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A NEW SUFFICIENT CONDITION FOR THE DENSENESS OF NORM ATTAINING OPERATORS

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ABSTRACT. We give a new sufficient condition for a Banach space Y to satisfy Lindenstrauss's property B, namely the set of norm-attaining operators from any other Banach space X into Y is dense. Even in the finite-dimensional case, our result gives new examples of Banach spaces with property B.

Introduction. As a special case of the Bishop-Phelps theorem [**5, 6**], the set of norm-attaining functionals on a Banach space is dense in the dual space for the norm topology. In their earlier paper [**5**], Bishop and Phelps addressed the question of what Banach spaces might play the role of the scalar field in their theorem. Intensive research on this question was initiated by J. Lindenstrauss [**15**], who introduced the so-called property B. Given two Banach spaces X and Y, let us consider the Banach space $L(X,Y)$ of bounded linear operators from X into Y, and let us denote by $NA(X,Y)$ the set of norm-attaining operators. Thus, $T \in NA(X,Y)$ means that there is some $x \in S_X$ (the unit sphere of X) such that $||Tx|| = ||T||$. The Banach space Y is said to satisfy property B if $NA(X,Y)$ is dense in $L(X,Y)$ for all Banach spaces X. Therefore, the Bishop-Phelps theorem gives that the scalar field has property B. We are not concerned in this paper with the corresponding property A (when Y is taken as the domain space) also introduced by Lindenstrauss in [**15**]. The interested reader may consult the papers by J. Bourgain [**7**], C. Stegall [**19**] and W. Schachermayer [**18**]. Unlike property A, present knowledge of property B is far from being satisfactory. Let us give a brief outline of known results and some open problems.

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Lindenstrauss gave a sufficient condition for property B which was later called property β. The Banach space Y has *property* β if there is a set

$$\{(y_\lambda^*, y_\lambda) : \lambda \in \Lambda\} \subset Y^* \times Y$$

and a constant $\rho < 1$ satisfying

(i) $||y_\lambda^*|| = ||y_\lambda|| = y_\lambda^*(y_\lambda) = 1$ for $\lambda \in \Lambda$.

(ii) $|y_\lambda^*(y_\mu)| \leq \rho$ for λ, $\mu \in \Lambda$, $\lambda \neq \mu$.

(iii) $||y|| = \sup\{|y_\lambda^*(y)| : \lambda \in \Lambda\}$ for $y \in Y$.

For instance, the space c_0 clearly satisfies β (with $\rho = 0$) and a finite-dimensional real normed space has property β if and only if its unit ball is a polyhedron. The proof by J. Lindenstrauss that β implies B is a successful application of the Bishop-Phelps theorem. As shown by J. Partington, property β entrains no isomorphic restriction on a Banach space, since *any* Banach space can be equivalently renormed to satisfy β [**16**]. On the other hand, by refining the arguments in a celebrated paper of J. Bourgain [**7**], R. Huff proved that any Banach space failing the Radon-Nikodym property can be equivalently renormed to fail B [**11**]. Actually, apart from the obvious one-dimensional case, no Banach space is known to satisfy property B in every equivalent renorming. For instance, the "irritating" question [**13**] if $\mathbf{R}^2$ with Euclidean norm has property B seems to be still open. Thus, even in the finite-dimensional case, to decide if a concrete Banach space has property B may be a hard task. Concerning negative results, Lindenstrauss already proved that norm-attaining operators from c_0 into a strictly convex Banach space must have finite rank, so a strictly convex Banach space fails B as soon as there is a noncompact operator from c_0 into it [**15**]. The proof by W. Gowers that l_p fails property B for $1 < p < \infty$ runs in a similar way, the space c_0 is useless in this case but Gowers finds a suitable substitute in a predual of a Lorentz space [**10**]. A slight refinement of Gowers' result can be found in [**2**]. The question if every finite-dimensional space satisfies property B is probably the most relevant open question concerning norm-attaining operators.

