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SMALL SALEM NUMBERS, EXCEPTIONAL UNITS, AND LEHMER'S CONJECTURE

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ABSTRACT. Lehmer's conjecture says that there is an $\varepsilon > 0$ so that if an algebraic integer α is not a root of unity, then its Mahler measure $M(\alpha)$ is greater than $1+\varepsilon$. This suggests that if $M(\alpha) > 1$ is small, then α should behave like a root of unity. For example, there might be many small values of n such that $1-\alpha^n$ is a unit; that is, such that α^n is an exceptional unit.

The smallest Mahler measures currently known occur for Salem numbers, and Boyd has constructed a table of small Salem numbers. We verify experimentally that many powers of the numbers in Boyd's table are exceptional units. We also show that if α is an algebraic integer of degree d, then at most $O(d^{1+\varepsilon})$ powers of α can be exceptional units. Finally, we consider the Mahler measure (canonical height) associated to arbitrary rational maps $\phi(x)$ and raise some questions related to ϕ-Salem numbers and the ϕ-Lehmer conjecture.

1. Heights and Mahler measure. Recall that the *Mahler measure* of an algebraic integer α is the quantity $M(\alpha)$ defined by

$$M(\alpha) = \prod_{\sigma:\mathbf{Q}(\alpha)\hookrightarrow\mathbf{C}} \max\{|\sigma\alpha|, 1\}.$$

Here the product is over all of the embeddings of $\mathbf{Q}(\alpha)$ into $\mathbf{C}$. Clearly we always have $M(\alpha) \geq 1$, and an elementary result of Kronecker tells us when there is equality.

Theorem (Kronecker [**8**]). *$M(\alpha) = 1$ if and only if α is a root of unity.*

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We also recall that the (absolute multiplicative) *height* of an algebraic number α is the quantity

$$H(\alpha) = \prod_{v\in M_K} \max\{\|\alpha\|_v, 1\}^{1/[K:\mathbf{Q}]},$$

where K is any number field containing α and M_K is an appropriately normalized set of absolute values on K. The height and the Mahler measure are related by the simple formula

$$M(\alpha) = H(\alpha)^{[\mathbf{Q}(\alpha):\mathbf{Q}]}.$$

2. Lehmer's conjecture. Kronecker's theorem leads naturally to the question of the minimum value of $M(\alpha)$ when α is not a root of unity. This question has applications to, among other things, finding multiplicative relations between units in number fields. It seems natural to suppose that one could find a sequence of algebraic integers $\alpha_1, \alpha_2, \ldots$ with $M(\alpha_n) > 1$ and $M(\alpha_n) \to 1$ as $n \to \infty$. Lehmer [**10**] raised the question of finding such a sequence, but the conjecture that bears his name says that this question has a negative answer.

Lehmer's conjecture. *There is an absolute constant $\varepsilon > 0$ so that if α is an algebraic integer that is not a root of unity, then*

$$M(\alpha) \geq 1 + \varepsilon.$$

This conjecture is still open. It is clear that for all αs of degree d, there is a lower bound with $\varepsilon = \varepsilon(d)$ depending on d. This assertion, which follows from the fact that there are only finitely many algebraic numbers of bounded degree and height, gives a terrible bound for $\varepsilon(d)$. The following finer results are known, where we write: α is an algebraic integer which is not a root of unity, $d = [\mathbf{Q}(\alpha) : \mathbf{Q}]$ is the degree of α, $c > 0$, an absolute (and effectively computable) constant.

(1) (Blanksby-Montgomery [**2**])

$$M(\alpha) \geq 1 + \frac{c}{d\log d}.$$

The proof uses a Fourier averaging technique.

(2) (Stewart [**19**])

$$M(\alpha) \geq 1 + \frac{c}{d \log d}.$$

This is the same estimate as that obtained by Blanksby-Montgomery, but the proof uses transcendence techniques. More precisely, the proof involves the construction of an auxiliary polynomial and extrapolation of zeros.

