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# ON CONVERGENCE OF CONDITIONAL EXPECTATION OPERATORS

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ABSTRACT. Given an operator $T : U_X(\Sigma) \to Y$ or $T : C(H, X) \to Y$, one may consider the net of conditional expectation operators $(T_\pi)$ directed by refinement of the partitions $\pi$. It has been shown previously that $(T_\pi)$ does not always converge to $T$. This paper gives several conditions under which this convergence does occur, including complete characterizations when $X = \mathbf{R}$ or when $X^*$ has the Radon-Nikodým property.

**1. Introduction.** It is well known that if $T : U_X(\Sigma) \to Y$ is a bounded linear operator, where $U_X(\Sigma)$ is the uniform closure of the $X$-valued $\Sigma$-simple functions, then there is a unique finitely additive set function $m : \Sigma \to L(X, Y)$ with finite semi-variation such that $T(f) = \int f\, dm$ for all $f \in U_X(\Sigma)$. Also, if $T : C(H, X) \to Y$, there is a unique weakly regular $m : \beta(H) \to L(X, Y^{**})$ such that $T(f) = \int f\, dm$ for $f \in C(H, X)$. In each case $\tilde{m}(H) = ||T||$. Given such an operator $T$, a finite partition $\pi$ of $H$, and a measure $\mu$ on $H$, a conditional expectation operator $T_\pi$ can be defined. It was shown in [**1**] that the net $(T_\pi)$ directed by refinement does not always converge to $T$ in the operator norm. Conditions under which this convergence does occur are discussed herein.

Throughout, $X$ and $Y$ are Banach spaces. The closed unit ball of $X$ is denoted by $B_X$. We will use $H$ for a compact Hausdorff space and $C(H, X)$ for the space of continuous functions from $H$ to $X$. An arbitrary $\sigma$-algebra of subsets of some universal space $\Omega$ will be represented by $\Sigma$, and when $\Omega = H$, we will use $\Sigma = \beta(H)$, the Borel sets of $H$, without further mention. An additive set function $m : \Sigma \to X$ will be called a vector measure, while by a measure we mean a countably additive set function $\mu : \Sigma \to [0, \infty)$.

For a vector measure $m : \Sigma \to X$, we define the variation of $m$ as usual and the scalar semi-variation $||m||$ of $m$ as in [**8**]. If $m : \Sigma \to L(X, Y)$,

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the semi-variation $\tilde{m}$ of $m$ is as in [**10**] (recall that if $X = \mathbf{R}$, then $\tilde{m}(A) = ||m||(A)$, and if $Y = \mathbf{R}$, then $\tilde{m}(A) = |m|(A)$). We write $m \leftrightarrow T$ to mean $m$ corresponds to $T$ as in the two theorems stated at the beginning of this section.

If $m : \Sigma \to X$ is a vector measure, $\mu$ a positive, bounded, finitely additive measure on $\Sigma$, and $\pi$ a partition of $\Omega$, then the conditional expectation $m_\pi$ of $m$ by $\pi$ and $\mu$ is given by

$$m_\pi(B) = \sum_{A\in\pi} \frac{\mu(A\cap B)}{\mu(A)} m(A), \quad \text{observing } \frac{0}{0} = 0.$$

If $m \leftrightarrow T : U_X(\Sigma) \to Y$ or $m \leftrightarrow T : C(H,X) \to Y$ is a bounded linear operator, $\pi$ is a partition, and $\mu$ is a measure on $\Sigma$ $(\beta(H))$, then the conditional expectation operator $T_\pi$ is given by

$$T_\pi(f) = \sum_{A\in\pi} m(A)\left(\frac{\int_A f\,d\mu}{\mu(A)}\right), \quad \text{again observing } \frac{0}{0} = 0.$$

If $m \leftrightarrow T : U_X(\Sigma) \to Y$, then $m_\pi \leftrightarrow T_\pi$ for each partition $\pi$. Also, if $m \leftrightarrow T : C(H,X) \to Y$ and $\mu$ is a regular measure on $\beta(H)$, then $m_\pi \leftrightarrow T_\pi$; see [**5**, Lemma 2.2]. We reserve $T_\pi \to T$ to mean convergence in operator norm. For other terminology not defined here, see [**8**] or [**4**].

**2. The results.** We now turn our attention to convergence of $T_\pi$ to $T$ in operator norm. Lemma 1 gives a necessary condition for that convergence.

