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CLOSED GEODESICS ON IDEAL POLYHEDRA OF DIMENSION 2

C. CHARITOS

1. Introduction. In this paper we are concerned with ideal polyhedra of dimension 2. These spaces consist of ideal hyperbolic triangles which are glued together by isometries along their sides. It is important to define and begin the study of ideal polyhedra in the context of negatively curved polyhedra introduced by Gromov in [**7**]. Ideal polyhedra of dimension 2 appear naturally as the 2-skeleton of 3-manifolds obtained by gluing together ideal tetrahedra. Topologically, simplicial complexes with vertices removed are examples of spaces which are homeomorphic to ideal polyhedra.

There is a consistent way to define a length pseudo-distance d_K on an ideal polyhedron K. Theorem 1 gives necessary conditions for this length pseudo-distance to be a geodesic metric. This result is analogous to a theorem of Bridson for geometric complexes (see [**1**, Theorem 1.1] or [**9**, Theorem 3.6]).

By introducing the concept of the developing surface along a curve γ in K we prove the existence of a closed geodesic in the free homotopy class of a closed curve which is not homotopic to a point or to a cusp in K. We prove this by elementary methods; we don't use any shortening process [**4**, Chapter 10, **5**] but we reduce the proof to the case of surfaces.

In Propositions 2 and 3 we prove simple properties of the universal covering $\tilde{K}$ of K. In particular, in Proposition 3 we prove that every local geodesic of $\tilde{K}$ is a geodesic; this is not true in general for two-dimensional simply connected polyhedra of negative curvature.

Finally, Propositions 2 and 3 permit us to establish the uniqueness of a closed geodesic in its free homotopy class. This is Theorem 2 and is the main application of the method outlined above.

2. Ideal polyhedra of dimension 2. We begin with the precise

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definition of ideal polyhedra of dimension 2. Note that all ideal polyhedra that we consider in the following are of dimension 2.

Definition. Let E be a disjoint union of ideal hyperbolic triangles, and let $\sim$ be an equivalence relation on $\cup E$ such that: For every T, T' in E, $D_{T',T} = \{x \in T : \exists y \in T', x \sim y\}$ is a (possibly empty) union of sides of T, and the restriction of $\sim$ to $T \times T'$ is the graph of a homeomorphism from $D_{T';T}$ to $D_{T,T'}$ which is an isometry on every side of $D_{T',T}$.

An *ideal polyhedron* is the quotient space $\cup E/\sim$.

An ideal polyhedron K Is said to be *locally finite* if every triangle of K meets only finitely many triangles of K.

Let K be an ideal polyhedron of dimension 2. A map from $[a,b]$ to K is called a broken geodesic if there is a subdivision $a = t_0 < t_1 < \cdots < t_{n+1} = b$ such that $f([t_i, t_{i+1}])$ is contained in some ideal triangle and the restriction of f to $[t_i, t_{i+1}]$ is a geodesic inside this ideal hyperbolic triangle for $i = 1, 2, \ldots, n$. Then define the length $l(f)$ of the broken geodesic map f to be

$$\sum_{i=0}^{n} l(f|_{[t_i,t_{i+1}]}) = \sum_{i=0}^{n} |f(t_i) - f(t_{i+1})|$$

the length inside an ideal triangle being measured with respect to the hyperbolic metric $|\quad|$. K is connected if every two points x, y in K can be joined by a broken geodesic.

Define $d(x,y)$ for every two points x, y in K to be the lower bound on the lengths of broken geodesics from x to y. This is clearly a symmetric map, satisfying the triangle inequality, called the *length pseudo-distance*.

We next recall a few definitions to fix notation and terminology:

Let (X, ϱ) be a metric space.

A *geodesic segment* is an isometric map $f : I \to X$ where I is some closed interval of R.

A *geodesic arc* is a map $f : I \to X$ such that for every t in I there exists an $\varepsilon > 0$ such that the restriction of f to $[t - \varepsilon, t + \varepsilon] \cap I$ is a geodesic segment.

A *geodesic*, respectively *local geodesic*, is a map $f : R \to X$ such that for every closed subinterval I of R, $f/I : I \to R$ is a geodesic segment, respectively geodesic arc.