(3) (Dobrowolski [**5**])

$$M(\alpha) \geq 1 + c\left(\frac{\log \log d}{\log d}\right)^3.$$

The proof uses Stewart's technique. The key new idea is to replace the (trivial) Liouville lower bound with something larger by using Frobenius. Thus for each prime p, the number α^p is congruent to one of the conjugates of α. Hence if $F(\alpha) = 0$ for some $F \in \mathbf{Z}[X]$, then $F(\alpha^p) \equiv 0 \pmod{\mathfrak{p}}$ for each prime over p, so one obtains $N_{\mathbf{Q}(\alpha)/\mathbf{Q}}\big(F(\alpha^p)\big) \equiv 0 \pmod{p^d}$.

Another approach is to try to prove that Lehmer's conjecture holds for certain collections of α's, as in the following two results.

(4) (Smyth [**18**]). If α is not a reciprocal number, then

$$M(\alpha) \geq \beta_1 \approx 1.324718,$$

where β_1 is the real root of $x^3 - x - 1 = 0$. (Recall that α is called *reciprocal* if α^{-1} is a Galois conjugate of α. Equivalently, α is reciprocal if its minimal polynomial $f(X) \in \mathbf{Z}[X]$ satisfies $f(X) = X^d f(X^{-1})$.) Smyth's result generalizes Siegel's theorem [**15**] to the effect that β_1 is the smallest PV number.

(5) (Silverman). If there exist primes $\mathfrak{p}_1, \dots, \mathfrak{p}_d$ in $\mathbf{Q}(\alpha)$ satisfying $N\mathfrak{p}_i \leq \sqrt{d \log d}$, then $M(\alpha) \geq 1 + c$. In particular, this is true if there exists a rational prime $p \leq \sqrt{d \log d}$ which splits completely in $\mathbf{Q}(\alpha)$.

3. Salem numbers. The smallest value of $M(\alpha) > 1$ currently known was found by Lehmer. It is the largest real root α_1 of the polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

The Mahler measure of α_1 is

$$M(\alpha_1) = \alpha_1 \approx 1.1762808.$$

Notice α_1 is reciprocal. Further, α_1 is an example of a Salem number according to the following definition:

Definition. Let α be an algebraic integer of degree d, and write its conjugates as $\alpha = \alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(d)}$. Then α is a *Salem number* if

(i) $\alpha \in \mathbf{R}$ and $\alpha > 1$.

(ii) $|\alpha^{(i)}| \le 1$ for all $i = 2, 3, \dots, d$.

(iii) $|\alpha^{(i)}| = 1$ for some i.

Example. David Boyd [**13**] developed a technique for constructing Salem numbers and conducted a computer search to find small Salem numbers. The first 24 small Salem numbers from Boyd's list together with their degrees over $\mathbf{Q}$ are listed in Table 1.

4. Exceptional units. Consider the following "syllogism."

(1) If $M(\alpha)$ is close to 1, then Kronecker's theorem says that α is "almost" a root of unity.

(2) If ζ is a root of unity, then $1 - \zeta^n$ "tends to be" a unit.

(3) Ergo, if $M(\alpha)$ is close to 1, then $1 - \alpha^n$ "should be" a unit for many values of n.

Definition. A unit u in a number field is called an *exceptional unit* if $1 - u$ is also a unit.

The syllogism suggests that if α is one of the small Salem numbers from Table 1, that many powers of α will be exceptional units. Looking at Column A of Table 2, we see that this is indeed the case. For

TABLE 1. List of small Salem numbers from D. Boyd's table.

