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RINGS OF ENDOMORPHISMS OF SEMIGROUP-GRADED MODULES

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ABSTRACT. We analyze various types of endomorphism rings of graded modules over rings graded by finite semigroups, and show connections between such rings and certain skew rings. In the specific case of group-graded rings our results will yield the isomorphism theorems of Albu and Năstăsescu.

0. Introduction. In this article we continue the investigation of *rings graded by semigroups* which was begun in [**1**] and [**3**]. Specifically, we investigate various rings of endomorphisms associated with certain types of graded modules. We briefly sketch our course of study in the next few paragraphs.

For a ring R graded by the semigroup S, and graded left R-module ${}_RN$, we define the graded module $U(N)$ to be a direct sum of graded modules of the form $N(f)$ (where $f \in S^* = S - \{z\}$). Our interest in such modules stems from their importance within the category $R-gr$ of S-graded left R-modules; for instance, as shown in [**3**, Corollary 3.2], modules of the form $U(R)$ are often progenerators for $R-gr$. We will analyze the three rings

$$\mathrm{End}_{R-gr}(U(N)), \quad \mathrm{END}_R(U(N)) \quad \text{and} \quad \mathrm{END}_R^{-1}(U(N)).$$

These are, respectively, the ring of graded endomorphisms of $U(N)$, the direct sum of the groups of endomorphisms of $U(N)$ of degree f (where f ranges in S^*), and the direct sum of the groups of endomorphisms of $U(N)$ of degree f^{-1} (where f again ranges in S^*). The

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main thrust of this article is to show that, for a large class of semigroups and rings graded by such semigroups, the rings $\mathrm{END}_R(U(N))$ and $\mathrm{END}_R^{-1}(U(N))$ may be realized as skew semigroup rings over $\mathrm{End}_{R-gr}(U(N))$. These results appear as Theorems 3.3 and 3.6. As one consequence, we obtain generalizations of the results of Albu and Năstăsescu for groups, especially [**4**, Theorem 3.6(1)].

The extension of the group-graded results of Albu and Năstăsescu to the semigroup situation is not a perfect one. In the course of our investigation we will encounter semigroups and/or modules for which the "expected" generalization of the corresponding group-graded result does not hold. As we shall see, a variety of phenomena which are not apparent in the group-graded case come to light in this more general setting. For instance, in the finite group-graded case we have $\mathrm{END}_R(U(N)) = \mathrm{End}_R(U(N)) = \mathrm{END}_R^{-1}(U(N))$, and each is a normalizing extension of the ring $\mathrm{End}_{R-gr}(U(N))$ (see, e.g., [**4**]). For finite semigroup-graded rings, these three endomorphism rings may be different, and they need not in general have the analogous normalizing extension property. Finally, in the group-graded case each of these three endomorphism rings is isomorphic to a full matrix ring over $\mathrm{End}_R(N)$; for semigroup-graded rings these rings yield certain (possibly distinct) types of matrix subrings.

Throughout this article S will denote a semigroup. If S contains a zero, we will always denote it by z, and in this case we denote $S-\{z\}$ by S^*. For simplicity of exposition, *we will always assume that S is finite*; however, the reader will observe that a number of these results remain true more generally. The opposite semigroup of S will be denoted by S^{op}. Unless otherwise indicated, all functions and morphisms will be composed from left to right, so that $f \circ g$ (or simply fg) will mean "first f, then g." The letter A will denote an associative unital ring. We denote by $E(A)$ the collection of ring endomorphisms of A (we do *not* assume that such a function preserves the identity of A), while $\mathrm{Aut}\,(A)$ denotes the ring automorphisms of A. The category of all (unitary) left A-modules will be denoted by $A-mod$. Unless otherwise indicated, the word "module" will always mean "left module."

The semigroup S is called l.i. (for "local identities") in case S^* contains a set of orthogonal idempotents E such that for each $g \in S^*$ there exist $e, e' \in E$ with $ege' = g$. In this case we sometimes denote e by e_g and e' by e'_g. We call S *right* (resp. *left*) $*$-*cancellative* in case for any three elements $f, g, h \in S^*$, if $fh = gh$ (resp. $hf = hg$) $\in S^*$ then $f = g$; S is called $*$-*cancellative* if it is both left and right $*$-cancellative. We call S a *category* in case S is l.i., and for any three elements $f, g, h \in S^*$, if $fg \in S^*$ and $gh \in S^*$, then $fgh \in S^*$.

If A is a unital subring of the unital ring B, then we say that B is a (*finite*) *normalizing extension* of A in case there exists a (finite) set I and a subset $\{b_i\}_{i\in I}$ of B such that $\sum_{i\in I} Ab_i = B$ and $Ab_i = b_iA$ for each $i \in I$.

We make two comments about the presentation of results in this article. First, the proofs of a number of the results are fairly straightforward, and hence will be omitted. Second, at the germane places we will try to explicitly point out the significant differences between the approach taken for group-graded rings and our more general approach.

1. Preliminaries.

We begin by reminding the reader of the definitions of the basic objects under consideration in this article.

Definitions 1.1. Let S be a semigroup. We say that the ring R is *graded by* S (or that R is S-*graded*) if there is a family $\{R_s \mid s \in S\}$ of additive subgroups of R such that $R = \oplus_{s\in S} R_s$ and for each pair $s, t \in S$ we have $R_s \cdot R_t \subseteq R_{st}$. We say that R is $*$-*graded* by S in case R is graded by S, and $R_z = \{0\}$.

If R is a ring graded by S, we say that the left R-module M is *graded by* S (or that M is S-*graded*) if there is a family $\{M_s \mid s \in S\}$ of additive subgroups of M such that $M = \oplus_{s\in S} M_s$ and for each pair $s, t \in S$ we have $R_s \cdot M_t \subseteq M_{st}$. We say that M is $*$-*graded* by S in case M is graded by S, and $M_z = \{0\}$.

If M and M' are S-graded left R-modules, the R-homomorphism $f : M \to M'$ is called a *graded homomorphism* in case $(M_s)f \subseteq M'_s$ for each $s \in S$.

When R is a ring graded by the semigroup S, we denote by $(R, S) - gr$ (or simply by $R - gr$) the category consisting of the S-graded left R-modules and S-graded homomorphisms. We denote by $R - gr_*$ the full subcategory of $R - gr$ consisting of the $*$-graded modules.

We denote by F the "ungrading" functor from $R - gr$ to $R - mod$. A submodule L of a graded left R-module M is a *graded submodule* in case L is a graded left R-module, and the canonical injection $L \to M$ is a graded morphism; i.e., $L_f = M_f \cap L$ for each $f \in S$.

Of course, any group is a semigroup; the above definition is easily seen to agree with the usual definition of group-graded rings and modules in this more specific setting. For additional examples of rings graded by semigroups, see [**2**].

Definitions 1.2. Let R be a ring graded by the finite semigroup S, let M and M' be graded R-modules, and let $f \in S$. An R-linear map $\Lambda : M \to M'$ is said to be a *morphism of degree* f (resp. f^{-1}) in case $(M_g)\Lambda \subseteq M'_{gf}$, respectively, $(M_h)\Lambda \subseteq \sum\{M'_g | gf = h\}$, for all $g, h \in S$. (The empty sum is interpreted as zero when appropriate.)

We denote the set of morphisms from M to M' of degree f by $\mathrm{HOM}_R(M, M')_f$, and we denote the set of morphisms from M to M' of degree f^{-1} by $\mathrm{HOM}_R^{-1}(M, M')_f$. These are easily shown to be additive subgroups of the group of all R-linear maps $\mathrm{Hom}_R(M, M')$. We denote $\sum_{f \in S^*} \mathrm{HOM}_R(M, M')_f$ by $\mathrm{HOM}_R(M, M')$, and set $\mathrm{END}_R(M) = \mathrm{HOM}_R(M, M)$; we denote $\sum_{f \in S^*} \mathrm{HOM}_R^{-1}(M, M')_f$ by $\mathrm{HOM}_R^{-1}(M, M')$, and set $\mathrm{END}_R^{-1}(M) = \mathrm{HOM}_R^{-1}(M, M)$.

We note that if S is an l.i. semigroup with local identities E and M, $M' \in R - gr_*$, then for $e \in E$ it is easy to show that a morphism of degree e or a morphism of degree e^{-1} from M to M' is in fact a graded homomorphism. We now describe a method which associates to any graded module L the $*$-graded module $G(L)$.

Definitions 1.3. Let R be a ring $*$-graded by the semigroup S, and let L be a graded R-module. Then L_z is a graded submodule of L. The quotient $G(L) = L/L_z$ is easily shown to be a $*$-graded left R-module, where for each $t \in S$, we set

$$G(L)_t = (L_t + L_z)/L_z.$$

We let $\pi_L : L \to G(L)$ be the canonical projection; it is easy to show that π_L is a morphism in $R - gr$. For any morphism $\alpha : L \to L'$ in $R - gr$ there is a unique morphism, which we denote by $G(\alpha)$, such that $\alpha \circ \pi_{L'} = \pi_L \circ G(\alpha)$. It is clear that $G(\alpha)$ is in fact a morphism in $R - gr_*$. The assignments $L \mapsto G(L)$ and $\alpha \mapsto G(\alpha)$ give rise to a functor $G : R - gr \to R - gr_*$.

If L is a graded R-module, then the subgroup $\sum_{t \in S^*} L_t$ of L need not be an R-module, as it need not be closed under the R-action (see, e.g., Example 2.2 below). In case $\sum_{t \in S^*} L_t \in R - mod$ we have $\sum_{t \in S^*} L_t \cong G(L)$ in $R - gr_*$, and $L = (\sum_{t \in S^*} L_t) \oplus L_z$ in $R - gr$. We now describe for semigroups a construction analogous to the skew group ring construction for groups. A fuller account of these definitions can be found in [**2**].

Definitions 1.4. Let A be an associative unital ring with identity 1, and let S be a semigroup.

(1) Suppose $\sigma : S^* \to E(A)$ has the property that $(gh)\sigma = (g)\sigma \circ (h)\sigma$ for any pair g, h in S^* for which $gh \in S^*$. Then we say that σ is an *action of S^* as endomorphisms* on A. For $a \in A$ and $h \in S^*$, we denote $(a)(h)\sigma$ by $a^{(h)\sigma}$.