A *closed geodesic* is a map $f : S^1 \to X$ such that for every t in S^1 there exists an $\varepsilon > 0$ such that the restriction of f to $[t-\varepsilon, t+\varepsilon] \cap S^1$ is a geodesic segment.

If every two points on (X, ϱ) can be joined by a geodesic segment, then X is called a *geodesic metric space* and ϱ a *geodesic metric.*

Note that the length pseudo-distance on an ideal polyhedron may not be a metric; see Example 1 below. Note also that even if the length pseudo-distance is a metric, it may not be geodesic metric; see Example 2 below.

To formulate our examples we will need the notion of gluing weight which we next define.

Let T_1, T_2 be two hyperbolic ideal triangles. We denote by C_k, $k = 1, 2$, the unique circle inscribed in T_k. Note that the existence and uniqueness of the inscribed circle in an ideal triangle follows easily from the fact that the group of isometries of hyperbolic plane H^2 operators transitively on the triples of points of the boundary of H^2. Let σ_1, σ_2 be two oriented sides of T_1, T_2, respectively, and let A_k be the point of tangency of C_k with the side σ_k, $k = 1, 2$. Let $I : \sigma_1 \to \sigma_2$ be an isometry. We use i to glue T_1 with T_2 along the faces σ_1 and σ_2. Let $A_1' = i(A_1) \in \sigma_2$. Then we define $m_{12} = +|A_1' - A_2|$ if the orientation of the side σ_2 coincides with that of the oriented arc $A_1'A_2$ and $m_{12} = -|A_1' - A_2|$ if the orientation of σ_2 coincides with that A_2A_1'.

Evidently $m_{12} = m_{21}$.

Definition. We call the real number m_{12} the *gluing weight* of σ_1 on σ_2 and the positive number $|m_{12}|$ the *gluing weight* of the triangles T_1, T_2.

Example 1. Let L be an ideal polyhedron, and let x, y be two points on two distinct edges σ_1, σ_2 of L. Suppose that $d_L(x, y) > 1$, where d_L is the length pseudo-distance on L.

Consider now a sequence of hyperbolic ideal triangles T_n, $n =$

$1, 2, \dots,$ and denote by s_{1n}, s_{2n} two sides of T_n. For each $n \in N$ we glue T_n to L by identifying isometrically the side s_{kn} of T_n with the edge σ_k of L, $k = 1, 2$. Moreover, for each $n \in N$, we choose the gluing weight of s_{kn} on σ_k, $k = 1, 2$, such that

$$|x - y| = 1/n \quad \text{on } T_n.$$

Then $d_L(x, y) = 0$, but $x \neq y$.

Example 2. Let L be an ideal polyhedron, and suppose that the length pseudo-distance d_L on L is a metric, for example L is finite. Now let σ_1, σ_2 be two edges of L, and let x, y be two points of L with $x \in \sigma_1$, $y \in \sigma_2$ and $d_L(x, y) > 2$.

For every $n \in N$ there is a hyperbolic ideal quadrilateral Q_n such that: If s_{1n}, s_{2n} are opposite sides of Q_n, the distance of s_{1n} from s_{2n} is $1 + 1/n$. Consider the points x', y' on s_{1n}, s_{2n}, respectively, which realize this distance $n = 1, 2, \dots$.

We now glue every Q_n, $n \in N$ on L by identifying isometrically s_{1n}, respectively s_{2n}, with σ_1, respectively σ_2, such that x', respectively y', is identified with x, respectively y. Let K be the ideal polyhedron obtained after gluing all Q_n, $n \in N$ on L. Then d_K is a metric but there is no geodesic segment joining x and y in K.

Given a curve γ on an ideal polyhedron K, we can construct, under a certain condition on γ, a surface by gluing the ideal triangles intersected by γ. We call this surface the developing surface along γ. Before giving the precise definition of the surface, we describe the condition we need γ to satisfy.

Definition. Let $f : [0, 1] \to K$ be a, possibly closed, curve in K. We say that f goes *back and forth* in K (see Figure 1) if there are t_1, t_2 in $[0, 1]$ such that:

(1) $f(t_1), f(t_2)$ belong to the same edge of K,

(2) $f((t_1, t_2))$ lies in the interior of a single ideal triangle of K.