k	d	α_k	minimal polynomial
1	10	1.176280818	$x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$
2	18	1.188368147	$\vdots$
3	14	1.200026523	$x^{14}-x^{11}-x^{10}+x^7-x^4-x^3+1$
4	14	1.202616743	$x^{14}-x^{12}-x^7-x^2+1$
5	10	1.216391661	$x^{10}-x^6-x^5-x^4+1$
6	18	1.219720859	$\vdots$
7	10	1.230391434	$x^{10}-x^7-x^5-x^3+1$
8	20	1.232613548	$\vdots$
9	22	1.235664580	$\vdots$
10	16	1.236317931	$\vdots$
11	26	1.237504821	$\vdots$
12	12	1.240726423	$x^{12}-x^{11}+x^{10}-x^9-x^6-x^3+x^2-x+1$
13	18	1.252775937	$\vdots$
14	20	1.253330650	$\vdots$
15	14	1.255093516	$x^{14}-x^{12}-x^{11}+x^9-x^7+x^5-x^3-x^2+1$
16	18	1.256221154	$\vdots$
17	24	1.260103540	$\vdots$
18	22	1.260284236	$\vdots$
19	10	1.261230961	$x^{10}-x^8-x^5-x^2+1$
20	26	1.263038139	$\vdots$
21	14	1.267296442	$x^{14}-x^{13}-x^8+x^7-x^6-x+1$
22	8	1.280638156	$x^8-x^5-x^4-x^3+1$
23	26	1.281691371	$\vdots$
24	20	1.282495560	$\vdots$

example, there are 22 values of n for which $1-\alpha_1^n$ is a unit, including $n=74$. Another interesting fact not listed in the table is that

$$\alpha_{20}, \alpha_{20}^2, \alpha_{20}^3, \alpha_{20}^4, \alpha_{20}^5, \alpha_{20}^6, \alpha_{20}^7, \alpha_{20}^8, \alpha_{20}^9, \alpha_{20}^{10} \quad \text{are all exceptional units.}$$

As far as I know, this is the current record for consecutive powers being exceptional units (other than roots of unity, of course).

This naturally raises the question of how many powers (or consecutive powers) of a given number can be exceptional units. The following general result of Evertse provides one answer.

Theorem (Evertse [**6**]). *Let $K/\mathbf{Q}$ be a number field of degree d. Then K contains at most 3×7^{3d} exceptional units.*

Looking again at Table 2, we see that for a few α_k's there are no powers which are exceptional units. It's clear what is happening. If $1-\alpha$ is not a unit, then $1-\alpha^n$ will never be a unit. More generally, in order for $1-\alpha^n$ to be a unit, every factor in the product

$$1-\alpha^n = \prod_{m|n} \Phi_m(\alpha)$$

must be a unit. Here $\Phi_m(x)$ is the mth cyclotomic polynomial. So possibly a better question is to ask for which ms is $\Phi_m(\alpha)$ a unit.

Looking at Columns C–F of Table 2, we see that an astonishingly large proportion of the $\Phi_m(\alpha_k)$s are units. For most of the α_ks, $\Phi_m(\alpha_k)$ is a unit for close to half of the ms less than 100. Although the frequency of such ms thins out for $100\le m\le 200$, the last column of Table 2 shows there are likely to be further ms greater than 200.

The condition that $1-\alpha^n$ be a unit is clearly stronger than merely counting all of the exceptional units, so one might expect that it is possible to improve on Evertse's estimate. This is indeed the case. Notice that if $1-\alpha^n$ is a unit, then $\Phi_n(\alpha)$ is a unit, so the following result also gives a bound for the number of ns such that α^n is an exceptional unit.

TABLE 2. Small Salem numbers and exceptional units.

