. *Compact perturbations of Fredholm operators are Fredholm. If A is a Fredholm operator on a Banach space $\mathbf{X}$, then $A^\sharp$ is a Fredholm operator on $\mathbf{X}^\sharp$, and $i(A^\sharp) = -i(A)$.*

Theorem 5.6. *For each $\alpha > 0$ and each w in $\mathbf{D}$, the operator $S_\alpha - w$ is surjective and its null space is the one-dimensional subspace spanned by k_w^α. The operator $M_z - w$ is injective, and its range is closed and has codimension one.*

Proof. The assertion concerning the null space of $S_\alpha - w$ is the result of an easy calculation. That $S - w$ is onto is immediate since for h in $\mathbf{A}$, if $f(z) = (1 - wz)^{-1}zh$, then $f \in \mathbf{A}$ and $(S - w)f = h$. Thus $S - w$ is Fredholm of index one. Since S_α is a compact perturbation of S, it follows that $S_\alpha - w$ is Fredholm of index one. The assertion concerning $M_z - w$ is now a consequence of Lemma 5.5. □

For $\alpha > 0$, $\mathcal{F}_\alpha$ is invariant under composition with Möbius transformations of $\mathbf{D}$ onto itself [**10**]. This result can be used to produce an alternative proof of the following corollary.

Corollary 5.7. *For $\alpha > 0$, if $F \in \mathcal{F}_\alpha$ and $F(w) = 0$ for some w in $\mathbf{D}$, then for some G in $\mathcal{F}_\alpha$ and all z in $\mathbf{D}$,*

$$F(z) = (z - w)G(z).$$

Proof. For w in $\mathbf{D}$ the range of $M_z - w$ is included in the set of functions that vanish at w, i.e., in the annihilator of k_w^α. Since that annihilator is a subspace of codimension one, the theorem implies that the range of $M_z - w$ is all of that subspace. □

Corollary 5.8. *For $\alpha > 0$, if ϕ is a multiplier of $\mathcal{F}_\alpha$ such that $\phi(w) = 0$ for some $w \in \mathbf{D}$ and $\psi(z) = \phi(z)/(z - w)$, then ψ is also a multiplier of $\mathcal{F}_\alpha$.*

Proof. If ϕ is a multiplier of $\mathcal{F}_\alpha$ that vanishes at a point w of $\mathbf{D}$ and $F \in \mathcal{F}_\alpha$, then ϕF vanishes at w. Corollary 5.7 now implies that $\psi F \in \mathcal{F}_\alpha$. Thus ψ is a multiplier of $\mathcal{F}_\alpha$. □

It follows from the corollary that if $\phi = B\psi$, where B is a finite Blaschke product, then ψ is a multiplier whenever ϕ is. Much stronger results have been obtained in the case $\alpha = 1$ in [**20**].

Theorem 5.9. *An operator on $\mathcal{F}_\alpha$ is a multiplication operator if and only if it commutes with M_z.*

Proof. Clearly, every multiplication operator commutes with M_z. Suppose that A is an operator on $\mathcal{F}_\alpha$ that commutes with M_z. Applying Theorem 5.6 and Lemma 5.5 to $M_z - w$ for $|w| < 1$, we see that $(M_z - w)^\sharp$ is surjective and has a one-dimensional null space which is invariant under $A^\sharp$. Thus, if we let K_w be the image of k_w^α in $\mathbf{A}^{\sharp\sharp}$ under the natural imbedding, then K_w is in the null space of $(M_z - w)^\sharp$ and $A^\sharp K_w = \phi(w)K_w$ for some complex number $\phi(w)$. We have

$$\begin{aligned} \phi(w) &= \langle 1, \phi(w)K_w \rangle = \langle 1, A^\sharp K_w \rangle \\ &= \langle A1, K_w \rangle = \langle k_w^\alpha, A1 \rangle_\alpha \\ &= (A1)(w). \end{aligned}$$