We denote by $S^* *_\sigma A$ the abelian group $\oplus_{s \in S^*} 1^{(s)\sigma} \cdot A_s$ where each $A_s = A$. For $s \in S^*$ and $a \in 1^{(s)\sigma} A$ we denote the element of $S^* *_\sigma A$ which is a in the s-component and zero elsewhere by $s[a]$, or simply sa. We define multiplication in $S^* *_\sigma A$ by setting $ga \cdot hb = (gh) a^{(h)\sigma} b$ whenever $gh \in S^*$, setting $ga \cdot hb = 0$ in case $gh = z$, and extending linearly to all of $S^* *_\sigma A$. Then $S^* *_\sigma A$ is an associative ring, which we call the *skew semigroup ring of S^* with coefficients* in A.

(2) Suppose $\gamma : S^* \to E(A)$ has the property that $(gh)\gamma = (h)\gamma \circ (g)\gamma$ for any pair g, h in S^* for which $gh \in S^*$. Then we say that γ is a *reversing action of S^* as endomorphisms* on A. For $a \in A$ and $h \in S^*$ we denote $(a)(h)\gamma$ by $a^{(h)\gamma}$.

We denote by $A *_\gamma S^*$ the abelian group $\oplus_{s \in S^*} A_s \cdot 1^{(s)\gamma}$ where each $A_s = A$. By defining multiplication in a manner similar to that given above, $A *_\gamma S^*$ becomes an associative ring, which we call the (*reversing*) *skew semigroup ring of S^* with coefficients in A.*

Analogous to the situation for groups, the rings $S^* *_\sigma A$ and $A *_\gamma S^*$ are the prototypical examples of rings graded by the semigroup S, where for each $f \in S^*$ we define the f-graded component by

$$(S^* *_\sigma A)_f = \{f[1^{(f)\sigma}a] \mid a \in A\}$$

and

$$(A *_\gamma S^*)_f = \{[a1^{(f)\gamma}]f \mid a \in A\}.$$

When S is a group and σ (resp. γ) is a map from S into $\mathrm{Aut}\,(A)$, it is easy to show that the rings $S^* *_\sigma A$ and $A *_\gamma S^*$ described above coincide with the usual definitions of skew group rings.

For any semigroup S and any ring A, we define $\iota : S^* \to E(A)$ by setting $(g)\iota = 1_{E(A)}$ for each g in S^*. We form the ring $S^* *_\iota A$, respectively $A *_\iota S^*$, and denote it simply by S^*A (resp. AS^*). This is the usual (contracted) semigroup ring of S with coefficients in A. In the particular setting where A is a field and S is a group, partially ordered set, or directed graph, the corresponding ring S^*A is the standard group algebra, incidence algebra, or path algebra, respectively.

Let S be a semigroup, and let A be a unital ring. If σ is any map from S^* to $E(A)$, we may also view σ as a map from $(S^{op})^*$ to $E(A)$. Then it is easy to show that $\sigma : S^* \to E(A)$ is an action of S^* as endomorphisms on A if and only if $\sigma : (S^{op})^* \to E(A)$ is a reversing action of $(S^{op})^*$ as endomorphisms on A. In particular, we may form $A *_\sigma (S^{op})^*$ whenever we can form $S^* *_\sigma A$.

To conclude this section we note that semigroups with local identities abound. For example, the semigroups arising from directed graphs, partially ordered sets, and the morphisms of a category are l.i. semigroups; in fact, each of these is an l.i. category. (For emphasis, we will often denote a category semigroup by the letter C, rather than the usual S.) As an example of an l.i. semigroup which is not a category, we offer

Example 1.5. Let $T = \{1, \alpha, 2, \beta, 3, z\}$, with $\{1, 2, 3\}$ as idempotents, $\alpha = 1\alpha 2$, $\beta = 2\beta 3$ and $\alpha\beta = z$. Then T is an l.i. semigroup; however, as $\alpha 2 \neq z$, $2\beta \neq z$, but $\alpha 2\beta = z$, T is not a category. This semigroup T may in fact be realized as the quotient $W/\langle\alpha\beta\rangle$, where W is the semigroup arising from the directed graph $\cdot 1 \to \cdot 2 \to \cdot 3$.

2. Shift modules. In this section we define and investigate properties of certain types of graded modules, and homomorphisms between graded modules. As will become apparent, we encounter many somewhat surprising differences between our results and the corresponding group-graded ones.

In certain settings we will need to impose restrictions on semigroups in order to produce germane homomorphisms between graded modules. The two most important such restrictions will be to require that S is right $*$-cancellative (used to define the functions p and j), and that S is a category (used to define the function i). We will attempt to point out, often and quite explicitly, where and why such restrictions are necessary.

Definitions 2.1. Let R be a ring $*$-graded by the semigroup S, and let $f \in S$. For any $N \in R - gr_*$ (specifically, $N_z = 0$) we define the graded left R-module $N[f]$ as follows. We set $N[f] = N$ as left R-modules. We then set

$$N[f]_g = \sum_{\substack{h \in S \\ hf = g}} N_h,$$

for each $g \in S$; we interpret the empty sum as 0. It is easy to show that this gives an S-grading on $N[f]$. We now define the $*$-graded module $N(f)$ by setting

$$N(f) = N[f]/N[f]_z = G(N[f])$$

where G is the functor described in Definitions 1.3. Specifically, for each $g \in S$,

$$N(f)_g = \frac{\sum_{\substack{h \in S \\ hf = g}} N_h + \sum_{\substack{l \in S \\ lf = z}} N_l}{\sum_{\substack{l \in S \\ lf = z}} N_l}.$$

We call $N(f)$ the *f-shift* of N.

We describe some properties of $N[f]$ and $N(f)$ which will be useful later. Let R be a ring $*$-graded by the semigroup S, let $N \in R - gr_*$, and let $f \in S^*$. Suppose that, for every pair $k, h \in S^*$, we have the property: $kh \neq z$ and $hf \neq z$ imply $khf \neq z$. Then it is easy to show that the subgroup $\sum_{t \in S^*} N[f]_t$ of $N[f]$ is closed under the R-action; hence, by an earlier observation, $\sum_{t \in S^*} N[f]_t$ is a graded R-module, isomorphic to $N(f)$. This directly yields that if S is a finite l.i. semigroup having local identities E, then for each $e \in E$ we have $N(e) \cong \sum_{te=t} N[e]_t$ and $N \cong \oplus_{e \in E} N(e)$ in $R-gr$. Finally, if S is a category, then $\sum_{t \in S^*} N[f]_t \in R-mod$ for every $f \in S^*$. However, we verify in the next example that, in general, $\sum_{t \in S^*} N[f]_t$ need not be an R-module.

Example 2.2. Let $T = \{1, \alpha, 2, \beta, 3, z\}$ denote the noncategory l.i. semigroup described in Example 1.5, and let k be any field. Recall that $\alpha 2 = \alpha$, $2\beta = \beta$, and $\alpha\beta = z$. The semigroup ring $R = kT^*$ is graded by T in the obvious way.

First, we note that the graded module $R[\beta]$ is not $*$-graded, as (for instance) $0 \neq k\alpha = R_\alpha \subseteq R[\beta]_z$. Furthermore, $R_2 \subseteq R[\beta]_\beta$ (as $2\beta = \beta$), and $\alpha 2 = \alpha$. Thus, we have $k\alpha = k\alpha \cdot k2 = k\alpha \cdot R_2 \subseteq R_\alpha \cdot R[\beta]_\beta$, so that $R_\alpha \cdot R[\beta]_\beta \neq \{0\}$. However, $R_\alpha \cdot R[\beta]_\beta \subseteq R[\beta]_{\alpha\beta} = R[\beta]_z \not\subseteq \sum_{g \in T^*} R[\beta]_g$. We conclude that $\sum_{g \in T^*} R[\beta]_g$ is not an R-submodule of $R[\beta]$.

Definition 2.3. Let R be a ring graded by the semigroup S. For each graded module $N \in R-gr$ we define

$$U(N) = \bigoplus_{g \in S^*} N(g).$$

In particular, $U(N)$ is the direct sum of S-graded R-modules, hence is S-graded. Clearly, $U(N) \in R-gr_*$ whenever $N \in R-gr_*$. Specifically, for each $k \in S^*$,

$$U(N)_k = \bigoplus_{g \in S^*} N(g)_k = \bigoplus_{g \in S^*} \frac{N[g]_k + N[g]_z}{N[g]_z}.$$

The results of the following proposition are not unexpected; the proof is straightforward and tedious, hence omitted.

Proposition 2.4. *Let R be a ring $*$-graded by the semigroup S.*

(1) *Let $N \in R{-}gr_*$. Then for every pair $f, g \in S$ we have $N(g)[f]_z = N[gf]_z/N[g]_z$, and the canonical projection $N/N[g]_z \to N/N[gf]_z$ yields an isomorphism $\eta_{g,f}(N) : N(g)(f) \to N(gf)$ in $R - gr_*$.*

(2) *Let $(N_i)_{i\in I}$ be a family of S-graded left R-modules. Then for each $f \in S$ we have $(\oplus_{i\in I} N_i)[f]_z = \oplus_{i\in I} N_i[f]_z$, and the canonical map $\oplus_{i\in I} N_i / \oplus_{i\in I} N_i[f]_z \to \oplus_{i\in I}(N_i/N_i[f]_z)$ yields an isomorphism $(\oplus_{i\in I} N_i)(f) \to \oplus_{i\in I} N_i(f)$ in $R - gr_*$.*

In the next few paragraphs we define certain graded homomorphisms which arise naturally in the investigation of modules of the form $U(N)$. These graded morphisms, which we denote by j and p, will play a fundamental role in our main results. In order to avoid possible confusion with the functions i and π which will be defined later in this section, we emphasize that the definitions of j and p are intimately related to the direct sum structure of the graded module $U(N)$. In particular, there is no analog of the functions j and p for arbitrary graded modules. (In contrast, the maps i and π will be defined for all graded modules.)