The original motivation for the research done in this paper came from a very simple observation. One can easily prove that property B is stable under c_0-sums, but a c_0-sum of spaces with β fails to satisfy β unless the constants ρ in the definition of β can be taken bounded away from 1. Thus, it was natural to look for a property weaker than β, still implying B and stable under c_0-sums. Of course, the new property should be satisfied by some spaces which are not isometric to c_0-sums of spaces with β. Let us explain how to find such a property, which we call *quasi-β*. By using the Hahn-Banach theorem and the "reversed" Krein-Milman theorem (see [**12**] for example) one sees that condition (iii) in the definition of β holds if and only if every extreme point of the unit ball B_{Y^*} lies in the w^*-closure of the set $\{ty_\lambda^* : \lambda \in \Lambda, |t| = 1\}$. Thus, we can get a weaker property if we only require a "local" version of condition (ii), more concretely, the estimate in (ii) is only required for those functionals y_λ^* which are actually needed to approximate each extreme point in B_{Y^*} (see below for the precise definition).

Our proof that quasi-β implies B, the main result in this paper, is a nontrivial improvement of the above mentioned argument by J. Lindenstrauss for property β. We use the result by V. Zizler that the set of operators between arbitrary Banach spaces whose adjoints attain their norm is dense [**20**], combined with the fact that the norm of a w^*-continuous operator between dual spaces must be attained at an extreme point provided that it is attained somewhere in the unit ball.

The well-known description of the extreme points in the unit ball of an l_1-sum allows an easy proof that property quasi-β is stable under c_0-sums. Moreover, there are Banach spaces with property quasi-β which are not isometric to c_0-sums of spaces with β. We give two examples of this kind. Surprisingly enough (at least it was surprising to the authors) there are even finite-dimensional spaces satisfying property quasi-β and not β. Furthermore, the space used by Gowers to show that l_p fails property B happens to be a natural example of an infinite dimensional space with property quasi-β which fails β and cannot be decomposed into a nontrivial c_0-sum of Banach spaces, actually its dual space has no nontrivial L-summands.

Let us start with the precise definition of property quasi-β:

Definition 1. We will say that a Banach space Y has *property quasi-β* if there exist a subset $A \subset S_{Y^*}$, a mapping $\sigma : A \to S_Y$ and a real-valued function ρ on A satisfying the following conditions:

(i) $y^*(\sigma(y^*)) = 1$ for $y^* \in A$.

(ii) $|z^*(\sigma(y^*))| \leq \rho(y^*) < 1$ for $y^*, z^* \in A$, $y^* \neq z^*$.

(iii) For every extreme point e^* in the unit ball of Y^*, there is a subset A_{e^*} of A and a scalar t with $|t| = 1$ such that te^* lies in the w^*-closure of A_{e^*} and $\sup\{\rho(y^*) : y^* \in A_{e^*}\} < 1$.

Sometimes, if it is necessary to be more precise, we say that Y satisfies property quasi-$\beta(A, \sigma, \rho)$.

If Y has property β (see the introduction) for a set $\{(y_\lambda^*, y_\lambda) : \lambda \in \Lambda\}$ in $Y^* \times Y$ and certain constant $0 \leq \rho < 1$, we know already that every extreme point of B_{Y^*} lies in the w^*-closure of the set $\{ty_\lambda^* : \lambda \in \Lambda, |t| = 1\}$, so Y has property quasi-$\beta(A, \sigma, \rho)$ where $A = \{y_\lambda^* : \lambda \in \Lambda\}$, $\sigma(y_\lambda^*) = y_\lambda$ for $\lambda \in \Lambda$ and ρ is constant. Note also that condition (iii) in the definition of property quasi-β clearly implies that the set A is norming, that is,

$$||y|| = \sup\{|y^*(y)| : y^* \in A\}, \qquad \forall\, y \in Y.$$

It follows, for any Banach space X and any operator $T \in L(X, Y)$, that

$$||T|| = \sup\{||T^* y^*|| : y^* \in A\},$$

a fact that will be used without comment. We can now state the main result in this paper.