Consequently, $\phi \in \mathcal{F}_\alpha$. Further, for $F \in \mathcal{F}_\alpha$,

$$\begin{aligned}(AF)(w) &= \langle k_w^\alpha, AF\rangle_\alpha = \langle AF, K_w\rangle \\ &= \langle F, A^\sharp K_w\rangle = \phi(w)\langle F, K_w\rangle \\ &= \phi(w)F(w),\end{aligned}$$

and hence $A = M_\phi$. □

6. Density of polynomials in $\mathfrak{M}_\alpha$. It will be shown that $\mathfrak{M}_\alpha$ is the dual space of a Banach space and that the polynomials are weak* dense in $\mathfrak{M}_\alpha$. The set $\mathfrak{M}_\alpha$ of multipliers of $\mathcal{F}_\alpha$ may be regarded as a subspace of the set $\mathcal{B}(\mathcal{F}_\alpha, \mathcal{F}_\alpha)$ of all bounded operators on $\mathcal{F}_\alpha$. Whenever $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces, the set of bounded operators $\mathcal{B}(\mathbf{X}, \mathbf{Y}^\sharp)$ from $\mathbf{X}$ to the dual space of $\mathbf{Y}$ may be regarded as the dual space of the projective tensor product $\mathbf{X} \otimes_p \mathbf{Y}$ of $\mathbf{X}$ and $\mathbf{Y}$ (see [**17**]). Thus $\mathcal{B}(\mathcal{F}_\alpha, \mathcal{F}_\alpha)$ is isomorphic to the dual space of $\mathcal{F}_\alpha \otimes_p \mathbf{A}$, and the duality is determined by the relation

$$\langle F \otimes h, T\rangle = \langle h, TF\rangle_\alpha$$

for $h \in \mathbf{A}$, $F \in \mathcal{F}_\alpha$ and $T \in \mathcal{B}(\mathcal{F}_\alpha, \mathcal{F}_\alpha)$. On bounded subsets of $\mathcal{B}(\mathcal{F}_\alpha, \mathcal{F}_\alpha)$ the weak* topology coincides with the topology generated by the seminorms $T \mapsto |\langle h, TF\rangle_\alpha|$ for $h \in \mathbf{A}$ and $F \in \mathcal{F}_\alpha$. It will be shown that $\mathfrak{M}_\alpha$ is weak* closed. Suppose $T \in \mathcal{B}(\mathcal{F}_\alpha, \mathcal{F}_\alpha)$ is the weak* limit of a net (ϕ_n) in $\mathfrak{M}_\alpha$. Then for $m \geq 0$ and $F \in \mathcal{F}_\alpha$,

$$\begin{aligned}\langle z^m, TzF\rangle_\alpha &= \lim_n \langle z^m, \phi_n zF\rangle_\alpha \\ &= \begin{cases} 0 & \text{if } m = 0 \\ \lim_n \langle z^{m-1}, \phi_n F\rangle_\alpha & \text{if } m > 0 \end{cases} \\ &= \langle z^m, zTF\rangle_\alpha.\end{aligned}$$

Thus, T commutes with multiplication by z, and it is a multiplication operator by Theorem 5.9. Hence, $\mathfrak{M}_\alpha$ is a weak* closed subspace of $\mathcal{B}(\mathcal{F}_\alpha, \mathcal{F}_\alpha)$, and it follows that $\mathfrak{M}_\alpha$ is itself a dual space. This establishes the following theorem.

Theorem 6.1. *For each $\alpha > 0$, $\mathfrak{M}_\alpha$ is a dual space.*

Theorem 6.2. *The polynomials in the unit ball of $\mathfrak{M}_\alpha$ are weak* dense in the unit ball of $\mathfrak{M}_\alpha$.*

Proof. For $\theta \in \mathbf{R}$ the mapping $V_\theta : \mathbf{A} \to \mathbf{A}$ defined by $V_\theta h(z) = h(e^{i\theta}z)$ is an isometry on $\mathbf{A}$. Clearly, $\theta \mapsto V_\theta h$ is continuous for each h in $\mathbf{A}$. Let $U_\theta = V_\theta^\sharp$. It is easy to see that $U_\theta F(z) = F(e^{i\theta}z)$ for each F in $\mathcal{F}_\alpha$. We claim that for $\phi \in \mathfrak{M}_\alpha$, $\theta \mapsto U_\theta M_\phi U_{-\theta}$ is continuous if $\mathfrak{M}_\alpha$ is given the weak* topology. Since the function $\theta \mapsto U_\theta M_\phi U_{-\theta}$ is bounded, it suffices to check that $\theta \mapsto \langle h, U_\theta M_\phi U_{-\theta} F\rangle$ is continuous for each $h \in \mathbf{A}$ and $F \in \mathcal{F}_\alpha$. If $w = e^{i\theta}$, then, writing $h_w(z) = h(wz)$, we see

$$\langle h, U_\theta M_\phi U_{-\theta} F\rangle_\alpha = \langle h_w, M_\phi F_{\bar{w}}\rangle_\alpha,$$

and continuity follows easily.