Lemma 1. *Let $g : [0, 1] \to K$ be a curve. We can freely homotop g in K to a curve $f_0 : [0, 1] \to K$ which does not go back and forth in K.*

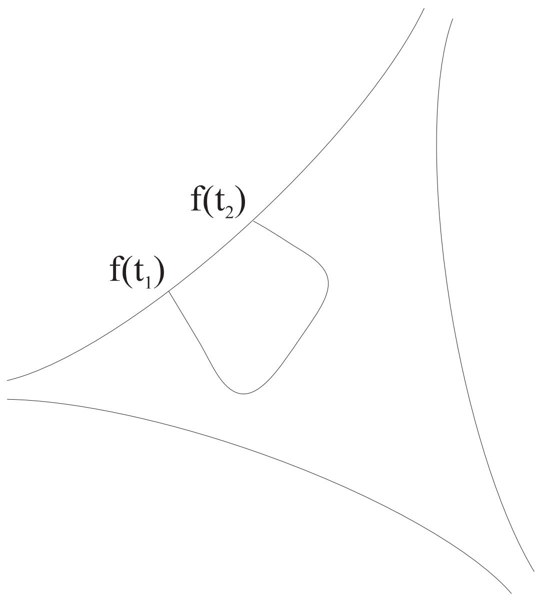

FIGURE 1.

Proof. By a small perturbation of g, we can find a C^∞-curve $f:[0,1]\to K$ which is homotopic to g and which intersects the edges of K transversely in a finite number of points.

Suppose that the curve $f:[0,1]\to K$ goes back and forth in K, and let t_1, t_2 be points of $[0,1]$ such that $f(t_1), f(t_2)$ belong to the same side of an ideal triangle T of K and $f((t_1,t_2))$ lies in the interior of T. Then we have two cases.

Case 1. For an $\varepsilon > 0$ small enough, the points $f(t_1-\varepsilon), f(t_2-\varepsilon)$ belong to the same ideal triangle of K.

Case 2. The points $f(t_1-\varepsilon), f(t_2-\varepsilon)$ belong to different triangles.

In Figure 2 we demonstrate, for each case, how we can do the homotopy from the curve f to a curve f_0 which does not go back and forth in K. $\square$

Now let $f:[0,1]\to K$ be a curve which does not go back and forth in K. Let $\gamma = f([0,1])$ be its image. Assume that f is transverse to

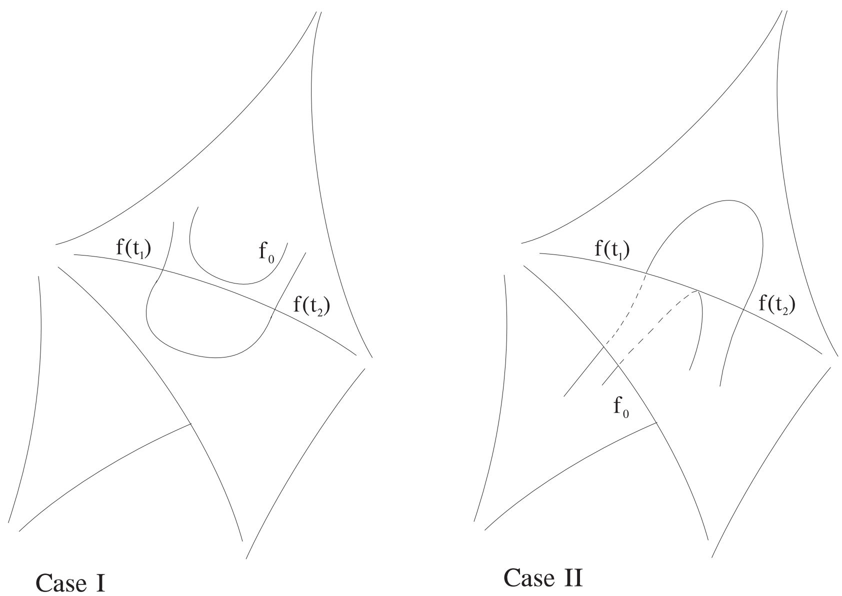

FIGURE 2.

the 1-skeleton of K. There is a finite number of points $t_1, t_2, \dots, t_n$ of $[0,1]$ such that the intervals t_i, t_{i+1}, $i = 1, \dots, n-1$, are disjoint and the points $f(t_i)$ are the only points which belong to the 1-skeleton of K.