Given $f, g \in S^*$, we have by Proposition 2.4(1) that $\eta_{g,f}(N) : N(g)(f) \to N(gf)$ is an isomorphism. We let $\bar{\eta}_{g,f}(N) : N(g)(f) \to U(N)$ denote the composition of $\eta_{g,f}(N)$ with the canonical injection from $N(gf)$ into $U(N)$. Now suppose that S is a right $*$-cancellative semigroup; in particular, the collection $\{gf\}_{g\in S^*}$ is distinct. So in this situation the sum $\sum_{g\in S^*} N(gf)$ is direct, and hence is a summand in $R-gr$ of $U(N)$. Therefore, using the maps $\{\bar{\eta}_{g,f}(N) \mid g \in S^*\}$ we get a split monomorphism $\oplus_{g\in S^*} N(g)(f) \to U(N)$. On the other hand, by Proposition 2.4(2) we have $U(N)(f) = (\oplus_{g\in S^*} N(g))(f) \cong \oplus_{g\in S^*} N(g)(f)$. We denote the split monomorphism which is the composition $U(N)(f) \cong \oplus_{g\in S^*} N(g)(f) \to U(N)$ of the two morphisms described above by

$$j(f, U(N)) : U(N)(f) \to U(N).$$

Specifically, the image of $j(f, U(N))$ is the summand $\oplus\{N(h) \mid h = gf \in S^* \text{ for some } g \in S^*\}$ of $U(N)$ in $R - gr$. We let

$$p(f, U(N)) : U(N) \to U(N)(f)$$

denote the canonical splitting map of $j(f, U(N))$ in $R - gr$. So, by definition, we have that $j(f, U(N)) \circ p(f, U(N))$ is the identity on $U(N)(f)$, and that $p(f, U(N)) \circ j(f, U(N))$ is the morphism from $U(N)$ to $U(N)$ which acts as the identity on the image of $j(f, U(N))$ (described above), and is zero elsewhere.

We now develop the definitions of the maps π and i. Unlike the maps p and j, these maps will be defined for general graded modules; however, we will be most interested in these morphisms as they relate to graded modules of the form $U(N)$.

In the group-graded situation, if ${}_RN \in R - gr$, then the graded modules ${}_RN$ and ${}_RN(f)$ are equal as modules; that is, $F(N) = F(N(f))$ where F denotes the "ungrading" functor. In particular, even though N and $N(f)$ need not be isomorphic in $R - gr$, there always exist R-linear isomorphisms between them. In the more general setting of semigroup-graded rings and modules, however, the graded modules N and $N(f)$ need not even be isomorphic as R-modules. As a consequence, there need not be any connection between two graded modules of the form $N(f)$ and $N(g)$. This, in turn, will, among other things, cause a dichotomy between category semigroups and more general semigroups, which will first surface in Proposition 2.5, and which will arise again on a number of occasions throughout this article. However, in certain important situations there are at least reasonable connections between such graded modules, which we discuss in the next proposition.

It will be helpful later to analyze the relationship between the abelian groups

$$\sum_{g \in S^*} N[f]_g = \sum_{\substack{h \\ hf \neq z}} N_h$$

and

$$\sum_{l \in S^*} N[e_f]_l = \sum_{\substack{l \\ le_f = l}} N_l.$$

In any l.i. semigroup S, if $hf \neq z$, then $he_f = h$. Thus, the first sum is always contained in the second. Conversely, if $le_f = l$, then for general semigroups it is possible to have $lf = z$ (e.g., $l = \alpha$, $f = \beta$ in the semigroup T of Example 1.5), so that the terms in the second sum need not appear in the first. However, in case S is a category, the two sums are equal, as $hf \neq z$ if and only if $he_f \neq z$.

Proposition 2.5. *Let R be a ring graded by the finite l.i. semigroup S, let $f \in S^*$, and let $N \in R - gr$. Then $N[e_f]_z \subseteq N[f]_z$, and $F(N(f))$ is an epimorphic image of $F((N(e_f))$. If S is a category, then $N[e_f]_z = N[f]_z$, and $F(N(f)) = F((N(e_f))$.*

Proof. For $h \in S$ having $he_f = z$ we get $hf = he_f f = zf = z$, from which the first inclusion follows. Now using the definition of shift modules, we have

$$N(f) = \frac{\sum_{g \in S^*} N[f]_g + N[f]_z}{N[f]_z}$$

and

$$N(e_f) = \frac{\sum_{l \in S^*} N[e_f]_l + N[e_f]_z}{N[e_f]_z}.$$

For $l \in S^*$ and $m \in N[e_f]_l$ we have $m \in \sum_{he_f = l} N_h = N_l$, which gives $m \in N[f]_{lf}$. We now consider the function $\rho : N(e_f) \to N(f)$, which is the linear extension of the function defined by setting, for each $l \in S^*$ and $m \in N[e_f]_l$,

$$\rho : m + N[e_f]_z \mapsto m + N[f]_z \in N[f]_{lf} + N[f]_z \subseteq N(f).$$

This is well-defined by the above remarks, and is easily shown to be an R-homomorphism. Thus, $F(N(f))$ is an epimorphic image of $F((N(e_f))$ in $R - mod$.

If S is a category, then for each $h \in S$ we have $hf = z$ if and only if $he_f = z$. This immediately yields that $N[e_f]_z = N[f]_z$. But the discussion prior to this proposition yields that the expressions $\sum_{g \in S^*} N[f]_g$ and $\sum_{l \in S^*} N[e_f]_l$ are equal in this case, from which we conclude that $F(N(f)) = F((N(e_f))$. □

When S is a category, the previous result says that for each $f \in S^*$ we have an R-isomorphism from $N(f)$ to $N(e_f)$, as they are equal as R-modules. The next example shows that, in more general settings, it can be the case that the only R-homomorphism from $N(f)$ to $N(e_f)$ is the zero homomorphism. This distinction is the major reason why we will be able to get stronger results for categories than for general semigroups.

Example 2.6. We again analyze the non-category l.i. semigroup $T = \{1, \alpha, 2, \beta, 3, z\}$, and consider the $*$-graded ring $R = kT^*$ for a field k. We note that the subsets $k1$, $k\alpha$ and $k\beta$ are in fact left ideals of R, while $k2$ and $k3$ are not. In addition, the subsets $k1 + k\beta + k3$ and $k1 + k\alpha + k\beta + k3$ are also left ideals of R, which we denote by I and J, respectively.

It is tedious but straightforward to verify that $R[\beta]_z = J$ and $R[\beta]_\beta = k2$, so that $R(\beta)_\beta = R/J$. Furthermore, as $R(\beta)_t = 0$ for all $t \neq \beta$, we conclude that $F(R(\beta)) = R/J$.

By the definition of T we have $2 = e_\beta$. It is easy to verify that $R[2]_z = I$. Also, as $R[2]_\alpha = k\alpha$ and $R[2]_2 = k2$, we have that $R(2)_\alpha = J/I$ and $R(2)_2 = (k2 + I)/I$. Since $R(2)_t = 0$ for the remaining three elements of T^*, we conclude that $F(R(2)) = J/I + (k2 + I)/I = R/I$.

We now claim that there is no nonzero R-homomorphism from $F(R(\beta))$ to $F(R(e_\beta))$; that is, $\mathrm{Hom}_R(R/J, R/I) = 0$. But an easy check yields that $\{r \in R \mid Jr \subseteq I\} \subseteq J$, from which the claim follows by a standard argument.

For future reference we note the following. By an argument similar to the one alluded to in the previous paragraph, one can also show that $\mathrm{Hom}_R(R/J, R) \cong k\beta$ as k-vector spaces via right multiplication. With this in mind, it follows that $\mathrm{HOM}_R^{-1}(R(\beta), R)_3 = k\beta$, which in particular yields $\mathrm{HOM}_R^{-1}(R(\beta), R)_\beta = \{0\}$.

We are now in position to give the definition of the maps π and i. Let R be a ring $*$-graded by the (arbitrary) semigroup S, let $L \in R - gr_*$, and let $f \in S^*$. We denote by $\pi(f, L) : L \to L(f)$ the "composition"

$$\pi(f, L) : L = L[f] \to L(f)$$

where the equality is as R-modules, and the map $L[f] \to L(f)$ is the canonical factor map $\pi_{L[f]}$ in $R - gr$. Then $\pi(f, L)$ is a morphism of degree f from L to $L(f)$. Specifically, if $x \in L_t$ then $(x)\pi(f, L) = x + L[f]_z \in (L_t + L[f]_z)/L[f]_z \subseteq (L[f]_{tf} + L[f]_z)/L[f]_z = L(f)_{tf}$.

Now let R be a ring $*$-graded by the category semigroup C, let $L \in R - gr_*$, and let $f \in C^*$. By Proposition 2.5, we have $F(L(f)) = F(L(e_f))$, and by the remarks after Definition 2.1 we have $L(e_f)$ is a direct summand of L in $R - gr_*$. We denote by $i(f, L) : L(f) \to L$ the "composition"

$$i(f, L) : L(f) = L(e_f) \to L$$

where the equality is as R-modules, and the map $L(e_f) \to L$ is the summand injection in $R - gr_*$. Then $i(f, L)$ is a morphism of degree f^{-1} from $L(f)$ to L. Specifically, if $g \in C^*$ and $x \in L(f)_g$, by definition we have $x = n + L[f]_z$ where $n \in \sum_{hf=g} L_h$. By Proposition 2.5 and the discussion prior to it, in the category case we may then write x (uniquely) as $x = n + L[e_f]_z \in \sum_{hf=g} L(e_f)_h$, where we view $n \in \sum_{he_f=h} L_h$. Then we have $(x)i(f, L) = n \in L$.

We reiterate that the definition of $i(f, L)$ is dependent on the fact that $F(L(f)) = F(L(e_f))$, which need not be true for rings and modules graded by non-category semigroups. (The map $\pi(f, L)$, on the other hand, always exists.) In fact, as $F(L(f))$ is in general a proper quotient of $F(L(e_f))$, there is no canonical map from $L(f) \to L$ which can be used to play the role of $i(f, L)$ in more general settings.