For each measure μ on $\mathbf{T}$, define $M_{\phi*\mu}$ by

$$M_{\phi*\mu} = \int_0^{2\pi} U_\theta M_\phi U_{-\theta}\, d\mu(e^{i\theta}),$$

where the integral exists in the weak sense:

$$\langle h, M_{\phi*\mu} F\rangle_\alpha = \int_0^{2\pi} \langle h, U_\theta M_\phi U_{-\theta} F\rangle_\alpha\, d\mu(e^{i\theta}).$$

Since $U_\theta M_\phi U_{-\theta}$ is multiplication by $\phi(e^{i\theta}z)$, it commutes with M_z, and it follows easily that $M_{\phi*\mu}$ also commutes with M_z. Thus, $M_{\phi*\mu}$ is a multiplication operator. Since

$$\begin{aligned}\langle z^m, M_{\phi*\mu} 1\rangle_\alpha &= \int_0^{2\pi} \langle z^m, U_\theta \phi\rangle_\alpha\, d\mu(e^{i\theta})\\ &= \int_0^{2\pi} \langle e^{im\theta} z^m, \phi\rangle_\alpha\, d\mu(e^{i\theta})\\ &= \hat{\phi}(m)\mu_m,\end{aligned}$$

it follows that it is multiplication by

$$\phi * \mu(z) = \sum_{n=0}^{\infty} \hat{\phi}(n)\mu_n z^n.$$

Also, from the defining relation for $M_{\phi*\mu}$,

$$\begin{aligned}|\langle h, M_{\phi*\mu} F\rangle_\alpha| &\le \int_0^{2\pi} |\langle h, U_\theta M_\phi U_{-\theta} F\rangle_\alpha|\, d|\mu|(e^{i\theta})\\ &\le ||h||\, ||M_\phi||_\alpha ||F||_\alpha ||\mu||,\end{aligned}$$

and thus

$$||M_{\phi*\mu}||_\alpha \le ||M_\phi||_\alpha ||\mu||.$$

If $d\mu = (1/2\pi)K_n dm$, where K_n is the Fejer kernel, then, since

$$||\mu|| = ||K_n||_{L^1(m)} = 1,$$

it follows from the preceding inequality that

$$||\sigma_n(\phi)|| \le ||M_\phi||_\alpha,$$

where $\sigma_n(\phi)$ is the nth Cesaro mean of ϕ. Thus, the sequence $\{\sigma_n(\phi)\}$ is bounded, and Alaoglu's theorem implies it has a weak* convergent subnet. But examination of Fourier coefficients shows that every convergent subnet of $\{\sigma_n(\phi)\}$ has the same limit ϕ, and thus $\{\sigma_n(\phi)\}$ converges to ϕ. □

For $\phi \in \mathfrak{M}_\alpha$ for $\alpha > 0$ and for $\beta > 1$, define polynomials $s_n(\phi;\beta)$ on **T** by

$$s_n(\phi;\beta)(z) = \sum_{k=0}^{n} \frac{A_{n-k}(\beta)}{A_n(\beta)} \hat{\phi}(k) z^k.$$

These polynomials were first studied in [**11**]. Note that $s_n(\phi;1) = s_n(\phi)$ and $s_n(\phi;2) = \sigma_n(\phi)$. With $\gamma > 0$, let K_n^γ be the quasipositive kernel defined in [**21**, p. 94] by

$$K_n^\gamma = \sum_{k=0}^{n} \frac{A_{n-k}(\gamma)}{A_n(\gamma+1)} D_k,$$

where D_k is the Dirichlet kernel. For $\beta > 1$, let $d\mu = K_n^{\beta-1}\, dm$. Then we have $s_n(\phi;\beta) = \mu * \phi$. This follows from a short calculation based on the relation

$$\sum_{k=0}^{m} A_k(\beta - 1) = A_m(\beta)$$

together with

$$\mu * \phi = \sum_{k=0}^{n} \frac{A_{n-k}(\beta-1)}{A_n(\beta)} s_k(\phi).$$

Since these kernels are uniformly bounded in L^1 norm for each $\beta > 1$, the polynomials $s_n(\phi;\beta)$ are uniformly bounded in $\mathfrak{M}_\alpha$. Their Fourier coefficients converge to those of ϕ, and thus it follows that they converge weak* in $\mathfrak{M}_\alpha$ to ϕ. We have proved the following corollary.