In the following, we will associate to the curve γ a surface S consisting of ideal triangles:

Let T_i, $i = 1, \dots, n-1$ be the distinct triangles of K such that $f([t_i, t_{i+1}]) \subset T_i$. We glue together these triangles along their sides as follows: We identify isometrically the side of T_i which contains the point $f(t_i)$ with the side of T_{i+1} which contains the point $f(t_i)$ such that these two points are identified after the gluing. Therefore, we form a surface S which is isometric to a hyperbolic ideal polygon if γ is not closed. If γ is a closed curve, then S is homeomorphic to a ring with a finite number of points removed from each component of its boundary; we call such a surface, S, a double crown (see Figure 3).

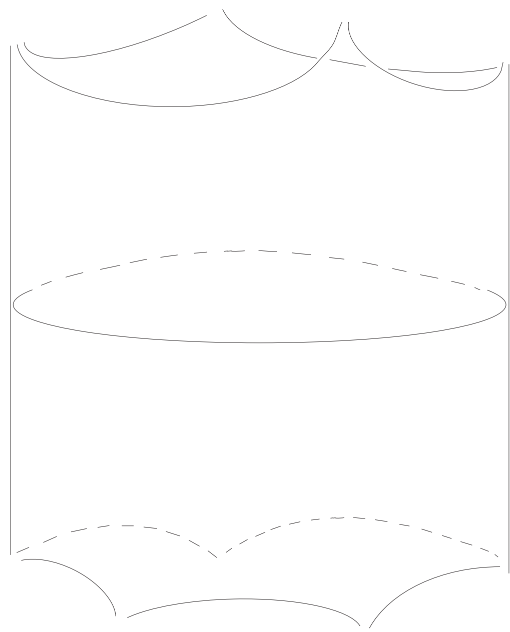

FIGURE 3.

Moreover, there is a natural projection

$$p : S \to K$$

which is a local isometry in the following sense: For every $x \in S$ there is a neighborhood U_x of x and a subset V_y of K with $y = p(x)$, both homeomorphic to a disk such that

$$p : U_x \to V_y$$

is an isometry.

Note that the arcs $f([t_i, t_{i+1}]) \subset Ti$, $i = 1, \ldots, n-1$, form a simple curve γ after the gluing of T_i with $p(\gamma) = \gamma$.

Definition. The surface S is called the *developing surface* associated to the curve γ of K.

We will prove now the following theorem:

Theorem 1. *Let K be an ideal polyhedron. Suppose that K is either locally finite or the gluing weight of the ideal triangles of K form a finite set. Then the length pseudodistance d of K is a geodesic metric.*

Proof. Let x, y points in K with $d(x, y) = 0$. We will prove that $x = y$. If one of the points x, y belongs in the interior of an ideal triangle T of K, say the point x, then there is an $\varepsilon > 0$ such that $B(x, \varepsilon) = \{x' \in K : d(x, x') < \varepsilon\}$ is isometric to a closed disk of the hyperbolic half-space H^2. So $y \in B(x, \varepsilon)$, and since d induced on $B(x, \varepsilon)$ is the usual hyperbolic metric, we get $x = y$.

If x and y belong to the 1-skeleton of K, then by hypothesis there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \cap T$ with the induced metric being isometric to H^2 for every ideal triangle T of K. Therefore, we also get that $x = y$; hence d is a metric.

Let K be locally finite. Then K is locally compact. A theorem of Cohn-Vossen asserts that every locally compact complete length space is geodesic. However, from the proof of this theorem (see [**6**]) it follows that the completeness assumption is only necessary for showing that the closed balls are compact. In our context this is immediate, since K is locally finite. Therefore, (K, d) is geodesic.