Now let R be a ring $*$-graded by the category C, let $L, L' \in R - gr_*$, and let $f \in C^*$. Then it is easy to show that the map $i(f, L) \circ \pi(f, L)$ is the identity on $L(f)$, and that the map $\pi(f, L) \circ i(f, L)$ is the identity on $\oplus\{L_t \mid tf \in C^*\}$ (and is zero on the remaining components of L). Furthermore, for every $\Lambda \in \mathrm{HOM}_R(L, L')_f$ we have $\pi(f, L) \circ i(f, L) \circ \Lambda = \Lambda$, and for each $\Delta \in \mathrm{HOM}_R^{-1}(L, L')_f$ we have $\Delta \circ \pi(f, L') \circ i(f, L') = \Delta$.

Expanding on the statements made subsequent to Proposition 2.4, we emphasize that the maps $i(f, L)$ and $\pi(f, L)$ have been defined for each $f \in S^*$ and each graded module L, whereas the maps j and p above are defined only for graded modules of the form $U(N)$. Much of the discussion in the remainder of this article will focus on the maps $i(f, L)$ and $\pi(f, L)$ *in the specific setting where* $L = U(N)$. This will allow us to investigate the interplay between maps of the form $j(f, U(N)), p(f, U(N)), i(f, U(N))$, and $\pi(f, U(N))$. As a first example, it is not hard to show that for each $f \in S^*$ and $N \in R - gr_*$ we have $\pi(f, U(N)) \circ j(f, U(N)) \circ p(f, U(N)) = \pi(f, U(N))$.

We now give the fundamental connections between graded homomorphisms, and morphisms of degree f for $f \in S^*$. Let R be a ring $*$-graded by the semigroup S, let $L, L' \in R - gr_*$, and let $f \in S^*$. We define the function $\Phi(f, L, L') : \mathrm{Hom}_{R-gr}(L(f), L') \to \mathrm{HOM}_R(L, L')_f$ (which we denote simply by Φ when the notation permits) by setting, for each $\alpha \in \mathrm{Hom}_{R-gr}(L(f), L')$,

$$(\alpha)\Phi = (\alpha)\Phi(f, L, L') = \pi(f, L) \circ \alpha.$$

It is straightforward to prove that $\Phi = \Phi(f, L, L')$ is an isomorphism of abelian groups. Moreover, if S is a category, then Φ^{-1} is given by the assignment $\Lambda \mapsto i(f, L) \circ \Lambda$.

Similarly, we define the function $\Theta(f, L, L') : \mathrm{HOM}_R^{-1}(L, L')_f \to \mathrm{Hom}_{R-gr}(L, L'(f))$ (which we denote simply by Θ when notation permits) by setting, for each $\Delta \in \mathrm{HOM}_R^{-1}(L, L')_f$,

$$(\Delta)\Theta = (\Delta)\Theta(f, L, L') = \Delta \circ \pi(f, L').$$

As expected, Θ possesses many properties analogous to those of the map Φ; however, even though Θ is defined in the context of any semigroup, many of these properties are valid only when the underlying semigroup is a category. Specifically, let R be a ring $*$-graded by the category C, let $L, L' \in R - gr_*$, and let $f \in C^*$. Then $\Theta = \Theta(f, L, L')$ is an isomorphism of abelian groups, and Θ^{-1} is given by the assignment $\alpha \mapsto \alpha \circ i(f, L')$. Moreover, using the appropriate definitions, it is tedious but routine to show that for any ring R which is $*$-graded by the semigroup S, and for any $L, L' \in R{-}gr_*$, the function $\Theta = \Theta(f, L, L')$ is injective. However, $\Theta(f, L, L')$ need not be an isomorphism for non-categories since, in general, it is not surjective. For example, let T denote the non-category l.i. semigroup of Example 2.6, and let $R = kT^*$. Then by setting $L = R(\beta)$ and $L' = R$ we have seen that $\mathrm{HOM}_R^{-1}(L, L')_\beta = \{0\}$, while $\mathrm{Hom}_{R-gr}(L, L'(\beta)) = \mathrm{Hom}_{R-gr}(R(\beta), R(\beta))$ is obviously nonzero.

With the functions π, i, j and p now in hand, we are finally in a position to describe the fundamental R-endomorphisms of graded modules of the form $U(N)$. These morphisms will provide the framework for the main results of this article (see Theorems 3.3 and 3.6), in which we realize rings of the form $\mathrm{END}_R(U(N))$ and $\mathrm{END}_R^{-1}(U(N))$ as skew semigroup rings over S. The dichotomy between categories and more general semigroups will be quite apparent here.

Let R be $*$-graded by the (arbitrary) right $*$-cancellative semigroup S, and let $N \in R - gr_*$. For each $f \in S^*$, we set

$$\hat{f}_N = \pi(f, U(N)) \circ j(f, U(N)) : U(N) \to U(N).$$

When the associated module N is clear from the context, we denote $\hat{f}_N$ simply by $\hat{f}$. Similarly, let R be $*$-graded by the right $*$-cancellative category C, and let $N \in R - gr_*$. For each $f \in C^*$, we set

$$\tilde{f}_N = p(f, U(N)) \circ i(f, U(N)) : U(N) \to U(N).$$

When the associated module N is clear from the context, we denote $\tilde{f}_N$ simply by $\tilde{f}$.

By using properties of the appropriate functions which have been developed throughout this section, it is easy to verify the following two lemmas.

Lemma 2.7. *Let R be $*$-graded by the right $*$-cancellative semigroup S, and let $N \in R - gr_*$.*

(1) *For each $f \in S^*$, for each $h \in S^*$ having $hf \neq z$, and for each element $x + N[h]_z$ of $N(h)$ we have $(x + N[h]_z)\hat{f}_N = x + N[hf]_z \in N(hf)$. Moreover, $\hat{f}_N$ is identically zero on $\oplus_{tf=z} N(t)$. In particular, this description yields that $\hat{f}_N$ is a morphism of degree f.*

(2) *Given $f, g \in S^*$ we have $\hat{f} \circ \hat{g} = \widehat{fg}$ whenever $fg \in S^*$, and $\hat{f} \circ \hat{g} = 0$ otherwise.*

Lemma 2.8. *Let R be $*$-graded by the right $*$-cancellative category C, and let $N \in R - gr_*$.*

(1) *For each $f \in C^*$, for each $h \in C^*$ with the property that there exists $k \in C^*$ with $h = kf$, and for each element $x + N[h]_z$ of $N(h)$ we have $(x + N[h]_z)\tilde{f}_N = x + N[k]_z \in N(k)$. Moreover, $\tilde{f}_N$ is identically zero on $\oplus_t \{N(t) \mid t \neq lf \text{ for any } l \in C^*\}$. In particular, this description yields that $\tilde{f}_N$ is a morphism of degree f^{-1}.*

(2) *For each* $f, g \in C^*$, $\tilde{g} \circ \tilde{f} = \widetilde{f \circ g}$ *whenever* $fg \in C^*$, *and* $\tilde{g} \circ \tilde{f} = 0$ *otherwise.*

We now explicitly describe the compositions $\hat{f} \circ \tilde{f}$ and $\tilde{f} \circ \hat{f}$. We remind the reader that we need to assume that the underlying semigroup is a category in order to define $i(f, U(N))$, and that we need to assume that the underlying semigroup is right $*$-cancellative in order to define $j(f, U(N))$. Since these functions are utilized in the definitions of $\hat{f}$ and $\tilde{f}$, we will need to restrict our attention to right $*$-cancellative categories when analyzing $\hat{f}$ and $\tilde{f}$ simultaneously.

Proposition 2.9. *Let* R *be* $*$*-graded by the right* $*$*-cancellative category* C, *and let* $N \in R - gr_*$.

(1) *For each* $f \in C^*$ *we have* $\hat{f} \circ \tilde{f} = \pi(f, U(N)) \circ i(f, U(N))$; *this is the graded endomorphism of* $U(N)$ *which is the identity on the summand* $\oplus\{N(h) \mid hf \in C^*\}$, *and is zero on the remaining summands of* $U(N)$.

(2) *For each* $f \in C^*$ *we have* $\tilde{f} \circ \hat{f} = p(f, U(N)) \circ j(f, U(N))$; *this is the graded endomorphism of* $U(N)$ *which is the identity on the summand* $\oplus\{N(t) \mid t = lf \text{ for some } l \in C^*\}$, *and is zero on the remaining summands of* $U(N)$.

Proof. We prove (1); the proof of (2) is similar. By definition, we have

$$\begin{aligned}\hat{f} \circ \tilde{f} &= \pi(f, U(N)) \circ j(f, U(N)) \circ p(f, U(N)) \circ i(f, U(N)) \\ &= \pi(f, U(N)) \circ i(f, U(N)).\end{aligned}$$

The remaining statements follow by the definitions of π and i. □

Corollary 2.10. *Let* R *be* $*$*-graded by the right* $*$*-cancellative category* C, *let* $f \in C^*$, *and let* $N, N' \in R - gr_*$.

(1) *For each* $\Lambda \in \mathrm{HOM}_R(U(N), N')_f$ *we have* $\hat{f} \circ \tilde{f} \circ \Lambda = \Lambda$. *In particular,* $\hat{f} \circ \tilde{f} \circ \hat{f} = \hat{f}$.

(2) *For each* $\Delta \in \mathrm{HOM}_R^{-1}(N', U(N))_f$ *we have* $\Delta \circ \hat{f} \circ \tilde{f} = \Delta$. *In particular,* $\tilde{f} \circ \hat{f} \circ \tilde{f} = \tilde{f}$.

Proof. For (1) we use the remarks made prior to the definition of Φ together with Proposition 2.9 to get $\hat{f}\circ\tilde{f}\circ\Lambda=\pi(f,U(N))\circ i(f,U(N))\circ\Lambda = \Lambda$. The proof of (2) is similar. □

We now give a generalization of Corollary 2.10 to right $*$-cancellative semigroups. This general version will be useful in proving Theorem 3.3.