Corollary 6.3. *For every $\alpha > 0$ and $\beta \geq 1$ and every $\phi \in \mathfrak{M}_\alpha$, the polynomials $s_n(\phi;\beta)$ converge to ϕ in the weak* topology.*

Corollary 6.4. *For every $\alpha > 0$ and $\beta \geq 1$ and every $\phi \in \mathfrak{M}_\alpha$, the polynomials $s_n(\phi;\beta)$ converge uniformly on compact subsets of* **D** *to ϕ.*

Proof. This follows from

$$\begin{aligned}\phi(z) &= \langle k_z^\alpha, \phi\rangle_\alpha \\ &= \lim_{n\to\infty}\langle k_z^\alpha, s_n(\phi;\beta)\rangle_\alpha \\ &= \lim_{n\to\infty} s_n(\phi;\beta)(z)\end{aligned}$$

together with the uniform boundedness of the $s_n(\phi;\beta)$. □

7. Properties of $\mathfrak{M}_\alpha$. In this section various properties of the families $\mathfrak{M}_\alpha$ are discussed. In [**19**] Vinogradov proved that if $\phi \in \mathfrak{M}_1$, then the radial limit $\lim_{r\to 1-}\phi(re^{i\theta})$ exists for all θ. The same result is true for $\phi \in \mathfrak{M}_\alpha$, $\alpha > 0$, and the proof proceeds just as in [**19**]. From this fact it follows that there are functions in H^∞ that are not in $\mathfrak{M}_\alpha$ for any $\alpha > 0$.

In the case of $\alpha < 1$, a stronger assertion can be made. Let $\phi(z) = \sum_{k=0}^\infty c_k z^k$, where $c_k = 1/n^2$ if $k = 2^n$, $n = 1, 2, \ldots$ and $c_k = 0$ for all other k. Since $\sum_{n=0}^\infty 1/n^2 < \infty$, $\phi \in H^\infty$. Since $H^\infty \subset \mathcal{F}_1$, Theorem 3.5 implies that $\phi(z)/(1-z)^\alpha$ belongs to $\mathcal{F}_{\alpha+1}$. We will show that $\phi(z)/(1-z)^\alpha$ does not belong to any $\mathcal{F}_\beta$ with $\beta < 1$. Thus, even though ϕ is continuous on the closed disc, it does not multiply $\mathcal{F}_\alpha$ into itself, nor even into any $\mathcal{F}_\beta$ for $\beta < 1$. A calculation yields

$$\frac{\phi(z)}{(1-z)^\alpha} = \sum_{m=0}^\infty\left(\sum_{k=0}^m A_{m-k}(\alpha)c_k\right)z^m.$$

In particular, for $m = 2^N$, the coefficient of z^m in $\phi(z)/(1-z)^\alpha$ is

$$\begin{aligned}\sum_{k=0}^{m} A_{m-k}(\alpha)c_k &= \sum_{n=1}^{N} A_{m-2^n}(\alpha)\frac{1}{n^2}\\ &\geq \frac{1}{N^2}.\end{aligned}$$

Because of the necessary condition for membership in $\mathcal{F}_\beta$, to show that $\phi(z)/(1-z)^\alpha$ is not in $\mathcal{F}_\beta$, it is enough to show that $b_m \neq O(m^{\beta-1})$, where b_m is the mth Taylor coefficient of $\phi(z)/(1-z)^\alpha$. But the above estimate of b_m now yields

$$\begin{aligned}\frac{b_m}{m^{\beta-1}} \geq \frac{1/N^2}{m^{\beta-1}} &= \frac{m^{1-\beta}}{N^2}\\ &= (\log 2)^2\frac{m^{1-\beta}}{(\log m)^2},\end{aligned}$$

which is unbounded. Therefore, $\phi(z)/(1-z)^\alpha$ is not in $\mathcal{F}_\beta$.

The inclusion of the following theorem is not new (see [**11**]); that it is contractive is.