Suppose now that the gluing weights of the ideal triangles of K form a finite set. Let x and y be two points in the same connected component of K. Pick a sequence (f_n), $n \in N$, of broken geodesics between them whose lengths converge to $d(x, y)$. Then we may replace f_n by a shorter geodesic arc g_n between x and y for each $n \in N$. In fact, for every $n \in N$, let S_n be the developing surface associated to f_n and $p_n : S_n \to K$ the projection into K. Let f'_n be the broken geodesic in S_n with $p_n(f'_n) = f_n$. In S_n we replace f'_n by the unique geodesic segment g'_n with the same endpoints and let $g_n = p_n(g'_n)$. There exists an n_0 such that for $n > n_0$ the length $l(g_n)$ of g_n satisfies

$$d(x, y) - \varepsilon < l(g_n) < d(x, y) + \varepsilon$$

where ε is arbitrarily enough. Therefore, for $n > n_0$, each geodesic arc g_n belong to a finite number δ_n of ideal triangles of K, since the gluing weights form a finite set. For the same reasons all δ_n are less than a natural number δ; hence, the set $\{\delta_n, n = 1, 2, \dots\}$ is finite. Therefore, the lengths $l(g_n)$ for $n > n_0$ can take a finite number of values. Hence, there is a geodesic segment in K between x and y. $\square$

3. The main application. For the rest of this paper we make the assumption that K is a connected ideal polyhedron of dimension 2 which is either locally finite, or the gluing weights of the ideal triangles of K form a finite set.

We begin with three propositions needed in the proof of the main Theorem 2 and concerning properties of K and of its universal covering $\tilde{K}$.

Proposition 1. *The polyhedron K has curvature less than or equal to -1. (For the definition of the spaces of curvature less than or equal to -1, we refer the reader to* [**9**, Definition 2.9].)

Proof. By a cell C of K we mean either an ideal triangle or an edge of K. For $x \in C$, the link of x in C, denoted by $\mathrm{Link}\,(x, C)$, is the set of local geodesics $f : [0, \infty) \to K$ with $f(0) = x$, $f(t) \in C$ for t small enough. The distance between two geodesics is the angle they form at x. The link of x in K, denoted by $\mathrm{Link}\,(x, K)$, is the union of the $\mathrm{link}\,(x, T)$ for all triangles T of K, where for every common side F of triangles T, T' of K the $\mathrm{Link}\,(x, T)$, $\mathrm{Link}\,(x, T)$ are glued along $\mathrm{Link}\,(x, F)$. Therefore, if x is on the 1-skeleton of K, then $\mathrm{Link}\,(x, K)$ is isometric to the union of half circles of length π, glued along their pairs of endpoints. In particular, if x is on the 1-skeleton of K, then every simple closed curve in $\mathrm{Link}\,(x, K)$ has length equal to 2π. The same is clearly true if x belongs in the interior of an ideal triangle T of K. Therefore, we can adapt the demonstration of a theorem of Gromov (see [**7**, 4.2] or [**9**, Corollary 3.18]) which says that if every simple closed curve in $\mathrm{Link}\,(x, P)$, x in P, where P is an M_{-1} multipolyhedron, has length greater than or equal to 2π, then P has curvature less than or equal to -1 and conversely. $\square$

Let $\tilde{K}$ be the universal covering of K. The space $\tilde{K}$ is simply connected, geodesic and complete. Moreover, $\tilde{K}$ has curvature less than or equal to -1 since $\tilde{K}$ is locally isometric to K. Using the theorem of Cartan-Hadamard-Aleksadrov-Gromov, we deduce that $\tilde{K}$ is a convex space which satisfies the $\mathrm{CAT}(-1)$ inequality globally. Denote by $\partial\tilde{K}$ the boundary of $\tilde{K}$. (For all conclusions and notions just mentioned as well as the theorem of Cartan and Gromov, we refer the reader to [**7,**

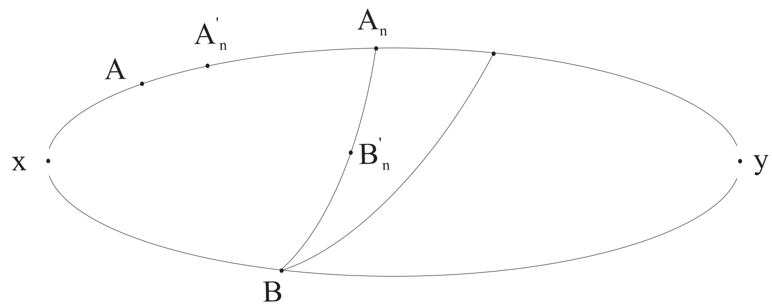

FIGURE 4.

9, 4].

We now need a proposition which is well known if $\tilde{K}$ is a hyperbolic manifold [**8**, Lecture I].