k	d	A	B	C	D	E	F
1	10	22	74	49	98	60	195
2	18	25	74	56	100	71	186
3	14	20	74	51	100	64	190
4	14	20	74	49	98	63	198
5	10	16	43	43	92	49	170
6	18	19	91	48	100	59	183
7	10	11	39	40	94	48	186
8	20	13	73	50	96	59	198
9	22	22	91	49	99	60	170
10	16	14	67	45	100	55	180
11	26	17	98	49	98	62	192
12	12	11	47	43	96	48	150
13	18	0	0	45	98	56	174
14	20	0	0	44	98	54	192
15	14	16	41	42	99	50	192
16	18	15	47	38	84	51	184
17	24	9	27	44	100	55	186
18	22	16	61	45	99	53	200
19	10	13	46	34	94	39	156
20	26	19	74	44	95	58	198
21	14	13	59	42	90	45	126
22	8	8	23	31	88	35	140
23	26	0	0	47	96	55	198
24	20	10	41	44	96	54	174

d = degree of k^{th} Salem number α_k from Boyd table
A = number of $n \leq 200$ with $1 - \alpha_k^n$ a unit
B = largest $n \leq 200$ with $1 - \alpha_k^n$ a unit
C = number of $m \leq 100$ with $\Phi_m(\alpha_k)$ a unit
D = largest $m \leq 100$ with $\Phi_m(\alpha_k)$ a unit
E = number of $m \leq 200$ with $\Phi_m(\alpha_k)$ a unit
F = largest $m \leq 200$ with $\Phi_m(\alpha_k)$ a unit

Theorem (Silverman). *Let $\alpha \in \overline{\mathbf{Q}}^*$ be an algebraic unit of degree d which is not a root of unity. Then for any $\varepsilon > 0$,*

$$\#\{m \geq 1 : \Phi_m(\alpha) \text{ is a unit}\} \ll d^{1+\varepsilon},$$

where the $\ll$ constant depends only on ε.

(It is probably possible to replace the d^ε by $d^{c/\log\log d}$.)

Proof sketch. The proof involves a number of steps. Let $\alpha_1, \ldots, \alpha_d$ be the conjugates of α. Assuming that $\Phi_m(\alpha)$ is a unit, one begins by writing

$$\begin{aligned} 0 &= \log N_{\mathbf{Q}(\alpha)/\mathbf{Q}}\bigl(\Phi_m(\alpha)\bigr) \\ &= \phi(m)\log M(\alpha) + \sum_{i=1}^{d} \log\bigl|\Phi_m(\beta_i)\bigr|, \end{aligned}$$

where each $\beta_i \in \mathbf{C}$ is equal to either α_i or α_i^{-1} and satisfies $|\beta_i| \leq 1$. If none of the β_i's are close to a primitive mth-root of unity, then one uses a lower bound $\bigl|\Phi_m(\beta_i)\bigr| \gg m^{-\varepsilon}$ and a lower bound for $M(\alpha)$, e.g., Dobrowolski's, to bound m. Next one shows that the ms such that some β_i is close to a primitive mth-root of unity satisfy a sort of "super-gap" principle. This is combined with a linear-forms-in-logarithms lower bound for $|\beta_i - \zeta_m|$ to show that for each β_i, only a bounded number of ms need to be eliminated.

5. Canonical heights for rational maps. The Mahler measure M has the property that (for most αs)

$$M(\alpha^n) = M(\alpha)^n.$$

More precisely, this will be true provided that $\mathbf{Q}(\alpha^n) = \mathbf{Q}(\alpha)$. It is easier to work with the height $H(\alpha)$ which satisfies

$$H(\alpha^n) = H(\alpha) \quad \text{for all } \alpha \in \overline{Q}.$$

Note, however, that since the absolute height H involves taking a root, Lehmer's conjecture acquires an extra d^{-1}. Thus in terms of height functions, Lehmer's conjecture is the assertion that there is an absolute constant $\varepsilon > 0$ so that

$$H(\alpha) \geq 1 + \varepsilon d^{-1},$$

provided α is not a root of unity. As usual, $d = [\mathbf{Q}(\alpha) : \mathbf{Q}]$ is the degree of α.

The height H behaves nicely with respect to the polynomial $\phi(x) = x^n$, in the sense that $H(\phi(\alpha)) = H(\alpha)^n$. We now want to define a height function which behaves nicely for an arbitrary rational function ϕ.

