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L^p MATRIX COEFFICIENTS FOR NILPOTENT LIE GROUPS

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ABSTRACT. We show that if G is a connected, simply connected nilpotent Lie group, then there is a fixed number p such that if π is any irreducible unitary representation of G, then some (equivalently, a dense set of) matrix coefficients are L^p functions on G mod the kernel of π.

1. Introduction. In this paper we undertake a study of one aspect of the asymptotic behavior of the matrix coefficients of irreducible unitary representations of Lie groups. The behavior at infinity of these matrix coefficients often gives important information about the structure of the irreducible representations themselves and about harmonic analysis in general. For example, detailed asymptotic estimates on the behavior of matrix coefficients of irreducible representations of semi-simple groups (real and p-adic) have played a central role in the work of Harish-Chandra, Langlands, and others. (Examples of fairly recent results for these groups are given in [**1, 3**] and [**11**].) A related example where the asymptotics of matrix coefficients plays a key role is the Kunze-Stein L^p convolution theorem [**9**]. Again, square integrable representations, or discrete series representations, which are characterized by the asymptotic behavior of their matrix coefficients (namely square integrability, see below) are a very important class of representations, since they are exactly the irreducible representations that appear as summands in the regular representation. (For a proof of this fact, see Section 14 of [**5**].) These representations are also the fundamental building blocks for all unitary representations of semi-simple groups, and they can play a similar role in other Lie groups as well; see [**10**].

For more general groups, it was proved in [**8**] that the matrix coefficients of any irreducible unitary representation π of a real algebraic

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group G vanish at ∞, mod the projective kernel P_π of π. In fact, it was also shown in [**8**] that some tensor power of π is a summand of the regular representation of G; this establishes a connection with square integrability. For simply connected nilpotent Lie groups (a special case of real algebraic groups), a simple characterization of when square integrable representations exist was proved in [**12**]; many other useful properties of these representations are also proved there. In [**7**], Howe studied some properties of matrix coefficients for general irreducible representations of matrix coefficients of nilpotent Lie groups. Thus the current study, which concerns pth power integrability of matrix coefficients for nilpotent Lie groups, is a continuation of themes in [**7, 9**] and [**12**].

We now give the basic definitions and results for this paper. Let G be a locally compact group and π an irreducible unitary representation of G on a Hilbert space H. We say that π has L^p matrix coefficients, $1 \leq p < \infty$, if there exist nonzero vectors $v, w \in H$ such that the matrix coefficients $f_{v,w} : G \to \mathbf{C}$, defined by

$$f_{v,w}(x) = \langle \pi(x), v, w \rangle$$

is in $L^p(G)$. (The case $p = \infty$, is of course, uninteresting.) Since $f_{v,w} \in L^p(g) \Leftrightarrow f_{\pi(y)v,w} \in L^p(G) \Leftrightarrow f_{v,\pi(y)w} \in L^p(G)$ for any $y \in G$, it is easy to see that if π has L^p matrix coefficients, then there are dense subspaces V, W of H such that $f_{v,w} \in L^p(G)$ for all $v \in V$ and $w \in W$. It is also obvious that if π has L^p matrix coefficients, then it has L^q matrix coefficients for all $q > p$.

If $z \in G$ is central, then $\pi(z)$ is scalar and $|f_{v,w}(zx)| = |f_{v,w}(x)|$ for all $x \in G$. Therefore π cannot have L^p matrix coefficients if its center is noncompact. More generally, π cannot have L^p matrix coefficients if its projective kernel $P_\pi = \{x \in G : \pi(x) \text{ is scalar}\}$ is noncompact. We therefore extend our definition in the customary way: we say that π has L^p matrix coefficients mod the center Z of G if for some nonzero $v, w \in h$, the function $|f_{v,w}(x)|$, which is constant on Z-cosets of G, is in $L^p(G/Z)$, and we define π to have L^p matrix coefficients mod the projective kernel P_π (or $\operatorname{mod}\operatorname{Ker}\pi$) analogously. If $P_\pi/\operatorname{Ker}\pi$ is compact (as it will be in the cases considered in this paper), then having L^p coefficients $\operatorname{mod} P_\pi$ and having L^p coefficients $\operatorname{mod}\operatorname{Ker}\pi$ are equivalent.

In this paper we consider real nilpotent Lie groups. Our main result is:

Theorem 1.1. *For any connected, simply connected real nilpotent Lie group G, there is a number $p < \infty$ such that for all $\pi \in \hat{G}$, π has L^p matrix coefficients* $\operatorname{mod}\operatorname{Ker}\pi$.

Our proof also gives a value for p (depending only on $\dim G$), but in general it is not the best possible.

We can also add one result about square integrable matrix coefficients for nilpotent Lie groups.

Theorem 1.2. *Let G be a connected, simply connected real nilpotent Lie group, and suppose that $\pi \in \hat{G}$ has square integrable matrix coefficients* $\operatorname{mod}\operatorname{Ker}\pi$. *Then for any $v, w \in H_\pi^\infty$ (the space of C^∞ vectors for π), $f_{v,w}$ is a function of Schwartz class on $S(G/\operatorname{Ker}\pi)$ on $G/\operatorname{Ker}\pi$. In fact, $(v,w) \mapsto f_{v,w}$ is continuous from $H_\pi^\infty \times H_\pi^\infty$ (with the usual C^∞ topology to $S(G/\operatorname{Ker}\pi)$).*

Theorem 1.2 has an obvious analog in the p-adic case: if G is an algebraic nilpotent group over a p-adic field, if $\pi \in \hat{G}$ has square integrable matrix coefficients $\operatorname{mod}\operatorname{Ker}\pi$, and if $v, w \in H_\pi$ are smooth vectors, then $f_{v,w}$ has compact support $\operatorname{mod}\operatorname{Ker}\pi$. This result is a theorem of van Dijk [**4**]. van Dijk also states in [**4**] that in the real case, he can show the existence of *one* matrix coefficient in $S(G/\operatorname{Ker}\pi)$. Roger Howe pointed out to us that one can then get the first part of Theorem 1.2 above by using Theorem 3.4 of [**7**]. Since the details of van Dijk's proof are not published, however, we have included a proof.

There is every reason to suppose that the theorems given here remain valid for more general classes of groups. For example, it is likely that if G is any connected algebraic solvable group over a local field of characteristic 0 and $\pi \in \hat{G}$, then there is a p such that π has L^p matrix coefficients $\operatorname{mod} P_\pi$. (The analogous result for nilpotent groups is our Theorem 3.1.) Quite possibly one can arrange for p to depend only on G when G is algebraic and solvable; this would be the theorem corresponding to our Theorem 1.1.