Proposition 2. *For every two points x, y in $\tilde{K} \cup \partial\tilde{K}$, there exists at most a unique geodesic joining these points.*

Proof. Existence of such a geodesic holds true for all proper negatively curved metric spaces. See, for example, [**2**, Chapter 2, Prop. 2.1]. For the uniqueness we distinguish 3 cases:

(1) The points $x, y \in \tilde{K}$. Then the claim is immediate since $\tilde{K}$ is convex.

(2) The point $x \in \tilde{K}$ and the point $y \in \partial\tilde{K}$. Let $g_i : [0, +\infty) \to \tilde{K}$ be two geodesic rays, parametrized by arc length, with $g_i(0) = x$, $g_i(+\infty) = y$, $i = 1, 2$.

Consider the positive function $r(s) = \bar{d}(g_1(s), g_2(s))$, $s \in [0, +\infty)$, where $\bar{d}$ denotes the metric on $\tilde{K}$. A theorem of Gromov [**9**, Theorem 2.19] asserts that this function r is convex. Moreover, $r(0) = 0$ and r is bounded because the geodesics g_1 and g_2 have the same end on $\tilde{K}$. Therefore, $r(s) = 0$ for each s in $[0, +\infty)$, hence g_1 coincides with g_2.

(3) The points $x, y \in \partial\tilde{K}$. Let $g_i : (-\infty, +\infty) \to \tilde{K}$ be two geodesics parametrized by arc length, with $g_i(-\infty) = x$, $g_i(+\infty) = y$, $i = 1, 2$.

Let $A = g_1(0)$, $B = g_2(0)$. We consider a sequence of points $A_n = g_1(s_n)$ with $s_n \to +\infty$. Denote by γ_n the geodesic segments joining B with A_n. Let $A'_n = g_1(s'_n)$, $B'_n = \gamma_n(s'_n)$ where the points s'_n are defined by the relation

$$\bar{d}(B, B'_n) = \bar{d}(A, A'_n) = 1/2 \min(\bar{d}(A, A_n), \bar{d}(B, A_n)).$$

Using the CAT(-1) inequality for the geodesic triangles ABA_n, we get that

$$(*) \qquad \bar{d}(A'_n, B'_n) \to 0 \quad \text{as } n \to +\infty$$

By the theorem of Ascoli, we can pick a subsequence γ_{n_k} of γ_n converging to a geodesic ray $\gamma : [0, +\infty) \to \tilde{K}$ with $\gamma(0) = B$, $\gamma(+\infty) = y$. By the previous case (2) we deduce that $\gamma(s) = g_2(s)$ for each s in $[0, +\infty)$. Therefore, we have

$$(**) \qquad \bar{d}(g_1(s'_n), g_2(s'_n)) \leq \bar{d}(g_1(s'_n), \gamma_n(s'_n)) + \bar{d}(\gamma_n(s'_n), \gamma(s'_n))$$

By $(*)$, $\bar{d}(g_1(s'_n), \gamma_n(s'_n)) \to 0$ as $n \to +\infty$, and since γ_{n_k} converges to γ uniformly on compact sets, it follows from $(**)$ that

$$\bar{d}(g_1(s'_n), g_2(s'_n)) \to 0 \quad \text{as } n \to +\infty.$$

Note now that the positive function $r(s) = \bar{d}(g_1(s), g_2(s))$, $s \in (-\infty, +\infty)$, is convex so it is equal to zero. This proves that $g_1(s) = g_2(s)$ for each s in $(-\infty, +\infty)$. □

Remark. In Proposition 2 we have used only that $\tilde{K}$ is convex and satisfies the CAT(-1) inequality.

We need another proposition which is valid because $\tilde{K}$ consists of ideal triangles.

Proposition 3. *If $\gamma : [0, +\infty) \to \tilde{K}$ is a local geodesic which joins the point $\gamma(0) = p$ with a point in the boundary of $\tilde{K}$, then γ is a geodesic ray of $\tilde{K}$. Therefore, every local geodesic of $\tilde{K}$ is a geodesic.*

Proof. Let $t_0 = \sup\{t \in [0, +\infty) : \gamma/[0, t] \text{ is a geodesic segment}\}$. If $t_0 = +\infty$, there is nothing to prove. If $t_0 < +\infty$, it is clear that $\gamma/[0, t_0]$

is a geodesic segment. Then there exists a sequence t_k converging from the right to t_0 such that $\gamma/[0,t_k]$ is not a geodesic segment.