Proposition 2.11. *Let R be $*$-graded by the right $*$-cancellative semigroup S. Let $N, N' \in R{-}gr_*$, $f \in S^*$ and $\Lambda \in \mathrm{HOM}_R(U(N), N')_f$. If Φ denotes $\Phi(f, U(N), N')$, then $\Lambda = \hat{f} \circ p(f, U(N)) \circ (\Lambda)\Phi^{-1}$.*

Proof. By using the equation given prior to the definition of Φ we have

$$\begin{aligned}\hat{f} \circ p(f, U(N)) \circ (\Lambda)\Phi^{-1} &= \pi(f, U(N)) \circ j(f, U(N)) \\ &\quad \circ p(f, U(N)) \circ (\Lambda)\Phi^{-1} \\ &= \pi(f, U(N)) \circ (\Lambda)\Phi^{-1} \\ &= \Lambda \quad \text{(by the definition of } \Phi). \qquad \square\end{aligned}$$

We note that the above proposition is indeed a generalization of Corollary 2.10(1), since if S is a category, then by a previous observation we have $\hat{f} \circ p(f, U(N)) \circ (\Lambda)\Phi^{-1} = \hat{f} \circ p(f, U(N)) \circ i(f, U(N)) \circ \Lambda = \hat{f} \circ \tilde{f} \circ \Lambda$.

We conclude this section by noting that the various functions defined throughout this section are easily shown to be isomorphisms in the case where S is a group. That is, if R is a ring graded by the finite group G, $f \in G$, and $L \in R - gr$, then the functions $\pi(f, L), i(f, L), p(f, L), j(f, L), \hat{f}_L$ and $\tilde{f}_L$ are isomorphisms between the respective abelian groups.

3. Rings of the form $\mathrm{END}\,(U(N))$ **and** $\mathrm{END}^{-1}(U(N))$ **as skew semigroup rings.** In this section we show how the results of the previous section may be used to concretely describe naturally occurring rings of endomorphisms as skew semigroup rings. We recall some notation: if R is graded by the semigroup S and $L \in R - gr$, then

$\mathrm{END}_R(L) = \mathrm{HOM}_R(L,L) = \sum_{f\in S^*} \mathrm{HOM}_R(L,L)_f$. If L is $*$-graded (i.e., $L_z = 0$), then it is easily verified that $\mathrm{END}_R(L)$ is a ring under composition. If, in addition, S is left $*$-cancellative, then the sum $\sum_{f\in S^*} \mathrm{Hom}_R(L,L)_f$ is direct, and this decomposition gives an S-grading of $\mathrm{END}_R(L)$.

In general, $\mathrm{END}_R(L)$ need not be unital, even under our standing assumption that S is finite. However, if S is a finite l.i. semigroup with identities E, then by the remarks prior to Example 2.2 it is easy to show that the identity on L (which we denote by 1_L) can be regarded as the sum of the morphisms $\{\eta_e \mid e \in E\}$, where η_e denotes the endomorphism of L which is the identity on the R-module $\sum_{t\in S^*} L[e]_t$ and is zero elsewhere. As each η_e is a morphism of degree e we conclude that $1_L \in \mathrm{END}_R(L)$.

Let R be a ring $*$-graded by the right $*$-cancellative semigroup S, and let $f \in S^*$. Then for each $\alpha \in \mathrm{End}_{R-gr}(U(N))$ we have $\alpha \circ \hat{f} \in \mathrm{HOM}_R(U(N),U(N))_f$. By the results of Section 2 we have an isomorphism of abelian groups $\Phi = \Phi(f,U(N),U(N)) : \mathrm{Hom}_{R-gr}(U(N)(f),U(N)) \to \mathrm{HOM}_R(U(N),U(N))_f$ given by $\gamma \mapsto \pi(f,U(N)) \circ \gamma$. Thus, we may consider the element $(\alpha \circ \hat{f})\Phi^{-1}$ of $\mathrm{Hom}_{R-gr}(U(N)(f),U(N))$.

Definition 3.1. Let R be $*$-graded by the right $*$-cancellative semigroup S, and let $N \in R - gr_*$. For each $f \in S^*$ we define the function

$$(f)\sigma : \mathrm{End}_{R-gr}(U(N)) \to \mathrm{End}_{R-gr}(U(N))$$

by setting $(\alpha)(f)\sigma = p(f,U(N)) \circ (\alpha \circ \hat{f})\Phi^{-1}$ for each $\alpha \in \mathrm{End}_{R-gr}(U(N))$. As is standard practice, we denote $(\alpha)(f)\sigma$ by $\alpha^{(f)\sigma}$.

It is easy to see that, for any ring R which is $*$-graded by the finite right $*$-cancellative semigroup S, and any $N \in R - gr_*$, we always have $1^{(f)\sigma} = p(f,U(N)) \circ j(f,U(N))$ (where 1 denotes the identity morphism in $\mathrm{End}_{R-gr}(U(N))$). Furthermore, we note that if S is a finite right $*$-cancellative category, then by the definition of $\tilde{f}$ and properties of Φ we have $\alpha^{(f)\sigma} = p(f,U(N)) \circ [i(f,U(N)) \circ (\alpha \circ \hat{f})] = \tilde{f} \circ \alpha \circ \hat{f}$ for any $\alpha \in \mathrm{End}_{R-gr}(U(N))$ and any $f \in S^*$. In particular, when S is a finite right $*$-cancellative category, we have $1^{(f)\sigma} = \tilde{f}\hat{f}$.

As $U(N)$ is a finite direct sum of modules, we can give a componentwise description of the map $\alpha^{(f)\sigma}$. For each $\beta \in \text{End}_{R-gr}(U(N)) = \text{Hom}_{R-gr}(\oplus_{l\in S^*} N(l), \oplus_{l\in S^*} N(l))$, we write $\beta = (\beta_{h,k})$ where, for each pair $h, k \in S^*$, $\beta_{h,k} \in \text{Hom}_{R-gr}(N(h), N(k))$. Then an easy check shows that for $\alpha \in \text{End}_{R-gr}(U(N))$, ${\alpha^{(f)\sigma}}_{h,k} = 0$ unless there exist $s, t \in S^*$ with $h = sf$, $k = tf$. Moreover, ${\alpha^{(f)\sigma}}_{sf,tf} : N(sf) \to N(tf)$ is given by ${\alpha^{(f)\sigma}}_{sf,tf} : n + N[sf]_z \mapsto (n + N[s]_z)\alpha_{s,t} \circ \pi_{t,tf}$, where $\pi_{t,tf} : N(t) = N/N[t]_z \to N/N[tf]_z = N(tf)$ is the map $x + N[t]_z \mapsto x + N[tf]_z$.

Proposition 3.2. *Let R be $*$-graded by the right $*$-cancellative semigroup S, let $N \in R{-}gr_*$, let $f, g \in S^*$, and let $\alpha, \beta \in \text{End}_{R-gr}(U(N))$. Then*

(1) $(\alpha \circ \beta)^{(f)\sigma} = \alpha^{(f)\sigma} \circ \beta^{(f)\sigma}$. *Consequently, each $f(\sigma)$ is a ring endomorphism of* $\text{End}_{R-gr}(U(N))$.

(2) $\alpha^{(fg)\sigma} = (\alpha^{(f)\sigma})^{(g)\sigma}$ *whenever $fg \neq z$.*

(3) $\hat{f} \circ \alpha^{(f)\sigma} = \alpha \circ \hat{f}$.

Proof. For ease of notation, we again let Φ denote $\Phi(f, U(N), U(N))$.

(1) By definition and a previous observation we have

$$\begin{aligned}
&\pi(f,U(N)) \circ (\alpha \circ \hat{f})\Phi^{-1} \circ p(f,U(N)) \circ (\beta \circ \hat{f})\Phi^{-1} \\
&\qquad = ((\alpha \circ \hat{f})\Phi^{-1})\Phi \circ p(f,U(N)) \circ (\beta \circ \hat{f})\Phi^{-1} \\
&\qquad = \alpha \circ \hat{f} \circ p(f,U(N)) \circ (\beta \circ \hat{f})\Phi^{-1} \\
&\qquad = \alpha \circ \pi(f,U(N)) \circ j(f,U(N)) \circ p(f,U(N)) \circ (\beta \circ \hat{f})\Phi^{-1} \\
&\qquad = \alpha \circ \pi(f,U(N)) \circ (\beta \circ \hat{f})\Phi^{-1} \\
&\qquad = \alpha \circ \beta \circ \hat{f} \\
&\qquad = \pi(f,U(N)) \circ (\alpha \circ \beta \circ \hat{f})\Phi^{-1}.
\end{aligned}$$

Since $\pi(f, U(N))$ is surjective we conclude that $(\alpha \circ \hat{f})\Phi^{-1} \circ p(f, U(N)) \circ$

$(\beta \circ \hat{f})\Phi^{-1} = (\alpha \circ \beta \circ \hat{f})\Phi^{-1}$, from which we get

$$\begin{aligned}\alpha^{(f)\sigma} \circ \beta^{(f)\sigma} &= (p(f, U(N)) \circ (\alpha \circ \hat{f})\Phi^{-1}) \\ &\quad \circ (p(f, U(N)) \circ (\beta \circ \hat{f})\Phi^{-1}) \\ &= p(f, U(N)) \circ (\alpha \circ \beta \circ \hat{f})\Phi^{-1} \\ &= (\alpha \circ \beta)^{(f)\sigma}.\end{aligned}$$

(2) This follows from a tedious, straightforward check, using the matrix interpretation of $\alpha^{(f)\sigma}$ given above.

(3)

$$\begin{aligned}\hat{f} \circ \alpha^{(f)\sigma} &= \hat{f} \circ p(f, U(N)) \circ (\alpha \circ \hat{f})\Phi^{-1} \\ &= \pi(f, U(N)) \circ j(f, U(N)) \circ p(f, U(N)) \circ (\alpha \circ \hat{f})\Phi^{-1} \\ &= \pi(f, U(N)) \circ (\alpha \circ \hat{f})\Phi^{-1} \\ &= ((\alpha \circ \hat{f})\Phi^{-1})\Phi = \alpha \circ \hat{f}. \qquad \square\end{aligned}$$

As a result of statements (1) and (2) of Proposition 3.2, we conclude that

$$\sigma : S^* \longrightarrow E(\mathrm{End}_{R-gr}(U(N)))$$

is an action of S^* as endomorphisms on $\mathrm{End}_{R-gr}(U(N))$. In particular, we may form the skew semigroup ring $S^* *_\sigma \mathrm{End}_{R-gr}(U(N))$. It is straightforward to show that if S is l.i. then σ satisfies the hypotheses of [**2**, Proposition 2.5], so that when S is l.i. the ring $S^* *_\sigma \mathrm{End}_{R-gr}(U(N))$ is in fact unital.