Theorem 2. *Property quasi-β implies property B. More explicitly, if X, Y are Banach spaces and Y satisfies property quasi-β, then the set of norm-attaining operators from X into Y is norm-dense in the operator space $L(X, Y)$.*

Proof. Let X and Y be Banach spaces and assume that Y has property quasi-$\beta(A, \sigma, \rho)$.

First of all, we apply the result by V. Zizler [**20**, Proposition 4] that the set

$$\{T \in L(X, Y) : T^* \in NA(Y^*, X^*)\}$$

is dense in $L(X, Y)$. Thus, we are left with showing that for every T in this set, with $||T|| = 1$, and arbitrary $\varepsilon > 0$ there is an operator $S \in NA(X, Y)$ such that $||T - S|| < \varepsilon$.

By a result due to T. Johannesen (see [**14**, Theorem 5.8]), T^* attains its norm at an extreme point e^* of B_{Y^*} and the definition of property quasi-β gives us a subset $A_{e^*} \subseteq A$ and a scalar t with $|t| = 1$ such that te^* lies in the w^*-closure of A_{e^*} and

$$r := \sup\{\rho(y^*) : y^* \in A_{e^*}\} < 1.$$

From this point on, our argument is analogous to the one used by Lindenstrauss for property β [**15**, Proposition 3]. We fix $0 < \gamma < \varepsilon/2$ satisfying

$$1 + r\left(\frac{\varepsilon}{2} + \gamma\right) < \left(1 + \frac{\varepsilon}{2}\right)(1 - \gamma)$$

and use that $1 = ||T^*(te^*)|| = \sup\{||T^*y^*|| : y^* \in A_{e^*}\}$ to find a $y_1^* \in A_{e^*}$ such that $||T^*y_1^*|| > 1 - \gamma$.

By the Bishop-Phelps theorem we can choose $z^* \in X^*$ attaining its norm and satisfying

$$||z^*|| = ||T^*y_1^*|| > 1 - \gamma, \qquad ||z^* - T^*y_1^*|| < \gamma.$$

Now we define the operator S by

$$S(x) = T(x) + \left[\left(1 + \frac{\varepsilon}{2}\right)z^*(x) - T^*y_1^*(x)\right]y_1, \qquad x \in X,$$

where $y_1 = \sigma(y_1^*)$. Then

$$||S - T|| \leq \frac{\varepsilon}{2}||z^*|| + ||z^* - T^*y_1^*|| \leq \frac{\varepsilon}{2} + \gamma < \varepsilon$$

and we will prove that S attains its norm. Since

$$S^*y^* = T^*y^* + y^*(y_1)\left(\frac{\varepsilon}{2}z^* + z^* - T^*y_1^*\right) \qquad \forall\, y^* \in Y^*,$$

for $y^* \in A$, $y^* \neq y_1^*$, we have

$$||S^*y^*|| \leq 1 + \rho(y_1^*)\left(\frac{\varepsilon}{2} + \gamma\right) \leq 1 + r\left(\frac{\varepsilon}{2} + \gamma\right)$$

while, for $y^* = y_1^*$, we get $S^* y_1^* = (1+\varepsilon/2) z^*$, so

$$\begin{aligned} ||S^* y_1^*|| &= \left(1+\frac{\varepsilon}{2}\right)||z^*|| \\ &> \left(1+\frac{\varepsilon}{2}\right)(1-\gamma) \\ &> 1 + r\left(\frac{\varepsilon}{2}+\gamma\right). \end{aligned}$$

Therefore, $||S^*|| = ||S^* y_1^*||$, but $S^* y_1^*$ is a multiple of z^*, so it attains its norm as a functional on X, hence S attains its norm. $\square$

In what follows we briefly discuss the stability of property quasi-β and provide some suggestive examples. The following simple observation motivated our search for such a property.