Definition. Let $\phi(x) \in \overline{\mathbf{Q}}(x)$ be a rational function of degree $n \geq 2$. The *ϕ-canonical height* of a number $\alpha \in \overline{\mathbf{Q}}$ is the quantity

$$(*) \qquad \hat{H}_\phi(\alpha) = \lim_{r \to \infty} H\big(\phi^r(\alpha)\big)^{1/n^r}.$$

Here $\phi^r = \phi \circ \phi \circ \cdots \circ \phi$ is the composition of ϕ with itself r times. Also, if ϕ is not a polynomial, then we treat it as a map $\phi : \mathbf{P}^1(K) \to \mathbf{P}^1(K)$ and define $H(\infty) = 1$.

This construction is due to Tate in the context of abelian varieties, and it has been observed over the years by various people that Tate's construction works also in this setting. For details and a proof of the following result, see, for example, Call-Silverman [**4**].

Theorem (aprés Tate).

(a) *The limit* $(*)$ *defining* $\hat{H}_\phi(\alpha)$ *converges.*

(b) $\hat{H}_\phi(\phi\alpha) = \hat{H}_\phi(\alpha)^n$.

(c) $\hat{H}_\phi(\alpha) \geq 1$ for all $\alpha \in \overline{\mathbf{Q}}$.

(d) $\hat{H}_\phi(\alpha) = 1$ *if and only if* α *is* pre-periodic *for* ϕ.

(Recall that α is called *pre-periodic for* ϕ if its forward orbit $\alpha, \phi\alpha, \phi^2\alpha, \ldots$ contains only finitely many points, or equivalently, if there are integers $i > j$ such that $\phi^i\alpha = \phi^j\alpha$.)

There is a natural generalization of Lehmer's conjecture to this more general setting. As far as I am aware, this question was first raised by

Moussa et al. [**12**] when ϕ is a polynomial map with integral coefficients, in which case they defined the ϕ-Mahler measure as a product of values of Green's functions. See also Moussa [**13**] for a further report.

Lehmer question for rational maps. *Let $\phi(x) \in \overline{\mathbf{Q}}(x)$ be a rational map of degree greater than or equal to 2. Does there exists a constant $\varepsilon(\phi) > 0$ so that if $\alpha \in \overline{\mathbf{Q}}$ is not pre-periodic for ϕ and has degree d, then*

$$\hat{H}_\phi(\alpha) \geq 1 + \varepsilon(\phi) d^{-1}?$$

It is not clear how to generalize the notion of Salem numbers for arbitrary rational maps, but there is a natural generalization for polynomial maps. It is easiest to give the definition using some of the basic concepts from the theory of dynamical systems.

Definition. Let $\phi(x) \in \overline{\mathbf{Q}}[x] \subset \mathbf{C}[x]$ be a polynomial of degree greater than or equal to 2, and let α be an algebraic integer of degree d. Then α is a *ϕ-Salem number* if

(i) α is in the attracting basin of ∞.

(ii) Every other conjugate of α is in the filled Julia set of ϕ.

(iii) Some conjugate of α is in the Julia set of ϕ.

These three conditions translate into:

(i) $|\phi^r \alpha| \to \infty$ as $r \to \infty$.

(ii) Every other conjugate α' of α has the property that the quantity $|\phi^r \alpha'|$ is bounded as $r \to \infty$.

(iii) Some conjugate α' of α has the property that for every neighborhood U of α',

$$\bigcup_{r \geq 1} \phi^r(U) = \mathbf{C}.$$

Example 1. Let $\phi(x) = x^n$ for any $n \geq 2$. Then the attracting basin of ∞ is $\{|z| > 1\}$, the Julia set is $\{|z| = 1\}$, and the filled Julia set is $\{|z| \leq 1\}$. So in this case a ϕ-Salem number is just an ordinary Salem number.