We now have the tools to state and prove the first main result of this article.

Theorem 3.3. *Let R be $*$-graded by the right $*$-cancellative semigroup S, let $N \in R - gr_*$, and let σ be the function described in Definition 3.1. Then there is a surjection of rings*

$$\kappa : S^* *_\sigma \mathrm{End}_{R-gr}(U(N)) \longrightarrow \mathrm{END}_R(U(N)).$$

If, in addition, S is left $*$-cancellative then κ is an isomorphism, and κ preserves the graded components of these graded rings. Moreover, if S is l.i., then these rings are unital, and κ is a unital ring homomorphism.

Proof. We let 1 denote the identity element of $\operatorname{End}_{R-gr}(U(N))$. By Proposition 3.2(3) we have $\hat{f} \circ 1^{(f)\sigma} = 1 \circ \hat{f} = \hat{f}$ for each $f \in S^*$. Thus we have a well-defined function $\kappa : S^* *_\sigma \operatorname{End}_{R-gr}(U(N)) \to \operatorname{END}_R(U(N))$, given as the linear extension of

$$\kappa : f[1^{(f)\sigma}\alpha] \longmapsto \hat{f} \circ 1^{(f)\sigma}\alpha = \hat{f}\alpha.$$

To show that κ is a ring homomorphism we note that for each pair $f, g \in S^*$ with $fg \neq z$ and each pair $\alpha, \beta \in \operatorname{End}_{R-gr}(U(N))$,

$$\begin{aligned}(f[1^{(f)\sigma}\alpha] \cdot g[1^{(g)\sigma}\beta])\kappa &= (fg[1^{(fg)\sigma}\alpha^{(g)\sigma}\beta])\kappa \\ &= \widehat{fg}\alpha^{(g)\sigma}\beta \\ &= \hat{f}\hat{g}\alpha^{(g)\sigma}\beta \qquad \text{(by Lemma 2.7(2))} \\ &= \hat{f}(\alpha\hat{g})\beta \qquad \text{(by Proposition 3.2(3))} \\ &= (\hat{f}\alpha) \circ (\hat{g}\beta) \\ &= (f[1^{(f)\sigma}\alpha])\kappa \circ (g[1^{(g)\sigma}\beta])\kappa.\end{aligned}$$

Now let $\Lambda \in \operatorname{END}_R(U(N))_f$. If Φ denotes $\Phi(f, U(N), U(N))$, then by Proposition 2.11 we have $\Lambda = \hat{f} \circ p(f, U(N)) \circ (\Lambda)\Phi^{-1}$, so that if we define $\alpha = p(f, U(N)) \circ (\Lambda)\Phi^{-1}$ we get $\Lambda = \hat{f} \circ \alpha = (f[1^{(f)\sigma}\alpha])\kappa$. Thus κ is surjective.

We now assume that S is left $*$-cancellative, so that $\operatorname{END}_R(U(N))$ is an S-graded ring. Using this, along with an easily verified result about homomorphisms between semigroup-graded abelian groups, the injectivity of κ will follow by demonstrating that κ is injective on each graded component of $S^* *_\sigma \operatorname{End}_{R-gr}(U(N))$. To this end, assume $0 = (f[1^{(f)\sigma}\alpha])\kappa = \hat{f}\alpha = \pi(f, U(N)) \circ j(f, (N)) \circ \alpha$. Then, as $\pi(f, U(N))$ is surjective, we get that $j(f, U(N)) \circ \alpha = 0$, and hence $1^{(f)\sigma}\alpha = p(f, U(N)) \circ j(f, U(N)) \circ \alpha = 0$, so that $f[1^{(f)\sigma}\alpha]$ is zero in $S^* *_\sigma \operatorname{End}_{R-gr}(U(N))$.

That κ preserves graded components is clear. □

Example 3.4. We show that the left $*$-cancellativity condition given in the previous theorem is necessary to ensure that the surjection κ is also an injection. Specifically, let W denote the l.i. semigroup with elements $W = \{1, 2, 3, h, f, g, j, z\}$, where $1, 2, 3$ are orthogonal idempotents, $1h2 = h$, $2f3 = f$, $2g3 = g$, $1j3 = j$, $hf = hg = j$, and all remaining products are z. Then W is right $*$-cancellative, but not left $*$-cancellative. Let R denote the semigroup ring kW^*, where k is a field. It is straightforward to show that

$$R(h) = \frac{k1 + R[h]_z}{R[h]_z}$$

and

$$R(f) = \frac{kh + R[f]_z}{R[f]_z} \bigoplus \frac{k2 + R[f]_z}{R[f]_z}.$$

We consider the map $\gamma : R(h) \to R(f)$ defined to be the linear extension of the map which takes $\gamma : a1 + R[h]_z \mapsto ah + R[f]_z$ for each $a \in k$. This map is well-defined as $R[h]_z \subseteq R[f]_z$ and is clearly an R-homomorphism. Furthermore, as $(R(h)_h)\gamma \subseteq R(f)_{hf} = R(f)_{hg}$, γ may be viewed both as a morphism of degree f and as a morphism of degree g.

Similar to the computation alluded to above, it is also straightforward to show that

$$R(hf) = R(hf)_{hf} = \frac{k1 + R[hf]_z}{R[hf]_z}.$$

We now define the map $\alpha : R(hf) \to R(f)$ to be the linear extension of the map $\alpha : a1 + R[hf]_z \mapsto ah + R[f]_z$; it is not hard to show that α is a well-defined, graded R-homomorphism.

Finally, if κ denotes the ring homomorphism described in the above theorem in this particular setting, then a straightforward computation yields $(f[1^{(f)\sigma}\alpha])\kappa = \gamma = (g[1^{(g)\sigma}\alpha])\kappa$, so that κ is not injective.

Corollary 3.5. *Let G be a finite group, let R be a ring graded by G, and let N be any G-graded R-module. Then $G *_\sigma \mathrm{End}_{R-gr}(U(N)) \cong \mathrm{END}_R(U(N))$.*

Proof. In the setting of groups we have that $\hat{f}$ and $\tilde{f}$ are isomorphisms. In turn, this property easily yields that each $(f)\sigma$ is an automorphism of $\mathrm{End}_{R-gr}(U(N))$. Therefore, the skew semigroup ring constructed here is identical to the construction for groups. The result follows now from Theorem 3.3. □

Thus, Theorem 3.3 is a nontrivial generalization of the corresponding group result given in [**4**, Theorem 3.6(1)].

We now continue the type of discussion presented above by focusing our attention on another collection of endomorphisms of $U(N)$. We again recall some notation: if R is graded by the semigroup S and $L \in R-gr$, then $\mathrm{END}_R^{-1}(L) = \mathrm{HOM}_R^{-1}(L,L) = \sum_{f \in S^*} \mathrm{HOM}_R^{-1}(L,L)_f$. If L is $*$-graded (i.e., $L_z = 0$), then it is easily verified that $\mathrm{END}_R^{-1}(L)$ is a ring under composition. Analogous to the observation made for rings of the form $\mathrm{END}_R(L)$, if S is left $*$-cancellative then the sum $\sum_{f \in S^*} \mathrm{HOM}_R^{-1}(L,L')_f$ is direct, and this decomposition gives an S^{op}-grading of $\mathrm{END}_R^{-1}(L)$. Similarly, $\mathrm{END}_R^{-1}(L)$ is unital whenever S is l.i..

We will now describe the ring $\mathrm{END}_R^{-1}(U(N))$ as a skew semigroup ring by a construction quite similar to that carried out above. If C is a category, then the function $\sigma : C^* \to E(\mathrm{End}_{R-gr}(U(N)))$ given in Definition 3.1 is described by $\alpha^{(f)\sigma} = \tilde{f}\alpha\hat{f}$. We view σ as $\sigma : (C^{op})^* \to E(\mathrm{End}_{R-gr}(U(N)))$, where σ satisfies the property $(fg)\sigma = (g)\sigma \circ (f)\sigma$ whenever $fg \neq z$ in C^{op}. Thus, we may form the skew semigroup ring $\mathrm{End}_{R-gr}(U(N)) *_\sigma (C^{op})^*$. An easy computation shows that the appropriate analog of [**2**, Proposition 2.5] applies, and yields that this ring is indeed unital. We are now ready to prove the second main result of this article.

Theorem 3.6. *Let R be a ring $*$-graded by the right $*$-cancellative category C, let $N \in R-gr_*$, and let σ be the function described in Definition* 3.1. *Then there is a surjection of unital rings*

$$\psi : \mathrm{End}_{R-gr}(U(N)) *_\sigma (C^{op})^* \longrightarrow \mathrm{END}_R^{-1}(U(N)).$$

If, in addition, C is left $*$-cancellative, then ψ is an isomorphism, and ψ preserves the graded components of these graded rings.

Proof. We let 1 denote the identity element of $\mathrm{End}_{R-gr}(U(N))$. By Corollary 2.10, we have $1^{(f)\sigma}\tilde{f} = \tilde{f}\hat{f}\tilde{f} = \tilde{f}$. Thus, we have a well-defined function $\psi : \mathrm{End}_{R-gr}(U(N)) *_\sigma (C^{op})^* \to \mathrm{END}_R^{-1}(U(N))$, given as the linear extension of

$$\psi : ([\alpha 1^{(f)\sigma}]f) \longmapsto \alpha 1^{(f)\sigma} \circ \tilde{f} = \alpha \tilde{f} \hat{f} \circ \tilde{f} = \alpha \tilde{f}$$

for each $f \in C^*$ and $\alpha \in \mathrm{End}_{R-gr}(U(N))$. We remind the reader that in the skew semigroup ring $\mathrm{End}_{R-gr}(U(N)) *_\sigma (C^{op})^*$ the semigroup operation is taking place in C^{op}, so that the expression $f \cdot g$ in fact denotes the product gf in C.