Theorem 7.1. *If $0 < \alpha < \beta$, then $\mathfrak{M}_\alpha \subset \mathfrak{M}_\beta$, and the injection from $\mathfrak{M}_\alpha$ into $\mathfrak{M}_\beta$ is a contraction.*

Proof. Let ϕ belong to $\mathfrak{M}_\alpha$. By Theorem 5.4, $\{\phi(z)/(1-\zeta z)^\alpha : |\zeta| = 1\}$ is a subset of the zero centered ball in $\mathcal{F}_\alpha$ of radius $||M_\phi||_\alpha$. Since the family $\{1/(1-\zeta z)^{\beta-\alpha} : |\zeta| = 1\}$ lies in the unit ball of $\mathcal{F}_{\beta-\alpha}$, it follows from Theorem 3.5 that $\{\phi(z)/(1-\zeta z)^\beta : |\zeta| = 1\}$ lies in the ball of radius $||M_\phi||_\alpha$ in $\mathcal{F}_\beta$. Another application of Theorem 5.4 shows that $\phi \in \mathfrak{M}_\beta$ and $||M_\phi||_\beta \leq ||M_\phi||_\alpha$. □

We conclude this section with the observation that the multipliers of $\mathcal{F}_{\alpha a}$ coincide with those of $\mathcal{F}_\alpha$ at least for $\alpha \geq 1$. This was stated in [**12**] for the case $\alpha = 1$.

Theorem 7.2. *The subspace $\mathcal{F}_{\alpha a}$ is invariant under every multiplier of $\mathcal{F}_\alpha$ for $\alpha \geq 1$. Every multiplier of $\mathcal{F}_{\alpha a}$ is a multiplier of $\mathcal{F}_\alpha$ for $\alpha > 0$.*

Proof. Suppose $\alpha \geq 1$ and $\phi \in \mathfrak{M}_\alpha$. Then $\phi \in H^\infty$, and consequently $\phi \in \mathcal{F}_{1a}$. By Theorem 3.4, $\phi \in \mathcal{F}_{\alpha a}$. It follows that $\phi z^n \in \mathcal{F}_{\alpha a}$ for every $n \geq 0$. Since every absolutely continuous measure is the limit in norm of its sequence of Cesaro means, it follows that every element of $\mathcal{F}_{\alpha a}$ is a norm limit of polynomials. Thus, if $f \in \mathcal{F}_{\alpha a}$ and polynomials p_n converge to ϕ in norm, then $\phi p_n \in \mathcal{F}_{\alpha a}$ and $\{\phi p_n\}$ converges to ϕf. Hence $\phi f \in \mathcal{F}_{\alpha a}$.

Conversely, suppose ϕ is a multiplier of $\mathcal{F}_{\alpha a}$, and let f be any member of $\mathcal{F}_\alpha$. By the closed graph theorem, multiplication by ϕ is a bounded operator on $\mathcal{F}_{\alpha a}$. By the observation at the end of the first section, f is a weak* limit of a bounded sequence of polynomials p_n. Thus the sequence $\{\phi p_n\}$ is bounded, and by Alaoglu's theorem, it contains a weak* convergent subsequence. We suppose $\{\phi p_n\}$ itself converges and g is its limit. For arbitrary w in $\mathbf{D}$,

$$\begin{aligned} g(w) &= \lim_{n\to\infty} \langle k_w^\alpha, \phi p_n \rangle_\alpha \\ &= \lim_{n\to\infty} \phi(w) p_n(w) = \phi(w) f(w). \end{aligned}$$

Thus, $\phi f \in \mathcal{F}_\alpha$ and ϕ is a multiplier of $\mathcal{F}_\alpha$. □

8. Coefficient conditions. Let ϕ be holomorphic in the unit disc, and suppose that $\phi(z) = \sum_{n=0}^\infty a_n z^n$. Vinogradov [**19**] proved that if $\sum_{n=0}^\infty |a_n| \log(n+2) < \infty$, then $\phi \in \mathfrak{M}_1$. It is of interest to find similar conditions on the coefficients of ϕ that imply that $\phi \in \mathfrak{M}_\alpha$ for $\alpha > 0$. Such conditions might suggest whether the inclusion in Theorem 7.1 is proper. We begin by considering the case $0 < \alpha < 1$.