Now let $\gamma_n : [0,l_n] \to \tilde{K}$ be the geodesic segment with

$$\gamma_n(0) = p, \qquad \gamma_n(l_n) = \gamma(t_n) \quad \text{where } l_n = d(\gamma(0),\gamma(t_n)).$$

We extend γ_n everywhere on $[0,+\infty)$ by defining $\gamma_n(t) = \gamma(t_n)$ for every $t > l_n$.

We apply Ascoli's theorem to obtain a subsequence γ_{n_k} of γ_n which converges uniformly on the compact sets to a map $\gamma_0 : [0,+\infty) \to \tilde{K}$, such that:

(i) $\gamma_0/[0,t_0] = \gamma/[0,t_0]$

(ii) $\gamma_0(t) = \gamma(t_0)$ for every $t \geq t_0$.

To establish property (i) it is sufficient to note that $\gamma/[0,t_0]$ is a geodesic segment for every n in N.

To prove property (ii) note that, for every $t, t > t_0$, there exists a natural number $n_0(t)$ such that:

$$\text{If } n > n_0(t) \quad \text{then } \gamma_n(t) = \gamma_n(l_n).$$

Therefore, for $t > t_0$,

$$\gamma_0(t) = \lim_k \gamma_{n_k}(t) = \lim_k \gamma_{n_k}(l_{nk}) = \lim_k \gamma(t_{n_k}) = \gamma(t_0).$$

These two properties imply that for every $\varepsilon > 0$ there exists a κ sufficiently large such that $\gamma_{n_\kappa}([0,l_{n_\kappa}])$ belongs to an ε-neighborhood of $\gamma([0,t_{n_\kappa}])$. In fact, for each $\varepsilon > 0$ there exists a μ such that $\gamma_{n_\mu}/[0,l_{n_\mu}]$ is ε-near to $\gamma_0/[0,l_{n_\mu}]$ because of the uniform convergence of the sequence γ_{n_k} to γ_0, on the compact sets. Finally we have seen (Property (ii) above) that $\gamma([0,t_0]) = \gamma_0([0,l_{n_k}])$ for each k in N. So if we put $\kappa = \max(\mu,\lambda)$ we get our claim.

Note that the local geodesic γ does not go back and forth in $\tilde{K}$. Therefore, we can associate to the curve $\gamma/[0,t_{n_\kappa}]$ a developing surface denoted by S. We remark also that $\gamma_{n_\kappa}/[0,l_{n_\kappa}]$ is a geodesic segment of $\tilde{K}$ so it does not go back and forth in $\tilde{K}$. As $\gamma_{n_\kappa}([0,l_{n_\kappa}])$ belongs to an ε-neighborhood of $\gamma([0,t_{n_\kappa}])$ for ε sufficiently small, we deduce that the developing surface associated to $\gamma_{n_\kappa}/[0,l_{n_\kappa}]$ is the same surface S.

If $p : S \to \tilde{K}$ is the projection into $\tilde{K}$, we denote by $\gamma([0, t_{n_\kappa}])$, respectively, $\gamma_{n_\kappa}([0, l_{n_\kappa}])$, the curve on S with $p(\gamma/[0, t_{n_\kappa}]) = \gamma/[0, t_{n_\kappa}]$, respectively, $p(\gamma_{n_\kappa}/[0, l_{n_\kappa}]) = \gamma_{n_\kappa}/[0, l_{n_\kappa}]$. But S is a part of H^2 so $\gamma/[0, t_{n_\kappa}]$ coincides with $\gamma_{n_\kappa}/[0, l_{n_\kappa}]$ and therefore the same is true for $\gamma/[0, t_{n_\kappa}]$ and $\gamma_{n_\kappa}/[0, l_{n_\kappa}]$.

Therefore, $\sup\{t \in [0, +\infty)$: is a geodesic segment$\}$ is strictly greater than t_0, and this gives a contradiction. So $t_0 = +\infty$. □

Definition. A closed curve γ in K is *homotopic to a cusp*, if it is not homotopic to a point and $\inf\{\text{length}(\alpha)$ where α varies over all curves which are freely homotopic to $\gamma\} = 0$.