Proposition 3. *Property* B *is stable under* c_0*-sums.*

Proof. Let Λ be an arbitrary nonempty set and $Y = (\oplus_{\lambda\in\Lambda} Y_\lambda)_{c_0}$ where the Banach spaces Y_λ have property B. For $T \in L(X,Y)$ with $||T|| = 1$, let us write $T_\lambda = P_\lambda T$ where P_λ is the natural projection of Y onto Y_λ for each λ. Given $\varepsilon > 0$, let $\alpha \in \Lambda$ be such that $||T_\alpha|| > 1 - \varepsilon/2$ and use property B of Y_α to find an operator $S_\alpha \in NA(X, Y_\alpha)$ such that $||S_\alpha|| = 1$ and $||S_\alpha - T_\alpha|| < \varepsilon$. If $S \in L(X,Y)$ is such that $P_\lambda S = T_\lambda$ for $\lambda \neq \alpha$ and $P_\alpha S = S_\alpha$, it is clear that $||S - T|| < \varepsilon$ and $S \in NA(X,Y)$. $\square$

Property quasi-β enjoys the same stability.

Proposition 4. *Property quasi-*β *is stable under* c_0*-sums.*

Proof. With the same notation as in the previous proof, let us now assume that each Banach space Y_λ actually satisfies property quasi-$\beta(A_\lambda, \sigma_\lambda, \rho_\lambda)$ for each $\lambda \in \Lambda$. Let I_λ be the natural embedding of Y_λ into Y and taking into account the standard identification $Y^* = (\oplus_{\lambda\in\Lambda} Y_\lambda^*)_{l_1}$ note that P_λ^* is in its turn the natural embedding of Y_λ^* into Y^*. An extreme point of B_{Y^*} must be of the form $P_\lambda^* e_\lambda^*$ for some

$\lambda \in \Lambda$ and some extreme point e_λ^* in $B_{Y_\lambda^*}$. With this observation in mind, it is straightforward to verify that Y has property quasi-$\beta(A, \sigma, \rho)$ where $A = \cup_{\lambda\in\Lambda} P_\lambda^* A_\lambda$ and

$$\sigma(P_\lambda^* y_\lambda^*) = I_\lambda(\sigma_\lambda(y_\lambda^*)), \qquad \rho(P_\lambda^* y_\lambda^*) = \rho_\lambda(y_\lambda^*)$$

for any $y_\lambda^* \in A_\lambda$ and $\lambda \in \Lambda$. □

It is not difficult to give an example of a c_0-sum of spaces with property β (even two-dimensional spaces) which fails β. We now give examples of spaces with property quasi-β which are not isometric to c_0-sums of spaces with β. The first one is even finite-dimensional.

Example 5. Let us write $a_n = (\sin \pi/2^n, \cos \pi/2^n, 0)$. Consider the set

$$A = \{a_n : n \in \mathbf{N}\} \cup \{(0,1,1), (0,1,-1)\} \subseteq \mathbf{R}^3,$$

and let Y be $\mathbf{R}^3$ provided with the norm such that $B_{Y^*} = \mathrm{co}\,(A \cup -A)$. Note that the set of extreme points in B_{Y^*} is $A \cup -A$ so that the third condition in the definition of property quasi-β will be trivially satisfied. Now one can easily verify that Y has property quasi-$\beta(A, \sigma, \rho)$ where

$$\sigma(a_n) = a_n \qquad \forall\, n \in \mathbf{N},$$
$$\sigma(0, 1, \pm 1) = (0, 1/2, \pm 1/2)$$

and

$$\rho(a_n) = \cos \pi/2^{n+1} \qquad \forall\, n \in \mathbf{N},$$
$$\rho(0, 1, \pm 1) = 1/2.$$

Also Y fails property β because its unit ball is not a polyhedron.

Let us remark that Y cannot be decomposed into a nontrivial direct sum with maximum norm, equivalently Y^* has no nontrivial L-summands. If $Y^* = U \oplus V$ with $||u+v|| = ||u|| + ||v||$ for all $u \in U$ and $v \in V$, then every extreme point of B_{Y^*} must be in $U \cup V$ so $A \subseteq U \cup V$. If U (say) were one-dimensional, then V would contain three linearly independent points of A, a contradiction.

The above example is clearly inspired in the well-known construction of a three-dimensional normed space such that the set of extreme points in its unit ball is not closed. Actually it is not difficult to show that a finite dimensional real normed space has property quasi-β if and only if the set of extreme points in the dual unit ball is a discrete topological space. In particular, quasi-β and β are equivalent for two-dimensional real spaces. Instead of going farther in this line, we prefer discussing a somehow more natural infinite dimensional example.