The techniques used here to prove Theorem 1.1 are brute force. For the idea behind the proof, consider the following simple example: $\mathfrak{g}$ is the Lie algebra spanned by X, Y_3, Y_2, Y_1 and Z, with $[X, Y_j] = Y_{j-1}$, $j = 2, 3$, and $[X, Y_1] = Z$, and G is the corresponding Lie group. Write a typical element of G as $(x, y, z) = (x, y_1, y_2, y_3, z) = \exp(y_1Y_1 + y_2Y_2 + y_3Y_3 + zZ) \exp X$. Then G has an irreducible representation π realized on $L^2(\mathbf{R})$ by

$$\pi(x, y, z)\phi(t) = \phi(x + t)e^{iP(y,t)+iz},$$

where

$$P(y, t) = \frac{y_3t^3}{3} + \frac{y_2t^2}{2} + y_1t.$$

(Indeed, for any irreducible unitary representation of G in general position, we can recoordinatize G so that π is of this form.) Thus, we look at $f = f_{\phi,\phi}$, where

$$f(x, y) = \int_{\mathbf{R}} \phi(x + t)\overline{\phi(t)}e^{iP(y,t)}\, dt.$$

Suppose that ϕ has compact support. Then f vanishes for large x, and we need only examine f as $y \to \infty$. Consider $P_t(y, t) = y_3t^2 + y_2t + y_1$. If we restrict y to a region where P_t has no roots (as a polynomial in t) near $\operatorname{supp}\phi$, then integrating by parts lets us estimate f. If P_t has roots near $\operatorname{supp}\phi$, then the method of stationary phase gives an estimate. (Stationary phase is not quite satisfactory because its estimates are not uniform when the roots are close together; we therefore use a somewhat different approach.) Combining these estimates gives the theorem.

In general we can regard π as acting on some $L^2(\mathbf{R}^k)$ in such a way that under a natural coordination of $G \approx \mathbf{R}^n$, we have

$$(\pi(x)\phi)(t) = e^{iP(x,t)}\phi(Q(x, t)), \qquad x \in \mathbf{R}^n, \quad t \in \mathbf{R}^k,$$

where P and Q are polynomials. Suppose that $k = 1$. If $P(x, t)$ and $Q(x, t)$, regarded as polynomials in t with coefficients in x, have bounded coefficients in some set S, then one can show that S is bounded $\operatorname{mod}\operatorname{Ker}\pi$. Thus we may assume that the coefficients in $P^2 + Q^2$ go to ∞ with x. When Q has large coefficients, the above modification of stationary phase gives an estimate for the matrix coefficient f. When the coefficients of P go to ∞, the simple considerations of the example do not generally suffice. However, we may restrict the vector $\phi \in L^2(\mathbf{R})$ to have support in $[-1,1]$, and we then show that $\{t : |t| \leq 1$ and $|P(x,t)| \leq 1\}$ has measure that is small when the coefficients of P are large. The proof is by induction on the degree of P (as a polynomial in t). If the constant term dominates the others, the result is clear. If not, $\partial P/\partial t$ has large coefficients and is of lower degree. But if $\partial P/\partial t$ is large, then $P(t)$ cannot stay small on a large interval in t. The details are given in Lemma 2.1.

Finally, we need to deal with the case $k > 1$. Write $t = (t_1, \dots, t_k) = (t, t')$ and assume that the function ϕ has support on $\{t : |t_j| \leq 1$, for all $j\}$. The case $k = 1$ suffices to prove Theorem 1.1 if the hypotheses apply to $P(x, t_1, t')$ as a polynomial in t_1, uniformly in $\{t' : |t_j| \leq 1, 2 \leq j \leq k\}$. We show that this is true after an appropriate change of variables. The (somewhat gory) result is Lemma 2.3.

These lemmas, together with fairly weak assumptions on the polynomials P and Q suffice to show that π has L^p matrix coefficients for some p. To get a uniform p for all $\pi \in \hat{G}$, we need further structural information on P and Q. This is done in Proposition 3.2.

For each $\pi \in \hat{G}$, the set of p such that π has L^p matrix coefficients is, as noted earlier, a half-infinite interval. In those cases where we have calculated it explicitly, the interval is open. It would be interesting to know if this is true in general. The simplest groups to consider (aside from those with square integrable representations) are the groups G_n with Lie algebras $\mathfrak{g}_n$ spanned by $X, Y_1, \dots, Y_n$, $n \geq 2$, and brackets $[X, Y_j] = Y_{j-1}$, $2 \leq j \leq n$, $[X, Y_1] = Z$. When $n = 2$ one can explicitly compute that there are matrix coefficients in $L^{A+\varepsilon}$ for any $\varepsilon > 0$. For $n = 3$ we can show that there are matrix coefficients in $L^{p+\varepsilon}$ for $p = 5 + \sqrt{7}$ and any $\varepsilon > 0$. We do not know if this is best possible, and the calculation does not give a hint of what an appropriate conjecture would be.

2. In this section we prove some technical lemmas about polynomials. We begin by fixing some notation.

We generally work with real-valued polynomials $P(x,t)$, $x \in \mathbf{R}^n$ and $t \in \mathbf{R}^k$. Fix Euclidean norms, $|\ |$, on these and any other finite-dimensional spaces that arise. We often regard $P(x,t)$ as a polynomial in $t = (t_1, \dots, t_k)$ with coefficients in $\mathbf{R}[x]$; then we write P, using multi-index notation, as

$$P(x,t) = \sum_{\alpha} P_\alpha(x) t^\alpha,$$

and define

$$||P(x)||^2 = \sum p_\alpha(x)^2.$$

In particular, we use this convention when $n = 0$ (so that the p_α are constants). When $k = 1$ we write $P'(x,t) = \partial P/\partial t$; for $k > 1$ we write $||P'(x)||$ for the largest of the $||\partial P/\partial t_j(x)||$, $1 \leq j \leq k$. When $Q(x,t)$ is a polynomial map to some $\mathbf{R}^s$, $Q = (Q_1, \dots, Q_s)$, $||Q(x)|| = \max\{||Q_j(x)|| : 1 \leq j \leq s\}$. We use $|\ |_\infty$ to denote the usual sup norm on $\mathbf{R}^k$.

Our first lemma is simple, but quite useful. For any polynomial $f \in \mathbf{R}[x]$ of one variable and any $K > 0$, set

$$E_f(K) = \{t \in \mathbf{R} : |t| \leq 1, |f(t)| \leq K\}, \qquad \mu_f(K) = \operatorname{meas}(E_f(K)).$$

Lemma 2.1. *For each integer $n \geq 0$, there is a constant C_n such that if $f \in \mathbf{R}[x]$ is a polynomial of one variable of degree n, then*

$$\mu_f(K) \leq C_n K^{1/(n+1)} (||f|| + K)^{-1/(n+1)}, \quad \forall K > 0.$$

Proof. It suffices to prove this for $K = 1$, since the general case follows by considering f/K. We proceed by induction on n. If $n = 0$, then $\mu_f(1) = 0$ once $||f|| > 1$, and we can take $C_0 = 2$.