The remaining details of the proof follow in a manner analogous to the proof of Theorem 3.3 and are therefore omitted. □

We will show in Example 3.12 below that Theorem 3.6 cannot be extended from categories to general l.i. semigroups; thus, there is a perhaps surprising lack of symmetry between rings of the form $\mathrm{END}_R(U(N))$ and $\mathrm{END}_R^{-1}(U(N))$, as regards their realization as skew semigroup rings. As an additional observation in this vein, suppose that S is any right and left $*$-cancellative semigroup. Then we may define $\varsigma : \mathrm{END}_R^{-1}(U(N)) \to \mathrm{End}_{R-gr}(U(N)) *_\sigma (S^{op})^*$ by setting $(\Delta)\varsigma = [\Delta \hat{f}]f$ and extending linearly (the left $*$-cancellativity of S is required here). That $[\Delta \hat{f}]f \in \mathrm{End}_{R-gr}(U(N)) *_\sigma (S^{op})^*$ follows from the fact that $\Delta \hat{f} = \Delta \hat{f} 1^{(f)\sigma}$; that ς is a ring homomorphism is easy to check. Moreover, we can use the injectivity of $\Theta(f, U(N), U(N))$ to get that ς is injective. Thus, for S right and left $*$-cancellative we have another result which can be viewed as a "dual" to Theorem 3.3; namely, that there is an injection of rings $\varsigma : \mathrm{END}_R^{-1}(U(N)) \to \mathrm{End}_{R-gr}(U(N)) *_\sigma (S^{op})^*$. In case S is also a category then ς is actually an isomorphism; in fact, it is easy to check that $\varsigma = \psi^{-1}$ in this situation.

We conclude this article by analyzing rings of the form $\mathrm{END}_R(U(R))$ and $\mathrm{END}_R^{-1}(U(R))$. As consequences we will see (among other things) that:

(1) the rings $\mathrm{END}_R(U(R))$ and $\mathrm{END}_R^{-1}(U(R))$ need not be isomorphic;

(2) the ring $\mathrm{END}_R^{-1}(U(R))$ is a normalizing extension of $\mathrm{End}_{R-gr}(U(R))$ (but this normalizing property is not necessarily valid for rings of the form $\mathrm{END}_R(U(R))$, nor for rings of the form $\mathrm{END}_R^{-1}(U(N))$ for an arbitrary graded module N); and

(3) we cannot extend Theorem 3.6 from categories to arbitrary l.i. semigroups.

Let S be an l.i. semigroup having a set of local identities E. If R is a ring $*$-graded by S, we call R *locally unital* in case for each $e \in E$ there exists $a_e \in R_e$ such that for each pair $e, e' \in E$, each $g \in S^*$ with $g = ege'$, and each $r \in R_g$ we have $a_e r = r = r a_{e'}$. If R is locally unital and E is finite, then R is necessarily unital, with $1_R = \sum_{e \in E} a_e$. The following lemma provides some useful computational information about the graded module $U(R)$ in case R is locally unital.

Lemma 3.7. *Let R be a locally unital ring graded by the* l.i. *semigroup S. Then for each pair $f, g \in S^*$, we have the following isomorphisms of abelian groups.*

(1)

$$\mathrm{Hom}_R(R(f), R(g)) \cong \begin{cases} \text{right multiplication by elements} \\ \quad a \in \oplus_y \{R_y \mid e_y = e_f \text{ and } e'_y = e_g\} \\ \quad \text{having } R[f]_z \cdot a \subseteq R[g]_z \\ 0 \quad \text{if no such } y \text{ exists} \end{cases}$$

(2)

$$\mathrm{HOM}_R(R(f), R(g))_h \cong \begin{cases} \text{right multiplication by the elements of} \\ \quad \oplus_y \{R_y \mid yg = fh\} \\ 0 \quad \text{if no such } y \text{ exists} \end{cases}$$

(3)

$$\mathrm{HOM}_R^{-1}(R(f), R(g))_h \cong \begin{cases} \text{right multiplication by elements} \\ \quad a \in \oplus_y \{R_y \mid f = ygh\} \\ \quad \text{having } R[f]_z \cdot a \subseteq R[g]_z \\ 0 \quad \text{if no such } y \text{ exists} \end{cases}$$

(4)

$$\mathrm{Hom}_{R-gr}(R(f), R(g)) \cong \begin{cases} \text{right multiplication by the elements of} \\ \quad \oplus_y \{R_y \mid f = yg\} \\ 0 \quad \text{if no such } y \text{ exists.} \end{cases}$$

Proof. The proofs of these results are straightforward, albeit tedious. As representative examples we prove (1) and (2). Let $f \in S^*$; then $R(f)$ is a cyclic R-module generated by $1 + R[f]_z = 1_{e_f} + R[f]_z$.

For (1), let $\lambda : R(f) \to R(g)$ be an R-homomorphism. Then for each $r \in R$ we have $(r1_{e_f} + R[f]_z)\lambda = r1_{e_f} \cdot (1_{e_f} + R[f]_z)\lambda$. Also, as $R(g)$ is cyclic there exists $a \in R$ with $(1_{e_f} + R[f]_z)\lambda = a1_{e_g} + R[g]_z$. Thus we get $(r1_{e_f} + R[f]_z)\lambda = r1_{e_f}a1_{e_g} + R[g]_z$ for each $r \in R$. Replacing a by $1_{e_f}a1_{e_g}$ if necessary, we conclude that for each $\lambda \in \mathrm{Hom}_R(R(f), R(g))$ there exists $a \in \oplus_y\{R_y \mid e_y = e_f \text{ and } e'_y = e_g\}$ such that for each $r \in R$, $(r + R[f]_z)\lambda = ra + R[g]_z$. Furthermore, as λ is a homomorphism we must have $R[f]_z \cdot a \subseteq R[g]_z$.

It is now straightforward to check that if $a \in \oplus_y\{R_y \mid e_y = e_f$ and $e'_y = e_g\}$ (so that $a = 1_{e_f}a1_{e_g}$) and a has the property that $R[f]_z \cdot a \subseteq R[g]_z$, then the map $\lambda_a : R(f) \to R(g)$ defined by

$$\lambda_a : r1_{e_f} + R[f]_z \longmapsto ra + R[g]_z$$

is a well-defined R-homomorphism. This correspondence between a and λ_a is easily shown to induce the isomorphism indicated in statement (1).

For (2), let $\Lambda \in \mathrm{HOM}_R(R(f), R(g))_h$. Then, proceeding as in the proof of (1) and using the definition of morphisms of degree h, we have an element $a \in R$ with $a1_{e_g} + R[g]_z \in R(g)_{fh}$. This is easily seen to imply the containment $a1_{e_g} \in \sum_{tg=fh} R_t$. We note that, as in (1), we also have $R[f]_z \cdot a \subseteq R[g]_z$. However, this condition need not be explicitly given here, as it is implied by the condition $a1_{e_g} \in \sum_{tg=fh} R_t$. (To see this, it is enough to note that if $lf = z$ and $yg = fh$, then $lyg = z$.)

The proofs of (3) and (4) are similar to the proofs of (1) and (2), respectively; specifically, the reason that the condition $R[f]_z \cdot a \subseteq R[g]_z$ need not be explicitly mentioned in the statement of (4) is similar to the reason given above in the proof of (2). $\square$

A result analogous to Lemma 3.7 for arbitrary graded modules of the form $U(N)$ is not available. For instance, if $0 \neq \alpha \in \mathrm{Hom}_{R-gr}(N(f), N(g))$, then the most we can say about the configuration of the morphisms is that $e'_f = e'_g$. In the case where $N = R$, however, we see from 3.7(4) that the elements of $\mathrm{Hom}_{R-gr}(R(f), R(g))$

are "well-known," and behave according to an S-grading. It is for this reason that the results presented here only for the specific case $U(R)$ cannot be extended to arbitrary S-graded R-modules.

Example 3.8. Let S be the semigroup arising from the partially ordered set $Y = \{a, b\}$ where $a \leq b$; we denote the elements of S^* by $\{1, \alpha, 2\}$. Let R denote the (S-graded, locally unital) ring kS^* where k is a field; R is in fact just the usual incidence ring of the partially ordered set Y, and can be viewed as the 2×2 upper triangular matrices over k.

As $U(R) = \oplus_{f \in S^*} R(f)$, $\mathrm{End}_R(U(R))$ is isomorphic to the ring of matrices whose rows and columns are indexed by the elements of S^*, with entries in the (f, g) coordinate taken from $\mathrm{Hom}_R(R(f), R(g))$. But by using Lemma 3.7(1) and the fact that this particular S is $*$-cancellative, and by listing the elements of S^* in the order $\{1, \alpha, 2\}$, a straightforward check gives

$$\mathrm{End}_R(U(R)) \cong \begin{pmatrix} k1 & k1 & k\alpha \\ k1 & k1 & k\alpha \\ 0 & 0 & k2 \end{pmatrix} \cong \begin{pmatrix} k & k & k \\ k & k & k \\ 0 & 0 & k \end{pmatrix},$$

where in the second isomorphism we have simply suppressed the underlying semigroup elements in kS^*. An analogous argument similarly yields

$$\mathrm{END}_R(U(R)) \cong \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix},$$

$$\mathrm{END}_R^{-1}(U(R)) \cong \begin{pmatrix} k & 0 & 0 \\ k & k & k \\ 0 & 0 & k \end{pmatrix},$$

and

$$\mathrm{End}_{R-gr}(U(R)) \cong \begin{pmatrix} k & 0 & 0 \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}.$$

In particular, we see immediately that the three rings $\mathrm{End}_R(U(R))$, $\mathrm{END}_R(U(R))$, and $\mathrm{END}_R^{-1}(U(R))$ are pairwise non-isomorphic. In contrast, we note that this behavior does not arise in the group-graded setting. Specifically, if G is a finite group which grades the ring R, and L is any G-graded R-module, then $\mathrm{END}_R(L) = \oplus_{f\in G}\mathrm{HOM}\,(L,L)_f = \oplus_{f^{-1}\in G}\mathrm{HOM}\,(L,L)_{f^{-1}} = \mathrm{END}_R^{-1}(L)$, and each is equal to $\mathrm{End}_R(L)$.