Example 2. Let $\phi(x) = T_n(x)$ be the nth Tchebycheff polynomial. This is the polynomial defined by the relation

$$T_n(2\cos z) = 2\cos(nz),$$

or, equivalently,

$$T_n(z + z^{-1}) = z^n + z^{-n}.$$

(We are using a slightly nonstandard normalization, the usual Tchebycheff polynomial would be $2^{-1}T_n(2x)$.) For example,

$$T_2(x) = x^2 - 2, \qquad T_3(x) = x^3 - 3x,$$

and

$$T_4(x) = x^4 - 4x^2 + 2.$$

The Julia set for every $T(x) = T_n(x)$ is the closed interval $[-2, 2]$, and a number $\alpha \in \overline{\mathbf{Q}}$ is pre-periodic for ϕ if and only if it has the form $\alpha = 2\cos(\pi t)$ for some $t \in \mathbf{Q}$. Using the fact that $z \mapsto z + z^{-1}$ maps the exterior of the unit circle onto the complement of $[-2, 2]$, we see that if α is a classical Salem number (that is, for x^n), then $\alpha + \alpha^{-1}$ will be a Tchebycheff-Salem number. In particular, we can use Boyd's list (Table 1) to produce small Tchebycheff-Salem numbers, such as

$$\alpha_1 + \alpha_1^{-1} \approx 2.02641795.$$

The Tchebycheff polynomials commute with one another, so they all give the same canonical height, which we will denote by $\hat{H}_T$. One can use the fact that $\hat{H}_T(\alpha) = 1$ if and only if α is pre-periodic for T to rederive another classical result of Kronecker.

Theorem (Kronecker [**8**]). *Let $\alpha \in \overline{\mathbf{Q}}$ be a totally real algebraic integer all of whose conjugates lie in the interval $[-2, 2]$. Then $\alpha = 2\cos(\pi t)$ for some $t \in \mathbf{Q}$.*

The fact that $T_p(x) \equiv x^p \pmod p$ allows one to apply the Dobrowolski-Stewart method to obtain an approximation to the Lehmer conjecture. Alternatively, one can easily show that

$$\hat{H}_T(\alpha) = H(\beta)^2 \quad \text{where } \beta \text{ satisfies} \quad \alpha = \beta + \beta^{-1}$$

and then apply Dobrowolski's result directly. In any case one obtains

$$\hat{H}_T(\alpha) \geq 1 + \frac{\varepsilon}{d} \cdot \left(\frac{\log\log d}{\log d} \right)^3 .$$

(Remember that the extra $1/d$ just reflects our use of absolute heights.)

Example 3. Consider the rational map

$$\phi(x) = \frac{(x^2-1)^2}{4x^3+4x}.$$

It is a classical fact that the Julia set for ϕ is all of $\mathbf{P}^1(\mathbf{C})$. This map corresponds to the duplication map on the elliptic curve

$$E : y^2 = x^3 + x.$$

In other words, we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{[2]} & E \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x} \\ \mathbf{P}^1 & \xrightarrow[\phi]{} & \mathbf{P}^1. \end{array}$$

Hence the canonical height associated to ϕ is the (exponential of the) usual canonical height on the elliptic curve E. For this rational map, or more generally for the rational map corresponding to multiplication on an elliptic curve with complex multiplication, Laurent has proven a Dobrowolski-style estimate.

Theorem (Laurent [**9**]). *Let E/K be an elliptic curve with complex multiplication given by a Weierstrass equation*

$$E : y^2 = x^3 + Ax + B,$$

let $m \geq 2$ *be an integer, and let* $\phi(x) \in K(x)$ *be the rational map satisfying*

$$x\bigl([m]P\bigr) = \phi\bigl(x(P)\bigr) \quad \textit{for all } P \in E(C).$$

There is a constant $\varepsilon(E) > 0$ *so that for all nontorsion points* $P \in E(\overline{K})$,

$$\hat{H}_\phi(P) \geq 1 + \frac{\varepsilon(E)}{d} \cdot \left(\frac{\log\log d}{\log d}\right)^3.$$

Up to now, Dobrowolski's method has not been successfully applied to general elliptic curves, so the resulting estimates are weaker.