We need the following lemma which amounts to a characterization of the set A appearing in the definition of property quasi-β. Recall that an element y^* in the unit sphere of a dual Banach space Y^* is said to be strongly w^*-exposed by $y \in S_Y$ if $y^*(y) = 1$ and $||y_n^* - y^*|| \to 0$ for any sequence $\{y_n^*\}$ in B_{Y^*} such that $y_n^*(y) \to 1$.

Lemma 6. *Let Y be a Banach space with property quasi-$\beta(A, \sigma, \rho)$. Then each $y^* \in A$ is strongly w^*-exposed by $\sigma(y^*)$. Conversely, if $y^* \in S_{Y^*}$ is strongly w^*-exposed, then there is a scalar t with $|t| = 1$ such that $ty^* \in A$.*

Proof. Since the absolutely convex hull $\tilde{A}$ of A is w^*-dense in B_{Y^*}, to prove the first assertion it is enough to show that, for given $y^* \in A$ and $\varepsilon > 0$, there is a $\gamma > 0$ such that $||z^* - y^*|| \leq \varepsilon$ whenever $z^* \in \tilde{A}$ and $|1 - z^*(\sigma(y^*))| < \gamma$. We check that $\gamma = \varepsilon(1 - \rho(y^*))/2$ works. Let us write $z^* = \sum_{k=1}^n t_k y_k^*$ with $y_k^* \in A$, $\sum_{k=1}^n |t_k| \leq 1$ and assume without loss of generality that $y_1^* = y^*$. Then, from

$$\begin{aligned}
\gamma &> |1 - z^*(\sigma(y^*))| \\
&= \left|(1 - t_1) - \sum_{k=2}^n t_k y_k^*(\sigma(y^*))\right| \\
&\geq |1 - t_1| - \rho(y^*) \sum_{k=2}^n |t_k| \\
&\geq |1 - t_1| - \rho(y^*)(1 - |t_1|) \\
&\geq |1 - t_1|(1 - \rho(y^*))
\end{aligned}$$

we deduce

$$||z^* - y^*|| \leq 2|1 - t_1| \leq \varepsilon.$$

For the converse, let $e^* \in S_{Y^*}$ be strongly w^*-exposed by $y \in S_Y$. Since e^* is an extreme point of B_{Y^*} and $e^*(y) = 1$, the definition of property quasi-β gives a sequence $\{y_n^*\}$ in A such that $|y_n^*(y)| \to 1$ and $r = \sup \rho(y_n^*) < 1$. Passing to a subsequence if necessary and up to a rotation, we can assume that $y_n^*(y) \to 1$ so that $||y_n^* - e^*|| \to 0$. However, for $n \neq m$, we have

$$||y_m^* - y_n^*|| \geq |1 - y_n^*(\sigma(y_m^*))| \geq 1 - r > 0$$

so the sequence $\{y_n^*\}$ is eventually constant, and $e^* \in A$ (up to rotation). $\square$

Example 7. We recall the definition of a Banach space considered by W. Gowers in [**10**]. For a sequence y of real numbers and each natural number n, let us write

$$\Phi_n(y) = \frac{1}{H_n} \sup \left\{ \sum_{j \in J} |y(j)| : J \subseteq \mathbf{N}, |J| = n \right\}$$

where $|J|$ is the cardinality of the set J and $H_n = \sum_{k=1}^{n} k^{-1}$. We will denote by G the Banach space of those sequences y satisfying that

$$\lim_{n \to \infty} \Phi_n(y) = 0$$

with norm given by

$$||y|| = \sup\{\Phi_n(y) : n \in \mathbf{N}\}, \qquad y \in G.$$

Let $\{e_n\}$ be the unit vector basis of G, and denote by $\{e_n^*\}$ the sequence of biorthogonal functionals, which is a basis for G^*. Actually, G^* can be identified with the Lorentz sequence space $d(\{1/n\}, 1)$, more concretely the norm of G^* is given by