We are now in a position to prove the theorem:

Theorem 2. *Let K be a complete ideal polyhedron of dimension* 2*. Suppose that either K is locally finite or the gluing weights of the ideal triangles of K form a finite set. Let α be a closed curve in K which is not homotopic to a point or to a cusp in K. Then there is a unique geodesic α_0 in the free homotopy class of α. Moreover, the length of α_0 realizes the minimum of the lengths in the free homotopy class of α.*

Proof. For the existence. Lemma 1 asserts that the curve α is freely homotopic to a curve γ which does not go back and forth in K. Let S be the developing surface associated to γ, and let $p : S \to K$ be the natural projection. Since γ is a closed curve, S is a double crown. We denote by γ' the simple closed curve on S with $p(\gamma') = \gamma$. The curve γ' is not homotopic to a cusp in S, otherwise the curve γ would be homotopic to a cusp in K. Therefore, there exists a unique closed geodesic α'_0 in S which is freely homotopic to γ'. Let $\alpha_0 = p(\alpha'_0)$. Since p is a local isometry, α_0 is a closed geodesic in K.

For the uniqueness. Let α_1 be another closed geodesic freely homotopic to α_0 in K, with free homotopy $F : S^1 \times I \to K$. Let $\bar{F} : R \times I \to \tilde{K}$ be the lifting of F to the universal covering $\tilde{K}$, and let $\bar{\alpha}_0 = \bar{F}(R \times 0)$, $\bar{\alpha}_1 = \bar{F}(R \times 1)$. The curves $\bar{\alpha}_0, \bar{\alpha}_1$ are local geodesics of $\tilde{K}$, and by Proposition 3 they are geodesics of $\tilde{K}$. Since S^1 is compact, there exists an upper bound c of the lengths of the arcs $F(z \times I)$, $z \in S^1$. Therefore, the arcs $\bar{F}(\bar{z} \times I)$, $\bar{z} \in R$ are also uniformly bounded by c,

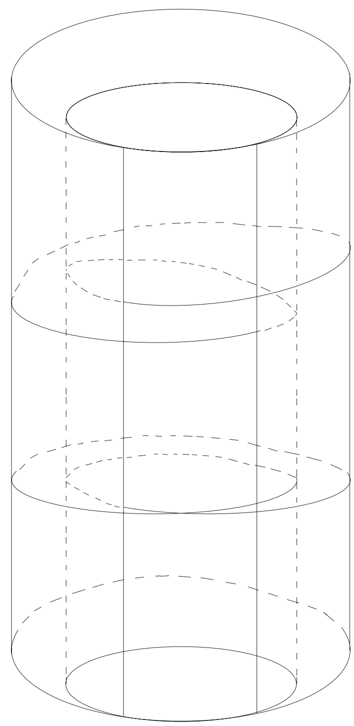

FIGURE 5.

hence $\bar{\alpha}_0, \bar{\alpha}_1$ join the same points on the boundary of $\tilde{K}$. Therefore, the geodesics $\bar{\alpha}_0, \bar{\alpha}_1$ coincide in $\tilde{K}$ and consequently α_0, α_1 coincide in K.

It remains to prove now that α_0 realizes the minimum of lengths in the free homotopy class of α. Let β be another curve, freely homotopic to α, with

$$\text{length}\,(\beta) < \text{length}\,(\alpha_0).$$

Then β is not a geodesic, otherwise β must coincide with α_0. Since β is not a geodesic then there exists a geodesic β_0, freely homotopic to β, with

$$\text{length}\,(\beta_0) < \text{length}\,(\beta)(< \text{length}\,(\alpha_0)).$$

But β_0 must coincide with α_0 and we get a contradiction. The proof of the theorem is now complete. □

Remark. It is possible (see Figure 5), that the unique geodesic α_0 which exists in the free homotopy class of a simple closed curve α in an ideal (more generally geometric) polyhedron K, is not necessarily simple. Therefore, the problem arises of studying the "complexity" of the geodesic which exists in the homotopy class of a simple closed curve in K (see also [**5, 3**]).

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Agricultural University of Athens, Department of Mathematics, 75 Iera Odos, Botanicos, 118 55 Athens, Greece