Assume the result for all integers $< n$, and let $f(t) = \sum_{j=0}^k a_j t^j$. If $|a_0| > 1 + \sum_{j=1}^k |a_j|$, then $E_f(1)$ is empty and the lemma surely holds for f. We may therefore assume that $|a_0| \leq 1 + \sum_{j=1}^k |a_j|$. Now assume that $||f|| \geq (n+2)||f'||$. Then

$$\sum_{j=0}^k a_j^2 \geq (n+2)^2 \sum_{j=1}^k (j a_j)^2 \geq (n+2)^2 \sum_{j=1}^k a_j^2,$$

or

$$a_0^2 > (n+1)^2 \sum_{j=1}^{k} a_j^2;$$

in particular,

$$|a_0| \geq \sum_{j=1}^{k} |a_j| + \max_{1 \leq j \leq n} |a_j|.$$

Hence if $||f|| \geq (n+2)||f'||$ and $|a_0| \leq 1+\sum_{j=1}^{k} |a_j|$, then $\max_{1\leq j\leq n} |a_j| < 1$ and $||f||$ is bounded. Since $\mu_f(1) \leq 2$, the lemma certainly holds for all these f if C_n is sufficiently large. We may therefore assume that $||f|| \leq (n+2)||f'||$.

We now consider $E_f(1)$. It is the union of at most n disjoint closed intervals, since between any two such intervals f must have an extremum. Fix a number $L \geq 1$ in a manner to be further specified below. Let $[a, b]$ be a typical interval in $E_f(1)$; divide it into maximal subintervals where $|f'(t)| > L$ and where $|f'(t)| \leq L$. the intervals where $|f'(t)| > L$ have length $< 2/L$, by the Mean Value theorem; there are at most $(n-1)$ intervals where $|f'(t)| \leq L$ and their total length is $\leq C_{n-1}L^{1/n}(||f'||+L)^{-1/n} \leq C_{n-1}L^{1/n}(||f'||+1)^{-1/n}$. Thus, there are at most n intervals in $[a, b]$ where $|f'(t)| > L$. So the total length of $[a, b]$ is at most $C_{n-1}L^{1/n}(||f'|| + 1)^{-1/n} + 2nL^{-1}$, and

$$\mu_f(1) \leq nC_{n-1}L^{1/n}(||f'|| + 1)^{-1/n} + 2n^2L^{-1}.$$

Set $L = (||f|| + 1)^{1/n+1}$ to complete the inductive step. □

We next prove a lemma which will serve as a substitute for stationary phase estimates in what follows.

Lemma 2.2. *For every integer $r \geq 1$, there is a number K_r with the following property: Let $f(x, t)$, $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$ be a C^1 function, let $P(x, t)$ be a polynomial of degree $\leq r$, and let m be a number such that*

(a) *for all x, $f(x, t)$ has support (as a function of t) in $|t| \leq 1$;*

(b) *for all x, the function $(f/P')'$ (derivatives with respect to t), which is defined except at a finite number of points, changes sign at most m times on $\mathbf{R}$.*

(c) $P(x,0)=0$ *for all* x.

Then

$$\left|\int_{\mathbf{R}} e^{iP(x,t)} f(x,t)\,dt\right| \le (m+r+2)K_r\|f\|_\infty(1+\|P(x)\|)^{1/r}. \tag{2.1}$$

Proof. Let $P'(x,t)=\sum_{j=0}^{r-1} p_j(x)t^j$. Then $\|P'\|$ and $\|P\|$ are of the same order of magnitude, from (c). So it suffices to prove (2.1) with P' for P on the right side of (2.1).

Assume that $P'(x,t)$ has h zeros within a fixed distance d of $[-1,1]$ and $(r-h-1)$ zeros elsewhere (we count multiplicity and fix x). Choose $\varepsilon>0$ in a way to be described below, and write the integral as $\int_{I_1}+\int_{I_2}$, where I_1, I_2 are unions of intervals, I_1 has total length $\le 2h\varepsilon$, and all zeros of $P'(x,t)$ are at least ε away from I_2. Then

$$\int_{|I_1|} \le \|f\|_\infty\cdot 2h\varepsilon \le \|f\|_\infty\cdot 2r\varepsilon. \tag{2.2}$$

Divide I_2 into intervals on which $(f/P')'$ never changes sign. Since I_1 is composed of at most r intervals, there are at most $(m+r+2)$ such intervals, and

$$\begin{aligned}\int_a^b e^{iP(x,t)}f(x,t)\,dt &= \frac{e^{iP(x,t)}}{P'(x,t)}f(x,t)\Big|_a^b \\ &\quad -\int_a^b (f/P')'(x,t)e^{iP(x,t)}\,dt,\end{aligned}$$

$$\begin{aligned}\left|\int_a^b e^{iP(x,t)}(f/P')'(x,t)\,dt\right| &\le \int_a^b |(f/P')'(x,t)|\,dt \\ &\le |(f/P')(x,b)|+|(f/P')(x,a)| \\ &\le \|f\|_\infty(|P'(x,a)|^{-1}+|P'(x,b)|^{-1}).\end{aligned}$$

From this and (2.2) we get

$$\left|\int_{\mathbf{R}} e^{iP(x,t)}f(x,t)\,dt\right| \le \|f\|_\infty(2r\varepsilon+4(m+r+2))\max_a |P'(x,a)|^{-1}, \tag{2.3}$$

where a ranges over the end points of the intervals in L_2.

Let $\alpha_1, \dots, \alpha_{r-1}$ be the roots of $P'(x,t)$; of course, the α_j depend on x. Suppose that $\alpha_1, \dots, \alpha_h$ are within distance d of $[-1,1]$. Then for a constant C_0 depending on d

$$|P'(x,a)| = |p_{r-1}(x)| \prod_{j=1}^{r-1} |a-\alpha_j| \geq C_0 |p_{r-1}(x)| \varepsilon^h \prod_{j=h+1}^{r-1} |\alpha_j|,$$

where C_0 is such that $\prod_{j=h+1}^{r-1} |a-\alpha_j| \geq C_0 \prod_{j=h+1}^{r-1} |\alpha_j|$ for all $a \in [-1,1]$. Furthermore,

$$\prod_{j=h+1}^{r-1} |\alpha_j| \geq C' |p_h(x)/p_{r-1}(x)|,$$

from the expression of polynomials as symmetric functions of their roots. For the same reason, $|p_h(x)|$ is of the same order of magnitude as $||P'(x)||$ (and hence as $||P(x)||$). Therefore,

$$|P'(x,a)| \geq C_1 |p_h(x)| \varepsilon^h \geq C_2 ||P(x)|| \varepsilon^h.$$

Combined with (2.2), this gives

$$\left| \int_{\mathbf{R}} e^{iP(x,t)} f(x,t)\, dt \right| \leq ||f||_\infty (2r\varepsilon + 4(m+r+2) C_2 ||P'(x)||^{-1} \varepsilon^{-h}).$$