Theorem 8.1. *Let $\phi(z) = \sum_{n=0}^\infty a_n z^n$ for $|z| < 1$, and suppose that $0 < \alpha < 1$. If*

$$\sum_{n=1}^\infty |a_n| (n+1)^{1-\alpha} \log(n+1) < \infty,$$

then $\phi \in \mathfrak{M}_\alpha$, and in this case

$$||M_\phi||_\alpha \leq c\left(|a_0| + \sum_{n=1}^\infty |a_n| (n+1)^{1-\alpha} \log(n+1) \right)$$

for some constant c independent of ϕ*.*

Proof. We first show that $\phi \in \mathcal{F}_\alpha$. Let $\phi_1(z) = \sum_{n=0}^{\infty}(a_n/A_n(\alpha))z^n$. It follows from Section 2 that $\phi \in \mathcal{F}_\alpha$ if and only if $\phi_1 \in \mathcal{F}_1$. Since $1/A_n(\alpha) = O(n^{1-\alpha})$, the given coefficient condition implies that $\phi_1 \in H^\infty$. Since $H^\infty \subset \mathcal{F}_1$, it follows that $\phi \in \mathcal{F}_\alpha$. By Lemma 4.4,

$$\sum_{n=0}^{\infty} |a_n|\,||S_\alpha^n|| \le c\bigg(|a_0| + \sum_{n=1}^{\infty} |a_n|(n+1)^{1-\alpha}\log(n+1)\bigg) < \infty,$$

and hence the series $\sum_{n=0}^{\infty} a_n S_\alpha^n$ converges to the bounded operator $\phi(S_\alpha)$ on $\mathbf{A}$. By Theorem 5.2, M_ϕ is bounded, and the norm estimate follows. □

Corollary 8.2. *If* $0 < \alpha < \beta < 1$*, then there is a function that belongs to* $\mathfrak{M}_\beta$ *but not* $\mathfrak{M}_\alpha$*.*

Proof. Suppose that $0 < \alpha < \beta < 1$, and let $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$, where

$$a_n = \begin{cases} n^{\beta-1}/(\log n)^3 & \text{if } n = 2^k \text{ for } k = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=1}^{\infty} |a_n|(n+1)^{1-\beta}\log(n+1) \le \sum_{k=1}^{\infty} \frac{2^{k(\beta-1)}}{(k\log 2)^3}(2^k+1)^{1-\beta}\log(2^k+1).$$

This series converges by comparison with $\sum_{k=1}^{\infty} 1/k^2$, and thus $\phi \in \mathfrak{M}_\beta$.

To obtain $\phi \notin \mathfrak{M}_\alpha$, we show that $\phi \notin \mathcal{F}_\alpha$. As in Section 2, a necessary condition for ϕ to belong to $\mathcal{F}_\alpha$ is that $a_n = O(n^{\alpha-1})$. For $n = 2^k$, we have

$$\frac{a_n}{n^{\alpha-1}} = \frac{n^{1-\alpha}n^{\beta-1}}{(\log n)^3} = \frac{n^{\beta-\alpha}}{(\log n)^3}.$$

Since the last quantity is unbounded, it follows that ϕ is not in $\mathcal{F}_\alpha$ and therefore not in $\mathfrak{M}_\alpha$. □

Let $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. An argument similar to the proof of Theorem 8.1, together with the result of Lemma 4.4 that

$||S_\alpha^n|| \le c\log(n+1)$ for $\alpha \ge 1$, shows that if $\sum_{n=1}^\infty |a_n|\log(n+1) < \infty$, then $\phi \in \mathfrak{M}_\alpha$. This is also a consequence of Theorem 7.1 and Vinogradov's coefficient condition for membership in $\mathfrak{M}_1$. A stronger result is needed to distinguish the families $\mathfrak{M}_\alpha$ for $\alpha \ge 1$. The next theorem deals with the case $\alpha = 2$. Since this paper was written, Hallenbeck, MacGregor and Samotij [**7**] have obtained the stronger result that, under the condition of the theorem, $\phi \in \mathfrak{M}_\alpha$ for all $\alpha > 1$.