**Lemma 1.** *Let $m \leftrightarrow T : C(H,X) \to Y$, respectively, $m \leftrightarrow T : U_X(\Sigma) \to Y$, and let $\mu$ be a regular measure, respectively a measure, on $\beta(H)$, respectively, $(\Sigma)$. If $T_\pi \to T$, then $m$ has relatively compact range.*

*Proof.* We have $m_\pi \to m$ in semi-variation norm since $T_\pi \leftrightarrow m_\pi$, using the regularity of $\mu$ in the $C(H,X)$ case. Hence, $m_\pi(A) \to m(A)$ uniformly in $A$. As the range of each $m_\pi$ is bounded and finite dimensional, we must have that the range of $m$ is totally bounded. $\square$

The converse is, in general, not true. A proof appears later. In the examples of [**1**], the measures (with possibly one exception) do not have relatively norm compact ranges. This lemma leads to the following when $X = \mathbf{R}$:

**Theorem 2.** *Let* $m \leftrightarrow T : C(H) \to Y$ *and suppose that* $m \ll \mu$. *The following are equivalent*:

(1) $m_\pi \to m$ *in semi-variation norm.*

(2) $m$ *has relatively compact range.*

(3) $T$ *is compact.*

*If, in addition,* $\mu$ *is regular, then* (1)–(3) *are equivalent to*

(4) $T_\pi \to T$.

*Proof.* The fact that (2) implies (1) in this setting follows from [**3**, Remark 5.2]. By Lemma 1, (1) and (2) are equivalent. The equivalence of (2) and (3) is in [**11**, p. 496]. The proposition follows. □

The conditions (1)–(4) are equivalent in the setting $m \leftrightarrow T : U(\Sigma) \to Y$. This theorem is applied frequently in the remainder of the paper.

Conditions under which $T_\pi \to T$ when $T$ takes its values in $\mathbf{R}$ are given after the following lemma. A martingale convergence theorem similar to the lemma can be found in [**7**].

**Lemma 3.** *The following are equivalent*:

(1) $X$ *has the Radon-Nikodým property* (*RNP*).

(2) *If* $(\Omega, \Sigma, \mu)$ *is a finite measure space,* $m : \Sigma \to X$ *is of bounded variation and* $m \ll \mu$, *then* $m_\pi \to m$ *in variation norm.*

*Proof.* That (1) implies (2) is seen from [**2**, Theorem 1] (and its proof) applied to the singleton $\{m\}$. Now suppose (2) holds and fix the finite measure space $(\Omega, \Sigma, \mu)$, and suppose $m : \Sigma \to X$ is of bounded variation and $m \ll \mu$. For each partition $\pi$, define $f_\pi : \Omega \to X$ by

$$f_\pi = \sum_{A \in \pi} \frac{m(A)}{\mu(A)} \chi_A.$$

Then $m_\pi(B) = \int_B f_\pi \, d\mu$ for all $B$, and $|m_\pi|(\Omega) = ||f_\pi||_1$ for each $\pi$. However, $(m_\pi)$ converges in variation norm by hypothesis; consequently, $(f_\pi)$ is Cauchy and converges to some $f \in L^1(\mu, X)$. Then for each $B \in \Sigma$, $\int_B f_\pi \, d\mu \to m(B)$, but also $\int_B f_\pi \, d\mu \to \int_B f \, d\mu$. Hence, $m(B) = \int_B f \, d\mu$ for all $B \in \Sigma$. □

Note that Lemma 3 provides a converse of [**2**, Theorem 1]. Consider applying the above theorem when $m \leftrightarrow T : C(H, X) \to Y$ or $m \leftrightarrow T : U_X(\Sigma) \to Y$; conditions under which $m_\pi \to m$ in variation norm are easily extracted. Also, if $Y = \mathbf{R}$ we have $||T|| = \tilde{m}(\Omega) = |m|(\Omega)$, which proves the following theorem.

**Theorem 4.** *The following are equivalent.*

(1) $X^*$ *has* RNP.

(2) $T_\pi \to T$ *whenever* $m \leftrightarrow T : C(H, X) \to \mathbf{R}$ *and* $m \ll \mu$.

(3) $T_\pi \to T$ *whenever* $m \leftrightarrow T : U_X(\Sigma) \to \mathbf{R}$ *and* $m \ll \mu$.

We shall now explore the general question for spaces whose duals possess RNP.

**Definition.** Let $m \leftrightarrow T : U_X(\Sigma) \to Y$ or $m \leftrightarrow T : C(H, X) \to Y$. For $y^* \in Y^*$, define $m_{y^*} : \Sigma \to X^*$ by $m_{y^*}(A)(x) = \langle y^*, m(A)(x) \rangle$.

Note that $|m_{y^*}|(\Omega) = \widetilde{m_{y^*}}(\Omega) \le ||y^*|| \tilde{m}(\Omega)$.