For $\varepsilon = ||P'(x)||^{-1/(h+1)}$, we get

$$\left| \int_{\mathbf{R}} e^{iP(x,t)} f(x,t)\, dt \right| \leq ||f||_\infty (m+r+1) C_3 ||P(x)||^{1/(h+1)}. \tag{2.4}$$

This implies the lemma, since $h \leq r+1$, and since all the constants introduced depend only on r, and since the integral is clearly bounded by $2||f||_\infty$ when $||P(x)||$ is small. □

The problem we face in using these lemmas is that they apply to functions defined on $\mathbf{R}$, while we will need to work with functions on $\mathbf{R}^k$. The purpose of the next lemma is to reduce the general case to $\mathbf{R}$.

Lemma 2.3. *Let r and s be integers greater than 0. There is a constant $C = C(r, s)$ with the following property: For every polynomial*

$$P(t) = \sum_{|\alpha| \le r} a_\alpha t^\alpha, \qquad t = (t_1, \ldots, t_s),$$

there is a matrix A such that

(a) *all coordinate entries of A and A^{-1} are bounded in absolute value by $2s$;*

(b) *under the change of variables $u = At$, $u = (u_1, \ldots, u_s) = (u_1, u')$, $u' \in \mathbf{R}^{s-1}$, we have*

$$P(A^{-1}u) = Q(u_1, u') = \sum_{k=0}^{r} q_k(u') u_1^k,$$

and for some k, $|q_k(u')| \ge C \cdot \max_{\alpha \ne 0} |a_\alpha|$ whenever $u = (u_1, u') = A(t)$ and $|t|_\infty \le 1$.

Proof. We may assume that $\max_\alpha |a_\alpha| = |a_{\alpha_0}| = 1$. Fix a number $K > 1$ to be determined below, and call the multi-index α special if $|a_\beta| \le K^{|\alpha| - |\beta|} |a_\alpha|$ for all β with $|\beta| > |\alpha|$. Choose a special α with $|\alpha|$ minimal with $|a_\alpha|$ maximal among such α. Then $|\alpha| \ge |\alpha_0|$ and one can see that $|a_\alpha| \ge K^{|\alpha_0| - |\alpha|} \ge K^{-r}$. Suppose that $|\alpha| = k$. Write $P_k(t) = \sum_{|\beta| = k} a_\beta t^\beta$, and let $\alpha = (\alpha_1, \ldots, \alpha_s)$. By perhaps permuting the variables, we may assume that $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_s$. Write $\beta \approx \alpha$ if $\beta = (\beta_1, \ldots, \beta_s)$ with $|\beta| = |\alpha|$ and $\beta_j = \alpha_j$ for $j \ne 1, s$; of course β is then determine by β_1 (or β_s).

Set $m = 3r^2$, and let l run through the integers $\le m/3$ in absolute value. Consider

$$\sum_{\beta \approx \alpha} a_\beta t_1^{\beta_1} \cdots t_{s-1}^{\beta_{s-1}} (t_s + l t_1/m)^{\beta_s}.$$

The coefficient of $t_1^{\alpha_1 + \alpha_s}$ in this expression is

$$\sum_{j=0}^{r} a_{\beta(j)} (l/m)^j,$$

where $\beta(j)$ is determined by $\beta(j)_s = j$, $\beta(j) \approx \alpha$. From Lemma 2.1 the set S of points y such that $|y| \le 1/3$ and $|\sum_{j=0}^r a_{\beta(j)} y^j| \le \kappa |a_\alpha|$ is the disjoint union of at most $(r+1)$ intervals and has measure $\le C'|\kappa|^{1/(r+1)}(1+\kappa)^{-1/(r+1)}$, where C' is a constant depending only on r. If one chooses κ small enough (in a manner depending only on r), this set has measure $\le 1/(3r+1)$. Thus, each interval in S has $\le r$ points of the form l/m, and there is some number $p_1 = l/m$ such that under the transformation $A_1 : t'_1 = t_1, \ldots, t'_{s-1} = t_{s-1}$, $t'_s = t_s - p_1 t_1$, we have

$$P_k(A_1^{-1} t') = \sum_{|\beta|=k} a'_\beta t'^\beta,$$

and $|a'_\beta| \ge \kappa |a_\alpha|$ for $\beta = (\alpha_1 + \alpha_s, \alpha_2, \ldots, \alpha_{s-1}, 0)$.

Now regard $P_k \circ A_1^{-1}$ as a polynomial in $t'_1 = t_1, \ldots, t'_{s-1}$, with coefficients in $\mathbf{R}[t'_s]$. The terms of highest degree all have coefficients in $\mathbf{R}$, and one coefficient is $\ge \kappa|a_\alpha|$ in absolute value. We now iterate the argument on the terms of degree k in $P_k \circ A_1^{-1}$. After $(s-2)$ more steps, we have transformations $A_1, \ldots, A_{r-1}$ such that $A = A_1 A_2 \cdots A_{k-1}$ and A^{-1} have all coefficients $\le 2s$ in absolute value and such that if $u = A_t$, $u = (u_1, u')$, then

$$P_k(A^{-1}u) = \sum_{|\beta|=k} b_\beta u^\beta,$$

where, for $\beta_0 = (k, 0, \ldots, 0)$, we have $|b_{\beta_0}| \ge |a_\alpha| \kappa^{s-1}$, κ depending only on r, s.