Theorem. *Let* E/K, $m \geq 2$, *and* ϕ *be as in the statement of Laurent's theorem, except that we no longer require* E *to have complex multiplication.*

(a) [**11, 20**]. *There is a constant* $\varepsilon(E) > 0$ *so that for all nontorsion points* $P \in E(\overline{K})$,

$$\hat{H}_\phi(P) \geq 1 + \frac{\varepsilon(E)}{d^3(\log d)^2}.$$

(b) [**7**]. *If* $j(E)$ *is nonintegral, then there is a constant* $\varepsilon(E) > 0$ *so that for all non-torsion points* $P \in E(\overline{K})$,

$$\hat{H}_\phi(P) \geq 1 + \frac{\varepsilon(E)}{d^2(\log d)^2}.$$

For each of the preceding examples, there is a group variety underlying the given rational map. If such a group variety is not present, then the dynamics are much harder to analyze and little is known of the arithmetic. Of course, one can find a lower bound for $\hat{H}_\phi(\alpha)$ by using an elementary bound for the number of points in $\mathbf{P}^1(\overline{\mathbf{Q}})$ of bounded degree and height. This leads to the following horrible estimate.

Trivial bound. *Let* $\phi(x) \in \overline{\mathbf{Q}}(x)$ *be a rational map of degree* $n \geq 2$. *There is a constant* $\varepsilon(\phi) > 0$ *so that if* $\alpha \in \overline{\mathbf{Q}}$ *has degree* d *and is not* *pre-periodic for ϕ, then*

$$\hat{H}_\phi(\alpha) \geq 1 + \frac{\varepsilon(\phi)}{n^{2^6 d^2}}.$$

Neither the transcendence theory methods of Dobrowolski-Stewart nor the Fourier averaging techniques of Blanksby-Montgomery are directly applicable to this general situation, so it would be interesting if one could prove a lower bound for $\hat{H}_\phi(\alpha)$ even as weak as $1+C(\phi)^{-d}$.

Example 4. An interesting example to consider is $\phi(x) = x^2 + 1/4$, or equivalently $\phi(x) = x^2 + x$. The Julia set is then connected, and its complement consists of two connected components. Thus the picture in some ways resembles the case x^2, so possibly there is some hope of analyzing the dynamics closely enough to obtain a good lower bound for $\hat{H}_\phi$. The inherent instability of the situation is illustrated by the fact that the Julia set for (say) $\phi(x) = x^2 + x + 1/1000$ is totally disconnected!

Some other examples with interesting Julia sets which might provide good testing grounds for studying $\hat{H}_\phi$ include

$$\phi(x) = x^2 - 3/5, \qquad \phi(x) = x^2 - 3/4,$$
$$\phi(x) = x^2 - 1, \qquad \phi(x) = x^2 - 7/4.$$

Question 1. Let $\phi(x)$ be one of the quadratic polynomials described in Example 4. Are there any ϕ-Salem numbers? Are there infinitely many? Does every polynomial $\phi(x) \in \mathbf{Q}[x]$ have at least one ϕ-Salem number? infinitely many?

If a polynomial $\phi(x) \in \mathbf{Q}[x]$ has a filled Julia set whose interior is non-empty, then we can also define ϕ-PV numbers. Thus we will say that $\alpha \in \overline{\mathbf{Q}}$ is a *ϕ-PV number* if α is in the attracting basin of ∞ and every other conjugate of α is in the interior of the filled Julia set. Notice for $\phi(x) = x^n$ we get the usual PV-numbers.

Question 2. Let $\phi(x) \in \mathbf{Q}[x]$ be a polynomial whose filled Julia set has a nonempty interior. Does there always exist at least one ϕ-PV number? Is the set of ϕ-PV numbers closed and countable? If so, what is the smallest ϕ-PV number for the polynomials in Example 4, such as $x^2 + x$ or $x^2 - 1$?

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