$$||y^*|| = \sup_{\pi} \left\{ \sum_{n=1}^{\infty} \frac{|y^*(e_{\pi(n)})|}{n} \right\}$$

where the supremum is taken over all permutations π of $\mathbf{N}$. We will show that G has property quasi-β. Let A be the set of those $y^* \in S_{G^*}$ which are of the form

$$y^* = \frac{1}{H_n} \sum_{j \in J} s_j e_j^* \tag{1}$$

where $J \subseteq \mathbf{N}$, $n = |J|$, $s_j = \pm 1$ for $j \in J$ and $s_k = 1$ for $k = \min J$. The set of extreme points of B_{G^*} is $A \cup -A$ (see [**9**, Section 3, Theorem 1], for example). We define a mapping σ on A by

$$\sigma(y^*) = \frac{H_n}{n} \sum_{j \in J} s_j e_j$$

provided that $y^* \in A$ is given by (1). It is easy to check that

$$y^*(\sigma(y^*)) = ||\sigma(y^*)|| = 1 \qquad \forall\, y^* \in A,$$

and we are left with showing that $\rho(y^*) < 1$ for every $y^* \in A$, where

$$\rho(y^*) = \sup\{|z^*(\sigma(y^*))| : z^* \in A, z^* \neq y^*\}.$$

So let $y^* \in A$ be given by (1), and let

$$z^* = \frac{1}{H_m} \sum_{i \in I} t_i e_i^*$$

be the analogous expression for $z^* \in A$ with $z^* \neq y^*$. Then

$$|z^*(\sigma(y^*))| \leq \frac{H_n}{nH_m} |I \cap J|$$

and it is not difficult to deduce from this inequality that

$$|z^*(\sigma(y^*))| \leq \frac{H_n}{H_{n+1}}$$

provided that $I \neq J$. If $I = J$ we have $s_k = t_k = 1$ for $k = \min J$ and $s_j = -t_j$ for some $j \in J$, so

$$|z^*(\sigma(y^*))| \leq \frac{n-2}{n} \leq \frac{H_n}{H_{n+1}}.$$

Thus $\rho(y^*) = H_n / H_{n+1} < 1$ as required.

Next we see that G fails property β. Suppose on the contrary that G has property β for a certain subset $\{(y_\lambda^*, y_\lambda) : \lambda \in \Lambda\} \subseteq Y^* \times Y$ and some constant $0 \leq \rho < 1$. By the above lemma, we must have

$\{\pm y_\lambda^* : \lambda \in \Lambda\} = A \cup -A$. Then, for $y^*, z^* \in A$ with $z^* \neq y^*$ we can find $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, such that $y^* = \pm y_\lambda^*$ and $z^* = \pm y_\mu^*$. In any case we have

$$||y^* - z^*|| = ||y_\lambda^* \pm y_\mu^*|| \geq |y_\lambda^*(y_\mu) \pm 1| \geq 1 - \rho > 0.$$

However, for any natural number n we can take

$$y^* = \frac{1}{H_{n+1}} \sum_{k=1}^{n+1} e_k^*, \qquad z^* = \frac{1}{H_n} \sum_{k=1}^{n} e_k^*$$

and we get

$$1 - \rho \leq ||y^* - z^*|| = \frac{1}{H_{n+1}} + \left(\frac{1}{H_n} - \frac{1}{H_{n+1}}\right)(H_{n+1} - 1) \to 0,$$

a contradiction.

Let us finally remark that G^* contains no nontrivial L-summands. Suppose that U and V are complementary L-summands in G^*, and recall that every extreme point of B_{G^*} must belong to $U \cup V$. If $e_1^* \in U$ (say) and $e_n^* \in V$ for some $n > 1$, we would have $||e_1^* + e_n^*|| = 2$, which is clearly false, so $e_n^* \in U$ for all n and $U = G^*$.

Paralleling the investigations on norm attaining operators, the problem of the denseness of numerical radius attaining operators has also received some attention in recent years [**8, 17, 3**]. Property β was found useful also in this context [**1**] and property quasi-β might play some role as well.

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