Now consider $P(A^{-1})u$. Write $P(t) = \sum_{j=0}^r P_j(t)$, where P_j is homogeneous of degree j. We have

$$P(A^{-1}u) = \sum_{i=0}^r q_i(u') u_1^i,$$

$$P_j(A^{-1}u) = \sum_{i=0}^r q_{i,j}(u') u_q^i,$$

say. We are interested in $q_k(u') = \sum_{j=0}^r q_{k,j}(u')$. For $k > j$, the definition of P_k implies that $q_{k,j}(u') \equiv 0$; $||q_{k,k}(u')|| = |b_{\beta_0}|$. For $k < j$, we estimate $||q_{k,j}(u)||$ as follows:

$$\begin{aligned}\sum_{|\beta|=j} |a_\beta| &\geq \sup_{|t|_\infty \leq 1} \sum_{|\beta|=j} |a_\beta t^\beta| \\ &\geq \sup_{|t|_\infty \leq 1} |P_j(t)| \\ &= \sup_{|A^{-1}u|\leq 1} \left| \sum_{i=0}^{r} q_{i,j}(u') u_1^i \right|.\end{aligned}$$

From Lemma 2.1 there exists K_0 such that $||q_{k,j}(u')|| \leq (\sum_{|\beta|=j} |a_\beta|)K_0$. There are fewer than $s^j \leq s^r$ multi-indices with $|\beta| = j$, and for all of them we have $|a_\beta| \leq |a_\alpha| K^{k-j}$. Thus

$$||q_{k,j}(u')|| \leq s^r K_0 K^{k-j} |a_\alpha| \quad \text{for } j > k \\ \text{and} \quad |A^{-1}u|_\infty \leq 1.$$

Now fix $K > 3(\kappa^{-1}s)^r K_0$, and make $\kappa < 1$, $K_0 > 1$. Then, for $j > k$,

$$||q_{k,j}(u')|| \leq 3^{k-j} \kappa^r |a_\alpha| \quad \text{when} \quad |A^{-1}u|_\infty \leq 1,$$

and so $||q_{k,j}(u')|| \geq \kappa^r |a_\alpha|/2$ when $|A^{-1}u|_\infty \leq 1$. Since all constants depend only on r and s, the lemma is proved. □

Applying this lemma to the situations of Lemmas 2.1 and 2.2, we immediately obtain

Corollary 2.4. *For each pair (n, s) of integers with $n \geq 0$, $s > 0$, there is a constant C such that if f is a polynomial in s variables of degree $< n$, then for any $K > 0$, $\{x \in \mathbf{R}^s : |x|_\infty \leq 1, |f(x)| \leq K\}$ has Lebesgue measure $\leq CK^{1/(n+1)}(||f|| + K)^{-1/(n+1)}$.*

Corollary 2.5. *For all integers $r, k \geq 1$, there exists a constant $K = K_{r,k}$ with the following property: Suppose that $f(x,t)$, $x \in \mathbf{R}^n$ and $t \in \mathbf{R}^k$ is a bounded C^1 function, that $P(x,t)$ is a polynomial of degree $\leq r$, and that m is an integer such that for all x,*

(a) *$f(x,t)$ has support in $\{|t|_\infty \leq 1\}$;*

(b) *On every line through the origin in* $\mathbf{R}^k$, *the directional derivative* $(f/P')'$ *in the direction of the line (which is defined except at a finite number of points on the line) changes sign at most* m *times on this line*;

$$P(x,0)=0.$$

Then

$$\left|\int_{\mathbf{R}^k} e^{iP(x,t)} f(x,t)\,dt\right| \le K(m+r+2)||f||_\infty(1+||P(x)||)^{-1/r}.$$

3. In this section we prove Theorem 1.1. It clearly suffices to consider the case in which $\operatorname{Ker}\pi$ is discrete; in that case G has a one-dimensional center. Let $\mathfrak{g}$ be the Lie algebra of G, Z the center of G and $\mathfrak{z}$ the center of $\mathfrak{g}$. Choose a (vector space) cross-section $\mathfrak{h}$ to $\mathfrak{z}$ in $\mathfrak{g}$, and let $H=\exp\mathfrak{h}$. If we coordinatize $\mathfrak{h}\cong\mathbf{R}^n$, then $H\cong\mathbf{R}^n$ via the exponential map (we shall, in fact, pick a different map shortly). Then, by using standard Kirillov theory, we may realize π on some $L^2(\mathbf{R}^k)$ so that for $x\in H$,

$$(\pi(x)\phi)(t)=e^{iP(x,t)}(\phi\circ Q)(x,t), \tag{3.1}$$

where P,Q are polynomials over $\mathbf{R}$ and $P(x,t)$, when regarded as a polynomial in t with coefficients in $\mathbf{R}[x]$, has no constant term. We are interested in

$$\begin{aligned} f_{\phi,\psi}(x) &= \int_{\mathbf{R}^k}(\pi(x)\phi)\overline{\psi(t)}\,dt \\ &= \int_{\mathbf{R}^k} e^{iP(x,t)}\phi\circ Q(x,t)\overline{\psi(t)}\,dt. \end{aligned}$$

Let ϕ,ψ be real-valued C^1 functions with support in $|t|_\infty\le 1$ and with the property that the number of changes of sign of the function $((\phi\circ Q)\bar{\psi}/P')'$ on any line through the origin is bounded. (One can arrange this by letting both functions be polynomial on their support.) Then, by Corollary 2.5,

$$|f_{\phi,\psi}(x)|\le C_1(||P'(x)||+1)^{-\gamma_1}, \tag{3.2}$$

where $C_1 > 0$ and $\gamma_1 > 0$ are constants (and γ_1 depends only on $\deg P$). Next, Corollary 2.4 says that $||Q(x,t)|| \leq 1$ on a set of measure $\leq C_2(||Q(x)||1)^{-\gamma_2}$ for some $C_2, \gamma_2 > 0$ (γ_2 depending only on $\deg Q$). Since the integrand is 0 unless $||Q(x,t)|| \leq 1$, we get

$$(3.3) \qquad |f_{\phi,\psi}(x)| \leq C_3(||Q(x)|| + 1)^{-\gamma_2} \quad \text{for some } C_3 > 0.$$

Combining (3.2) and (3.3) gives

$$(3.4) \qquad |f_{\phi,\psi}(x)| \leq C_4(||P(x)|| + ||Q(x)|| + 1)^{-\gamma_3},$$

where $C_4 > 0$ and $\gamma_3 = \min(\gamma_1, \gamma_2)$, and where the constants C_3 and C_4 depend on the choice of ϕ and ψ. Since $P(x,t)$ has no constant term in t, $||P(x)||$ and $||P'(x)||$ have the same order of magnitude. It follows that if we can prove that

$$(3.5) \qquad ||P(x)|| + ||Q(x)|| + 1 \geq K(|x| + 1)^{\delta} \quad \text{for some } K_1, \delta > 0,$$

then $f_{\phi,\psi} \in L^p$ (for any $p > (\gamma_3\delta)^{-1}$).

It is in fact possible to prove (3.5) directly, and this gives a weak form of Theorem 1.1. We include the proof here because it may be useful in more general situations.