**Theorem 5.** *The following are equivalent.*

(1) $X^*$ *has* RNP.

(2) $T_\pi \to T$ *whenever* $m \leftrightarrow T : U_X(\Sigma) \to Y$, *where* $T$ *is compact and* $\{|m_{y^*}| : y^* \in B_{Y^*}\}$ *is uniformly absolutely continuous with respect to a positive, bounded, finitely additive* $\mu$.

(3) $T_\pi \to T$ *whenever* $m \leftrightarrow T : C(H, X) \to Y$ *where* $T$ *is compact,* $\mu$ *is as in* (2), *and* $\mu$ *is regular.*

*Proof.* Suppose (1) holds. Let $X, m$ and $T$ be as stated, and consider $T^* : Y^* \to U_X(\Sigma)^*$. Now $U_X(\Sigma)^* = \{\mu \mid \mu : \Sigma \to X^*$ has finite variation$\}$. Also, $T^*(y^*) \leftrightarrow m_{y^*}$. Since $T^*$ is compact, $\{m_{y^*} : y^* \in B_{Y^*}\}$ is relatively compact in variation norm. Let $\mu$ be as stated (existence is guaranteed by [**2**]). Applying [**2**] again, $(m_{y^*})_\pi \to m_{y^*}$ uniformly in $y^* \in B_{Y^*}$, in variation norm. Hence, $T^*(y^*)_\pi \to T^*(y^*)$ uniformly in $y^* \in B_{Y^*}$. However, for $f \in U_X(\Sigma)$, we have $T^*_\pi(y^*)(f) = T^*(y^*)_\pi(f)$. Thus, $T^*_\pi(y^*) \to T^*(y^*)$ uniformly in $y^* \in B_{Y^*}$, i.e., $T^*_\pi \to T^*$. Therefore, (2) holds.

For (2) implies (3), consider $\hat{T} : U_X(\beta(H)) \to Y$. As $T$ is compact, $m$ takes its values in $L(X,Y)$ and $\hat{T}$ is also compact. By hypothesis, $\widehat{T_\pi} \to \hat{T}$. As $T_\pi = \widehat{T_\pi}|_{C(H,X)}$ and $T = \hat{T}|_{C(H,X)}$, we have $T_\pi \to T$.

It remains to show that (3) implies (1). Let $m \leftrightarrow T : C([0,1],X) \to \mathbf{R}$ such that $m \ll \lambda =$ Lebesgue measure. Let $y \in Y$ with $||y|| = 1$ and define $T' : C([0,1],X) \to Y$ by $T'(f) = T(f)y$. Define $m' : \Sigma \to L(X,Y)$ by $m'(A)(x) = m(A)(x)y$, for all $A \in \Sigma$, $x \in X$. Then $m' \leftrightarrow T'$ and $|m'| \ll \lambda$. If $||y^*|| \leq 1$, then $|m'_{y^*}|(A) \leq |m'|(A)$, and consequently $\{|m_{y^*}| : y^* \in B_{Y^*}\}$ is uniformly absolutely continuous with respect to $\lambda$. Hence, by hypothesis, $T'_\pi \to T'$. But, it can readily be seen that $T'_\pi(f) = T_\pi(f)y$; thus $T_\pi \to T$. By Theorem 4, $X^*$ has RNP. □

For a related theorem, see [**6**, Theorem 2.3]. Recall now the following definition:

**Definition.** Suppose $m \leftrightarrow T : C(H,X) \to Y$. Define $T^\sharp : C(H) \to L(X,Y)$ by $T^\sharp(f)(x) = T(xf)$ for $f \in C(H)$. Define $m^\sharp : \Sigma \to L(\mathbf{R}, L(X,Y))$ by $m^\sharp(A)(r) = rm(A)$ for $r \in \mathbf{R}$, $A \in \Sigma$.

Note that if $m \leftrightarrow T : C(H,X) \to Y$ and $m$ is strongly bounded, then $m^\sharp \leftrightarrow T^\sharp$.