We now turn to questions regarding normalizing extensions. If S is a finite l.i. semigroup and R is a locally unital S-graded ring, then the *smash product* $R\#S^*$ is defined to be the collection of S^*-square matrices of the form $\sum_{\substack{f,h\in S^*\\ fh\neq z}} r_f e_{fh,h}$, where $r_f \in R_f$ and $e_{fh,h}$ denotes the standard matrix unit which is 1 in the (fh,h) coordinate and 0 elsewhere, under the usual matrix operations. (For additional information about $R\#S^*$ see [**2**].) In this case $R\#S^*$ is unital, with $1_{R\#S^*} = \sum_{l\in S^*} a_{e_l} e_{l,l}$. We show in [**2**, Section 4] that there is both a natural action and a natural reversing action of S^* as endomorphisms on $R\#S^*$.

There is an intimate connection between the ring $\mathrm{End}_{R-gr}(U(R))$ and the smash product ring $R\#S^*$. First, as a consequence of Lemma 3.7(4) and the definition of the smash product it is straightforward to verify that if R is a locally unital ring graded by the finite l.i. semigroup S, then $R\#S^* \cong \mathrm{End}_{R-gr}(U(R))$. Furthermore, under this isomorphism the action σ of S^* as endomorphisms on $\mathrm{End}_{R-gr}(U(R))$ is precisely the action ρ of S^* as endomorphisms on $R\#S^*$ described in [**2**, Section 4]. In particular, by invoking Theorems 3.3 and 3.6, respectively, these observations yield

Proposition 3.9. *Let R be a locally unital ring graded by the $*$-cancellative* l.i. *semigroup S. Then*

(1) $\mathrm{END}_R(U(R)) \cong S^* *_\rho (R\#S^*)$.

(2) $\mathrm{END}_R^{-1}(U(R)) \cong (R\#S^*) *_\rho (S^{op})^*$ *in case S is a category.*

The connections established in Proposition 3.9 allow us to rephrase some of the results of [**2**] in this "graded endomorphisms" setting.

Corollary 3.10. *Let R be $*$-graded by the finite right $*$-cancellative category C. Then*

(1) $\mathrm{END}_R^{-1}(U(R))$ *is a finite normalizing extension of* $\mathrm{End}_{R-gr}(U(R))$.

(2) $\mathrm{END}_R(U(R))$ *need not be a normalizing extension of* $\mathrm{End}_{R-gr}(U(R))$.

Proof. Statement (1) follows from [**2**, Proposition 4.8(2)], while statement (2) follows from [**2**, Example 4.1 and the final remarks of Section 1]. $\square$

Perhaps surprisingly, we now show that the normalizing extension behavior described in Corollary 3.10(1) does not extend to all graded modules of the form $U(N)$.

Example 3.11. Let S denote the semigroup arising from the partially ordered set $X = \{u, v, w, x\}$ having nontrivial relations $u \leq v \leq x$, $u \leq w \leq x$. Let k be a field, and let R denote the semigroup algebra $R = kS^*$. Then R is just $I(X, k)$, the incidence ring of X with coefficients in k. For notational convenience we denote the elements of S^* by $\{1, 2, 3, 4, f, g, h, i, j\}$, where $\leq_{u,u} = 1$, $\leq_{v,v} = 2$, $\leq_{w,w} = 3$, $\leq_{x,x} = 4$, $\leq_{u,v} = f$, $\leq_{v,x} = g$, $\leq_{u,w} = h$, $\leq_{w,x} = i$, and $\leq_{u,x} = j$.

Let N denote the left ideal $N = kf + kh$ of R. Then N is clearly S-graded. It is tedious but routine to show that $N(1) = N(4) = N(f) = N(h) = N(j) = 0$, $N(2) = N(2)_f = kf$, $N(3) = N(3)_h = kh$, $N(g) = N(g)_j = kf$, and $N(i) = N(i)_j = kh$. The map which takes $af \mapsto ah$ (resp. $ah \mapsto af$) for each $a \in k$ induces a left R-homomorphism from $N(g)$ to $N(i)$ (resp. $N(i) \mapsto N(g)$); by the above descriptions, this is easily seen to be a graded homomorphism. Thus, using the matrix description of $\mathrm{End}_{R-gr}(U(N))$ (and listing the germane elements of S in the order $2, 3, g, i$), we have

$$\mathrm{End}_{R-gr}(U(N)) = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & k \\ 0 & 0 & k & k \end{pmatrix}.$$

Again, using the above descriptions, the identity map from $N(g)$ to $N(2)$ (resp. $N(i)$ to $N(3)$) is easily seen to be a morphism of degree g^{-1} (resp. i^{-1}). Similarly, the map which takes $af \mapsto ah$ (resp. $ah \mapsto af$) induces a morphism from $N(g)$ to $N(3)$ (resp. $N(i)$ to $N(2)$) of degree

i^{-1} (resp. g^{-1}). With these observations, we conclude that

$$\mathrm{END}\,{}_R^{-1}(U(N)) = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ k & k & k & k \\ k & k & k & k \end{pmatrix}.$$

Using an argument similar to that given subsequent to [**2**, Example 1.11], it can now be shown that this second matrix ring is not a normalizing extension of the first; we outline the important details. We let T and A denote the rings $\mathrm{END}\,{}_R^{-1}(U(N))$ and $\mathrm{End}\,{}_{R-gr}(U(N))$, respectively. Now suppose $t = (t_{i,j}) \in T$ has the property that $tA = At$. Then, in particular, $(e_{3,4}+e_{4,4})t \in tA$. Thus there exists $a = (x_{i,j}) \in A$ with $(e_{3,4} + e_{4,4})t = ta$. Now using the form of elements of T and A, this equation yields in particular that $t_{4,1} = t_{3,1}x_{1,1}$. Similarly, $(e_{3,3}+e_{4,3})t \in tA$, so there exists $a' = (y_{i,j}) \in A$ with $(e_{3,3}+e_{4,3})t = ta'$. Arguing as above, we get that $t_{3,1} = t_{4,1}y_{1,1}$. We have $e_{4,3}t \in tA$; so there exists $a'' = (w_{i,j})$ with $e_{4,3}t = ta''$. On equating matrix entries, this yields $t_{3,1}w_{1,1} = 0$ and $t_{4,1}w_{1,1} = t_{3,1}$.

The equations $t_{4,1} = t_{3,1}x_{1,1}$ and $t_{3,1} = t_{4,1}y_{1,1}$ yield that $t_{4,1} \neq 0$ if and only if $t_{3,1} \neq 0$. But the equations $t_{3,1}w_{1,1} = 0$ and $t_{4,1}w_{1,1} = t_{3,1}$ yield that $t_{3,1} = 0$. Thus we have shown that any element $t = (t_{i,j})$ of T having the property that $tA = At$ must have $t_{3,1} = t_{4,1} = 0$. But then a straightforward computation yields that $\sum\{At \mid t \in T \text{ with } At = tA\} \neq T$, which implies that T is not a normalizing extension of A.

Using the description of $\mathrm{END}\,{}_R^{-1}(U(R))$ given in Lemma 3.7, it is not hard to show that for $\alpha \in \mathrm{End}\,{}_{R-gr}(U(R))$ and $f \in S^*$ that $\tilde{f} \circ \hat{f} \circ \alpha\tilde{f} = \alpha\tilde{f}$. It is this fact which is responsible for the normalizing extension behavior of $\mathrm{END}\,{}_R^{-1}(U(R))$ over $\mathrm{End}\,{}_{R-gr}(U(R))$ as given in Corollary 3.10(1). (This fact was not required here, however, as we have justified Corollary 3.10 by invoking results from [**2**].) The reason why the normalizing extension property could not be extended to rings of the form $\mathrm{END}\,{}_R^{-1}(U(N))$ for arbitrary N is because the above equation need not hold in this more general setting. Specifically, referring to Example 3.11, we let α denote the graded morphism from $N(g)$ to $N(i)$. Then the map $\alpha\tilde{\imath}$ is a nonzero morphism of degree i^{-1} from $N(g)$ to $N(i)$. However, $\tilde{\imath} \circ \hat{\imath} \circ \alpha\tilde{\imath} = 0 \neq \alpha\tilde{\imath}$.

We conclude this article by showing that Theorem 3.6 cannot be extended from categories to general semigroups.

Example 3.12. We again consider the left and right $*$-cancellative, l.i., non-category semigroup $T = \{1, \alpha, 2, \beta, 3, z\}$ of Example 1.5. Let k be a field, and let R be the semigroup algebra kT^*. Using the description of $\mathrm{END}_R^{-1}(U(R))$ given in Lemma 3.7 we have that

$$\mathrm{END}_R^{-1}(U(R)) \cong \begin{pmatrix} k1 & 0 & 0 & 0 & 0 \\ k1 & k1 & k\alpha & 0 & 0 \\ 0 & 0 & k2 & 0 & 0 \\ 0 & 0 & 0 & k2 & k\beta \\ 0 & 0 & 0 & 0 & k3 \end{pmatrix}.$$

Here we have listed the elements of T^* in the order $1, \alpha, 2, \beta, 3$. (We remark that the entry in the fourth row, third column is zero due to the fact that $\mathrm{HOM}_R^{-1}(R(\beta), R(2)) = 0$; indeed, we showed in Example 2.6 that $\mathrm{Hom}_R(R(\beta), R(2)) = 0$. It is this property which in some sense produces the non-extendability of Theorem 3.6 to this more general setting.) In particular, we see that $\mathrm{END}_R^{-1}(U(R))$ is an eight-dimensional k-algebra.

On the other hand, it is tedious but straightforward to show that the skew semigroup ring $\mathrm{End}_{R-gr}(U(R)) *_\sigma (T^{op})^*$ is in fact nine-dimensional. Thus the rings $\mathrm{END}_R^{-1}(U(R))$ and $\mathrm{End}_{R-gr}(U(R)) *_\sigma (T^{op})^*$ are not isomorphic for the $*$-cancellative l.i. semigroup T.

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