Theorem 3.1. *Let G be a connected, simply connected nilpotent Lie group, and let π be an irreducible unitary representation of G. Then there exists $p < \infty$ such that π has L^p matrix coefficients modulo its projective kernel P_π.*

Proof. As the discussion above shows, it suffices to prove (3.5). But for this purpose, it suffices to show merely that

$$(3.6) \qquad ||P(x)||^2 + ||Q(x)||^2 + 1 \longrightarrow \infty \quad \text{as } |x| \longrightarrow \infty$$

(see, e.g., [**6**, p. 367]. If (3.6) fails to hold, then there is a sequence $\{x_n\} \subset H$ converging to ∞ such that $||P(x)||^2 + ||Q(x)||^2$ is bounded; passing to a subsequence we may assume that in the polynomials

$$P(x,t) = \sum_{\alpha} p_\alpha(x) t^\alpha, \qquad Q(x) = \sum_{\alpha} q_\alpha(x) t^\alpha,$$

the sequences $\{p_\alpha(x_n)\}, \{q_\alpha(x_n)\}$ all converge as $n \to \infty$. Then, for any nonzero $\phi_0 \in L^2(\mathbf{R}^k)$, $\pi(x_n)\phi_0$ converges to a nonzero vector ϕ_1. Hence,

$$\langle \pi_n(x)_{\phi_0,\phi_1} \rangle \longrightarrow ||\phi_1||^2 \neq 0,$$

which contradicts the fact that matrix coefficients go to zero at ∞ (see Theorem 6.1 of [**8**]). □

For given P and Q, the number δ is effectively computable (see [**13**]). However, any estimate for p that Theorem 3.1 gives is not easily computable from the usual data for nilpotent Lie groups. We now get such an estimate by examining the polynomials P and Q.

Let l be an element in the Kirillov orbit of π so that $\pi = \pi_l$. Let m be a polarizing subalgebra for l, and set $M = \exp m$. We can find a weak Malcev basis $X_0, \dots, X_n$ for $\mathfrak{g}$ through m, where X_0 spans $\mathfrak{z}$; that is, for each j, $\mathfrak{g}_j = \operatorname{span}\{X_0, \dots, X_j\}$ is a subalgebra, and $m = \mathfrak{g}_j$ for some j. We can then coordinatize G by using the diffeomorphism

$$(t_0, t_1, \dots, t_n) \longmapsto (\exp t_0 X_0)(\exp t_1 X_1) \cdots (\exp t_n X_n)$$

of $\mathbf{R}^{n+1}$ with G and the image of all points $(0, \dots, 0, t_{j_0+1}, \dots, t_n)$, where $m = \mathfrak{g}_{j_0}$ gives a cross-section to M that lets one realize π on $L^2(\mathbf{R}^k)$, $k = n - j_0$. As before, π is given in this realization by an expression of the form (3.1).

Proposition 3.2. *We can choose the weak Malcev basis so that the polynomials $P = Q_0$ and $Q = (Q_1, \dots, Q_k)$ have the following property: for every j with $1 \le j \le n$, there is a multi-index β and an integer i, $0 \le i \le r$, such that if we write $Q_i(x,t) = \sum_\alpha q^i_\alpha(x)t^\alpha$, then $q^i_\beta(x) = x_j +$ a polynomial in $x_{j+1}, \dots, x_n$ without a constant term.*

Assume the proposition for the moment; we show how it implies Theorem 1.1. Suppose that the q^i_α are all of degree $\le m$ and that $R(x) = \sum_{\alpha,i} q^i_\alpha(x)^2 \le K^2$ for some K. Assume further that all coefficients appearing in R are $\le A$. Since some q^i_α is equal to x_n, we have $|x_n| \le K$. Another q^i_α is $x_{n-1}+$ a polynomial in x_n, and the polynomial in x_n certainly has absolute value $\le AmK^m$. Hence $|x_{n-1}| \le 2AmK^m$ for large K. Continuing inductively, we get

$|x_j| \leq CK^{m^j}$ for some constant C. That is, $R(x) \leq K^2|x| \leq C'K^{m^n}$, or $R(x)^{1/2} \geq C''|x|^{2/m^n}$. Since $m \leq n$, this gives a matrix coefficient in L^p for some p depending only on $n = \dim(G) - 1$.

Proof of Proposition 3.2. We use induction on $n = \dim(G) - 1$, the result being obvious for n small. As before, we may assume that $\operatorname{Ker}\pi$ is discrete. Let l be a functional in the orbit of π. Induce π from π_0 on a subgroup G_0 of codimension 1 and let $l_0 = l|_{\mathfrak{g}_0}$. Let K_0 be the connected component of $\operatorname{Ker}\pi_0$, and write $\overline{G}_0 = G_0/K_0$; similarly, write $\bar{\mathfrak{g}}_0 = \mathfrak{g}_0/\mathfrak{k}_0$, $\overline{X} = X \bmod \mathfrak{k}_0$, etc. Then l_0 is trivial on $\mathfrak{k}_0$, and thus gives $\bar{l}_0$ on $\bar{\mathfrak{g}}_0$. The representation π_0 yields $\bar{\pi}_0$ on $\overline{G}_0$, and we can, by the inductive hypothesis, find a weak Malcev basis $\overline{X}_0, \overline{X}_1, \dots, \overline{X}_{r-1}$ for $\overline{G}_0$ (with $\overline{X}_0$ central), passing through a polarizing subalgebra $\bar{\mathfrak{m}}$ for $\bar{l}_0$ and such that if we realize π_0 on $\mathbf{R}^{k-1}$ by means of this basis, we have

$$(\bar{\pi}_0(\bar{x}_0)\phi_0)(t_0) = e^{iR_0(\bar{x}_0,t_0)}\phi_0 \circ R(\bar{x}_0, t_0),$$

where R_0 and $R = (R_1, \dots, R_{k-1})$ are polynomials such that the proposition holds for $\overline{G}_0$. Lift these elements back to pre-images $X_0, \dots, X_{r-1}$ in $\mathfrak{g}$ with X_0 central, and let X_r be an element of $\mathfrak{g}$ not in $\mathfrak{g}_0$. We look for a weak Malcev basis $Y_1, \dots, Y_s$ of $\mathfrak{k}_0$ such that the weak Malcev basis of $\mathfrak{g}$ given by $X_0, Y_1, \dots, Y_s, X_1, \dots, X_r$ gives polynomials with the required property. Note that the preimage $\mathfrak{m}$ of $\bar{\mathfrak{m}}$ is polarizing for l and that the above basis passes through $\mathfrak{m}$.

Write a typical element of G (mod the center $\exp \mathbf{R}X_0$) as (x, y, w), where the x coordinates refer to $X_1, \dots, X_{r-1}$, the y coordinates to $Y_1, \dots, Y_s$, and w to X_r; write the coordinates of $\mathbf{R}^k$ as (t_0, u) where $t_0 \in \mathbf{R}^{k-1}$ and $u \in \mathbf{R}$. We have

$$(\pi(x, y, w)\phi)(t_0, u) = e^{iQ_0(x,y,w,t_0,u)}\phi \circ Q(x, y, w, t_0, u),$$

where Q, Q_0 are related to R, R_0 as follows: let $\overline{H}(x, y, u)$ be the result of conjugating $(x, y, 0) \in G_0$ by $\exp uX_r$ and projecting to $\overline{G}_0$. (In particular, $\overline{H}(x, y, 0) = x$.) Then

$$\begin{aligned} Q_0(x, y, w, t_0, u) &= R_0(\overline{H}(x, y, u), t_0), \\ Q_j(x, y, w, t_0, u) &= R_j(\overline{H}(x, y, u), t_0), \qquad 1 \leq j \leq k-1, \\ Q_r(x, y, w, t_0, u) &= u + w, \end{aligned}$$

as a straightforward calculation shows. (In realizing the induced representation, we deal with functions which are covariant under a left G_0-action, and π acts by right translation.)