Theorem 8.3. *Let $\phi(z) = \sum_{n=0}^\infty a_n z^n$ for $|z| < 1$. If $\sum_{n=0}^\infty |a_n| < \infty$, then $\phi \in \mathfrak{M}_2$, and in this case $||M_\phi||_\alpha \le c\sum_{n=0}^\infty |a_n|$ for some constant c independent of ϕ.*

Proof. As in the proof of Theorem 8.1, we show that $\sum_{n=0}^\infty |a_n|\,||S_2^n|| < \infty$, and for this it clearly suffices to show that $\{||S_2^n||\}$ is uniformly bounded. For a function $h(z) = \sum_{n=0}^\infty \hat{h}(n)z^n$ in $\mathbf{A}$,

$$z^n S_2^n h(z) = \sum_{k=n}^\infty \frac{A_{k-n}(2)}{A_k(2)}\hat{h}(k)z^k = \sum_{k=n}^\infty \frac{k-n+1}{k+1}\hat{h}(k)z^k.$$

Let

$$\psi_n(e^{i\theta}) = \sum_{k=-\infty}^\infty c_k e^{ik\theta},$$

where

$$c_{-k} = c_k = \begin{cases} 1 & \text{if } 0 \le k < n \\ n/(k+1) & \text{if } k \ge n. \end{cases}$$

Except for finitely many terms, the sequence $\{c_k\}_{k=0}^\infty$ is nonnegative, monotonically decreasing and convex. It also converges to zero. Therefore, by Theorem (1.5) of Chapter V in [**21**], ψ_n is in $L^1(m)$. It will be shown later that $\{||\psi_n||_1\}$ is bounded. Here, and for the remainder of this proof, $||\cdot||_1$ means the norm in $L^1(m)$. Note that

$$z^n S_2^n h(z) = h(z) - (\psi_n * h)(z),$$

and thus

$$||S_2^n h|| = ||z^n S_2^n h|| \le (1 + ||\psi_n||_1)||h||.$$

It follows that $||S_2^n|| \le 1 + ||\psi_n||_1$, which implies the result.

It remains to estimate $||\psi_n||_1$. Observe that, for $k \ge n$, the Fourier coefficients of ψ_n form a nonnegative, monotonically decreasing, convex sequence that converges to zero. Therefore, as in the proof of Theorem (1.5) on page 183 of [**21**], one can employ summation by parts twice to obtain

$$\begin{aligned}\psi_n = &-\frac{n-1}{n+1}K_{n-2} + \frac{2n}{(n+1)(n+2)}K_{n-1}\\ &+\sum_{k=n}^{\infty}\frac{2n}{(k+2)(k+3)}K_k,\end{aligned}$$

where K_n is the Fejer kernel. The L^1 norms of the three terms on the right can be estimated separately. The first two are obviously bounded. As for the third, since all terms are positive, term by term integration yields

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{2n}{(k+2)(k+3)} &= 2n\sum_{k=n}^{\infty}\left(\frac{1}{k+2}-\frac{1}{k+3}\right)\\ &= \frac{2n}{n+2},\end{aligned}$$

and the proof is complete. □

Corollary 8.4. *There is a function that belongs to* $\mathfrak{M}_2$ *but not* $\mathfrak{M}_1$.

Proof. It follows from the theorem that every function with absolutely summable Taylor coefficients is in $\mathfrak{M}_2$. But Corollary 1 on page 16 of [**19**] implies that there exists such a function not in $\mathfrak{M}_1$. Here is a short construction of one such function. Choose an increasing sequence $\{N_k\}$ such that $N_{k+1} > 2N_k$ and $\sum_{n=1}^{N_k} 1/n \ge k^3$. Let a_n be $1/k^2$ if $n = N_k$ and 0 otherwise. If $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$, then $\phi \in \mathfrak{M}_2$. We claim that $f(z) = \phi(z)/(1-z) = \sum_{n=0}^{\infty} c_n z^n$ is not in $\mathcal{F}_1$, and consequently, ϕ is not in $\mathfrak{M}_1$. Note that $c_n = 0$ for $n < N_1$ and $c_n = \sum_{j=1}^{k} 1/j^2$ if $N_k \le n < N_{k+1}$. Thus $\{c_n\}$ is a slowly increasing sequence with a jump of $1/k^2$ at N_k for each k. The polynomials $p_k(z) = \sum_{j=1}^{N_k}(1/j)(z^{N_k+j} - z^{N_k-j})$ are uniformly bounded (see [**21**, p. 182]). If f were in $\mathcal{F}_1$, then the sequence $\{\langle p_k, f\rangle_1\}$ would be bounded.

But,

$$\begin{aligned}\langle p_k, f\rangle_1 &= \sum_{j=1}^{N_k} \frac{1}{j}\langle z^{N_k+j} - z^{N_k-j}, f\rangle_1\\ &\geq \sum_{j=1}^{N_k} \frac{1}{j}\frac{1}{k^2}\\ &\geq k.\end{aligned}$$

This completes the proof. □

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