**Corollary 6.** *Suppose that $m \leftrightarrow T : C(H,X) \to Y$, where $X^*$ has* RNP. *If $T$ is compact, then so is $T^\sharp$.*

*Proof.* Since $T$ is compact, it is strongly bounded (see [**4**, Theorem 4.2]). Then there is a regular measure $\mu$ on $\Sigma$ such that $\tilde{m} \ll \mu$. By

Theorem 5, $T_\pi \to T$. Consequently, $m$ (and hence $m^\sharp$) has relatively compact range by Lemma 1. Finally, an application of Theorem 2 yields the compactness of $T^\sharp$. $\square$

In [**13**], Saab and Smith showed that $T$ nuclear implies $T^\sharp$ nuclear if and only if $X^*$ has RNP. It is not known if the converse of Corollary 6 is true. However, we do have the following example:

**Example 7.** $T$ is compact need not imply $T^\sharp$ is compact.

*Demonstration.* Let $m : \beta([0,1]) \to L(C([0,1],\mathbf{R}) = C([0,1])^*$ be given by $m(A)(f) = \int_A f\,d\lambda$, where $\lambda$ is Lebesgue measure. Then $||m(A)|| \leq \lambda(A)$, and therefore $m$ is a dominated representing measure (see [**4**, Theorem 2.8]). Let $T : C([0,1], C([0,1])) \to \mathbf{R}$ be given by $T(f) = \int f\,dm$. Then $T$ is certainly compact; however, the range of $m$ is not relatively compact. To see this, consider the sets $A_n = \cup_{i=1}^{2^{n-1}} C_{n,2i-1}$ where the $C_{n,i}$ are the dyadic intervals. Let $r_n$ be the $n$th Rademacher function. Then for $k < n$, we have $||m(A_n) - m(A_k)|| \geq 1/2$. By Theorem 2, $T^\sharp$ is not compact.

The next proposition illustrates another use of Theorem 2. It is a special case of [**12**, Proposition 3].

**Proposition 8.** *Suppose $L(X,Y)$ has the weak Radon-Nikodým property* (wRNP) *and $m \leftrightarrow T : C(H,X) \to Y$ where $m$ is of bounded variation. Let $f$ be the Pettis-integrable Radon-Nikodým derivative of $m$, and let $f_\pi$ be as in Lemma 3, where $\mu = |m|$. Then $f_\pi \to f$ in Pettis norm.*

*Proof.* By the observation of Stegall on the range of an indefinite Pettis integral (see, e.g., [**9**]) and Theorem 2, $m_\pi^\sharp \to m^\sharp$ in semi-variation = scalar semi-variation norm. Therefore, $m_\pi \to m$ in scalar semi-variation norm. Thus, $|x^* m_\pi - x^* m|(\Omega) \to 0$, uniformly in $x^* \in B_{L(X,Y)^*}$. However, we have that $x^* f_\pi$ is the Radon-Nikodým derivative of $x^* m_\pi$ and $x^* f$ is the Radon-Nikodým derivative of $x^* m$.

Thus,

$$\int_\Omega |x^* f_\pi - x^* f| \, dm = |x^* m_\pi - x^* m|(\Omega),$$

and $x^* f_\pi \to x^* f$ uniformly over $x^* \in B_{L(X,Y)^*}$. Therefore, $f_\pi \to f$ in Pettis norm. □

We now show that the converse of Lemma 1 is, in general, false.

**Proposition 9.** *If* $m \leftrightarrow T : C(H,X) \to \mathbf{R}$, *then* $T_\pi^\sharp \to T^\sharp$ *need not imply that* $T_\pi \to T$.

*Proof.* Let $X$ be a Banach space such that $X^*$ has wRNP but not RNP (for example, $X$ can be the James tree space; see [**14**, p. 87] and [**8**, p. 214]). Let $H = [0,1]$. Again, by Stegall's observation, $T^\sharp$ is compact, for all $T : C(H,X) \to \mathbf{R}$. Let $\mu$ be Lebesgue measure, and let $m \ll \mu$ where $m \leftrightarrow T : C(H,X) \to \mathbf{R}$. We then have $T_\pi^\sharp \to T^\sharp$. If this implied that $T_\pi \to T$, then by Theorem 4, we would have that $X^*$ has RNP, a contradiction. □

Now consider the first three statements of Theorem 2 in the setting $m \leftrightarrow T : C(H,X) \to Y$ with $m \ll \mu$, $\mu$ regular. The only implication that remains true in this setting is (1) implies (2), provided by Lemma 1. The proof of Proposition 9 above shows that (2) and (3) together need not imply (1), and the fact that (3) does not imply (2) was treated in Example 7. To see that (1) and (2) do not imply (3), let $H$ be a singleton and let $X$ and $Y$ be such that there is a noncompact operator $f : X \to Y$. Let $m(H) = f$ and $m(\phi) = 0$. Let $T \leftrightarrow m$. Then $C(H,X) \cong X$, and $T : C(H,X) \to Y$ is actually $f : X \to Y$. Hence, $T$ is not compact. However, $m_\pi = m$ for the unique partition $\pi$ of $H$ and $m$ certainly has relatively compact range.

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