The above formulas give

$$Q_j(x, y, w, t_0, 0) = R_j(x, t_0) \quad \text{for } 1 \leq j \leq k-1.$$

This, plus the formula for Q_k, shows that the inductive step holds for $x_1, \ldots, x_{q-1}, w$ regardless of how we choose the Y_j. The real work lies in choosing the Y_j so that the inductive step holds for the y_j.

Let $\mathfrak{j}_0$ be the largest $\mathfrak{g}_0$-ideal contained in $\mathfrak{m}$, $J_0 = \exp \mathfrak{j}_0$. It is easy to see that an element X of $\mathfrak{g}_0$ is in $\mathfrak{j}_0$ if and only if $R(\exp \overline{X}, t_0)$, viewed as a vector in $\mathbf{R}^{k-1}$, is equal to t_0. Set

$$\mathfrak{j}_1 = \{X \in \mathfrak{j}_0 : [X_r, X] \in \mathfrak{j}_0\} = \{X \in \mathfrak{j}_0 : [\mathfrak{g}, X] \in \mathfrak{j}_0\},$$

and inductively define

$$\mathfrak{j}_{i+1} = \{X \in \mathfrak{j}_1 : [X_r, X] \in \mathfrak{j}_i\}.$$

One checks easily that the $\mathfrak{j}_i$ are ideals in $\mathfrak{g}_0$. Let $s_1 = \dim(\mathfrak{k}_0 \cap \mathfrak{k}_1)$, and note that $\mathfrak{k}_0$ is an ideal in $\mathfrak{m}$, so that $\dim(\mathfrak{k}_0 \cap \mathfrak{j}_0) = s$. Let $Y_{s_1+1}, \ldots, Y_s$ span a complement in $\mathfrak{k}_0$ to $\mathfrak{k}_0 \cap \mathfrak{j}_1$ in such a way that $\mathfrak{k}_0 \cap \mathfrak{j}_1, Y_{s_1+1}, \ldots, Y_j$ span an ideal in $\mathfrak{g}_0$ for all j. For $j = s_1+1, \ldots, s$, we have

$$[Y_j, X_r] \notin \mathfrak{j}_0.$$

Hence $\overline{H}(x, y, u)$ has a component in which the coefficient of u is linear in y_s. By adding multiples of the Y_j, $j < s$, to Y_s, we may assume that no other y_j appear in this component. The inductive hypothesis, applied to this component, says that for some j, $Q_j(x, y, w, t_0, u) = R_j(\overline{H}(x, y, u), t_0)$ has a term of the form

$$(c_s y_s + F(x_1, \ldots, x_{r-1}))ut_0^\alpha, \quad F \text{ a polynomial and } c_s \in \mathbf{R}.$$

By scaling Y_s we may make $c_s = 1$. Now the proposition is true for Y_s, and the same procedure can be applied to prove the proposition for all Y_j with $j > s_1$. We proceed similarly with a basis extending from $\mathfrak{j}_2 \cap \mathfrak{k}_0$ to $\mathfrak{j}_1 \cap \mathfrak{k}_0$, this time looking at terms in u^2. If we let $\mathfrak{n} = \cap_{i=1}^\infty \mathfrak{j}_i$, we see by an easy inductive argument that we can construct a basis

$Y_{s'+1}, \dots, Y_s$ (where $s' = \dim(\mathfrak{n} \cap \mathfrak{k}_0)$) from $\mathfrak{n} \cap \mathfrak{k}_0$ to $\mathfrak{j} \cap \mathfrak{k}_0$ such that the proposition holds for this basis (attached to $X_1, \dots, X_r$).

We are left with the problem of finding an appropriate basis for $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{k}_0$. (Note that $X_0 \in \mathfrak{n}$, but $X_0 \in \mathfrak{n}$, but $X_0 \notin \mathfrak{n}_0$.) Let

$$\mathfrak{n}_1 = \{X \in \mathfrak{n}_0 : [X, X_r] \in \mathfrak{n}_0\},$$

and inductively define

$$\mathfrak{n}_{j+1} = \{X \in \mathfrak{n}_j : [X, X_r] \in \mathfrak{n}_j\}.$$

Then $X \in \cap_{j=1}^{\infty} \mathfrak{n}_j \Leftrightarrow \exp sX \in \operatorname{Ker} \pi$ for all $s \in \mathbf{R}$, so that $\cap_{j=1}^{\infty} \mathfrak{n}_j = \{0\}$.

The procedure we follow is similar to the procedure given above. Since $\mathfrak{n}$ is an ideal, we see that if $Y \in \mathfrak{n}_0 - \mathfrak{n}_1$, then $[Y, X_r]$ must be a multiple of X_0. If the multiple is 0, then $\exp sY \in \operatorname{Ker} \pi$ for all $s \in \mathbf{R}$. Hence $\dim \mathfrak{n}_0 / n_1 = 0$ or 1. If it is zero, then $\mathfrak{n}_0 = \{0\}$ and we are done. If not, then let $Y_1 \in \mathfrak{n}_0 - \mathfrak{n}_1$. A similar argument gives us vectors $Y_2, \dots, Y_{s'}$ with $\mathfrak{n}_j = \mathfrak{n}_{j+1} + \mathbf{R} Y_{j+1}$ for $0 \leq j < s'$. We may assume by scaling and by adding appropriate linear combinations of $Y_1, \dots, Y_{j-1}$ to Y_j that $[X_r, Y_1] X_0$ and $[X_r, Y_j] = (j+1)! Y_{j-1}$ for $j > 1$. Then

$$\overline{H}(0, (y_1, \dots, y_{s'}, 0, \dots, 0)u) = (y_1 u + y_2 u^2 + \cdots + y_{s'} u^{s'}, 0, \dots, 0).$$

So if $l(X_0) = 1$ (as we may plainly assume), then

$$Q_0(0, (y_1, \dots, y_{s'}, 0, \dots, 0), 0, 0, u) = y_1 u + y_2 u^2 + \cdots + y_{s'} u^{s'}.$$

This proves the proposition for the y_j, since the coefficient of u^j in Q_0 is obviously of the desired type. This also completes the proof of Proposition 3.2. □

Remarks. 1. It is easy to check that $f_{\pi(\phi)v,w} = f_{v,w} * \phi^{\sim}$, where $\phi^{\sim}(x) = \phi(x^{-1})$. Thus $f_{\pi(\phi)v,w}$ is in L^p if $f_{v,w}$ is. In view of Theorem 3.4 of [**7**], this means that we can arrange for v to be an arbitrary C^{∞} vector. A similar argument applies to w. In the proof of Theorem 1.1, however, we needed to work with vectors that were not C^{∞}.

2. If π is induced from a character on a normal subgroup M of G, we can get a much better estimate for p. The technical difference is this: let M be normal and write coordinates in G as (x', x''), where $x' \in M$ and x'' is in a cross-section for $M\backslash G$ (the same one used to produce the representation space, for instance). Then we have, as before,

$$(\pi(x', x'')f)(t) = e^{iP(x',x'',t)}Q(x', x'', t),$$

but Q is in fact independent of x', while $P(x', x'', t)$ can be written as $P_1(x', t) + P_2(x'', t)$, where P_1 is linear in x'. This makes the analysis much easier. A simple estimate shows that p can be any number $> n^2 s$, where $\dim G = n + 1$ and $\dim G/M = s$.

4. We turn now to the proof of Theorem 1.2. We may assume that G has a one-dimensional center Z with Lie algebra $\mathfrak{z}$, and that π is nontrivial on Z. Write $\pi = \pi_l$, $l \in \mathfrak{g}^*$, and let $X_0 \in \mathfrak{z}$ satisfy $l(X_0) = 1$. Then $\dim G$ is odd. Let $X_0, \ldots, X_{2n}$ be a strong Malcev basis for $\mathfrak{g}$ (this means that $X_0, \ldots, X_j$ span an ideal of $\mathfrak{g}$ for every j). We then know that X_1 is in the second center of $\mathfrak{g}$ (i.e., its image is central in $\mathfrak{g}/\mathfrak{z}$); we may assume that $[X_1, X_j] = 0$ for $j < 2n$. Hence an arbitrary element of G can be written as $(x, y, w, x) = \exp zX_0 \exp yX_1 \ldots \exp(\sum_{j=2}^{2n-1} w_{j-1}X_j) \exp xX_n$; here, $w = (w_1, \ldots, w_{2n-2})$. We need to prove that, for C^∞ vectors v and v', the matrix coefficient $f_{v,v'}$ is in $S(\mathbf{R}^{2n})$, the Schwartz space, as a function of y, w, and x. For notational convenience we write (y, x, w) for $(0, y, x, w)$.

We prove the theorem by induction on n, the case $n = 0$ (or $\dim G = 1$) being trivial. In the general case, π on G is induced from π_0 on $G_0 = \exp \mathfrak{g}_0$, where $\mathfrak{g}_0$ is spanned by $X_0, \ldots, X_{2n-1}$. We may assume that π_0 is associated with $l_0 = l|_{\mathfrak{g}_0}$ and that $l(X_1) = 0$. Write

$$\pi_t(z, y, w, 0)f = \pi_0(0, 0, 0, t)(z, y, w, 0)(0, 0, 0, -t), \qquad f, t \in \mathbf{R};$$
$$(0, 0, 0, t)(0, 0, w, 0)(0, 0, 0, -t) = \alpha(w, t), \beta(w, t), \gamma(w, t), 0).$$

Then from the usual way of realizing induced representations we get

$$(\pi(y, w, x)\phi)(t) = e^{i(ty+\alpha(w,t))}(\pi_0(\gamma(w, t))(\phi(x + t))),$$

where $\phi : \exp \mathbf{R}X_{2n}(\cong \mathbf{R}) \to H(\pi_0)$. Both α and γ are polynomial maps and, for fixed t, $w \mapsto \gamma(w, t)$ has polynomial inverse. It is therefore clear that if $f : \mathbf{R} \to S(\mathbf{R}^{n-2})$ is C^∞, then

$$g : \mathbf{R} \longrightarrow S(\mathbf{R}^{n-2}), \qquad g(t)(w) = e^{i\alpha(w,t)} f(t)(\gamma(w,t))$$

is C^∞ (and rapidly vanishing in w for fixed t); also if f vanishes rapidly at ∞, then so does g.

Let $\phi, \psi \in S(\mathbf{R}^n) = H_\pi^\infty$, the space of C^∞ vectors for π. We regard ϕ, ψ as elements of $S(\mathbf{R}, S(\mathbf{R}^{n-1}))$. By the inductive hypothesis the map T_0 on $S(\mathbf{R}^{n-1}) \times S(\mathbf{R}^{n-1})$ defined by $T_0(h,k)(w) = \langle \pi_0(w), h, k \rangle$ is a continuous map into $S(\mathbf{R}^{2n-2})$. Now let $\phi, \psi \in S(\mathbf{R}^n) = H_\pi^\infty$, the space of C^∞ vectors for π. We regard ϕ, ψ as elements of $S(\mathbf{R}, S^{n-1}(\mathbf{R}^{n-1}))$, and we want to examine

$$\langle \pi(y, w, x)\phi, \psi \rangle = \int_{\mathbf{R}} e^{iyt} \langle \pi_t(w)(\phi(x+t), \psi(t)\rangle \, dt. \tag{4.1}$$

Define $A : S(\mathbf{R}^n) \times S(\mathbf{R}^n) \to \mathbf{R}^{n-2} \times \mathbf{R} \times \mathbf{R}$ by

$$A(\phi, \psi)(w, x, t) = \langle \pi_t(w)\phi(x+t), \psi(t) \rangle.$$

As remarked above, $A(\phi, \psi)$ is Schwartz in w, t for fixed x. It is also C^∞ in all variables, since differentiating in x presents no problem. Furthermore, all derivatives vanish rapidly at ∞ with x as easy estimates show (since $\phi(x+t)$ and $\psi(t)$ decay rapidly at ∞). Thus A maps into $\mathbf{R}^{2n-2} \times \mathbf{R} \times \mathbf{R}$.

We need to prove that

$$T(\phi, \psi)(y, w, x) = \int_{\mathbf{R}} A(\phi, \psi)(w, x, t) e^{iyt} \, dt$$

is in $S(\mathbf{R}^{2n})$. It is clearly C^∞ in w, x and y, and it still vanishes rapidly in w and x. Differentiate the integrand with respect to t (and note that the integrand is still in $S(\mathbf{R}^{2n})$ to see that it vanishes rapidly in y as well.

To complete the proof we need to prove that T is continuous. The Closed Graph theorem shows that it is separately continuous in ϕ and ψ; it is also bilinear. A standard application of the Banach-Steinhaus theorem (see, e.g., [**14**, p. 54]) gives continuity.

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