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ON MAHLER'S CLASSIFICATION IN LAURENT SERIES FIELDS

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ABSTRACT. In 1932, K. Mahler introduced his famous classification for complex numbers in disjoint sets A, S, T, U [**9, 10**]. In 1978, P. Bundschuh introduced a similar classification for the field of formal Laurent series over a finite field K and gave some explicit series in the class U. Here we consider the case of an arbitrary field K and prove the existence of U-numbers whose continued fractions verify additional properties.

0. Introduction. T. Schneider's book [**16**, Chapter 3] is a complete introduction to the subject, whereas A. Baker's book [**2**, Chapter 8] offers a general outlook. For a polynomial $P = c_n x^n + \cdots + c_0$ in $\mathbf{Z}[X]$ with $c_n \neq 0$, we define the degree $d(P)$ and the height $h(P)$ by

$$d(P) = n, \qquad h(P) = \operatorname{Max}\{|c_j|, 0 \le j \le n\}.$$

For natural numbers $n \ge 1$, $H \ge 1$, we consider

$$P_{n,H} = \{P \in \mathbf{Z}[X] : d(P) \le n, H(P) \le H\} \tag{1}$$

and for any complex number ξ, we define $w(n, H, \xi)$, $w_n(\xi)$, $w(\xi)$ by

$$\operatorname{Min}\{|P(\xi)| : P \in P_{n,H}\} = H^{-nw(n,H,\xi)}$$

and

$$\begin{aligned} w_n &= w_n(\xi) = \limsup_{H\to\infty} w(n, H, \xi);\\ w &= w(\xi) = \limsup_{n\to\infty} w_n(\xi)\\ v &= v(\xi) = \operatorname{Inf}\{n : w_n(\xi) = \infty\}\\ &\text{with } v = \infty \text{ if } w_n < \infty \text{ for all } n. \end{aligned} \tag{2}$$

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The classes A, S, T, U are defined by

$$(3)\qquad \begin{cases} \xi \text{ is an } A\text{-number} & \text{if } w(\xi)=0,\ v(\xi)=\infty \\ \xi \text{ is an } S\text{-number} & \text{if } 0<w(\xi)<\infty,\ v(\xi)=\infty \\ \xi \text{ is a } T\text{-number} & \text{if } w(\xi)=\infty,\ v(\xi)=\infty \\ \xi \text{ is a } U\text{-number} & \text{if } w(\xi)=\infty,\ v(\xi)<\infty. \end{cases}$$

In this case we say ξ is a U_v-number.

The A-numbers are the set of algebraic numbers. Almost all complex numbers and almost all real numbers are S-numbers (Lebesgue measure). Liouville numbers are U_1-numbers. The existence of U_v-numbers of any degree, v, was proved by LeVeque [**9**], and the existence of T-numbers was proved by W.M. Schmidt [**15**]. W.M. Schmidt gives a recursive construction to prove the existence of T^*-numbers (with Koksma's classification [**8**], based on approximation of ξ by algebraic numbers, which is equivalent to Mahler's classification). In the same paper W.M. Schmidt shows that, for any irrational number α which isn't Liouville, there exists an irrational β which is not Liouville such that α/β is Liouville.

The basic results of diophantine approximation give few results on the functions $w(n,h,\xi)$, $w_n(\xi)$, $w(\xi)$. Using Dirichlet's theorem one proves that $w_n \geq 1$, for any n and ξ transcendental, but also if ξ is algebraic of degree $d > n$. Roth's theorem gives $w_1 \leq 1$ for ξ algebraic and Schmidt's theorem gives $w_n \leq 1$ for $n < d$ and ξ algebraic of degree d. There is some correlation with the theory of continued fractions. A. Baker [**3**] proved that there are U-numbers and also T or S-numbers in the set of real numbers with bounded partial quotients and that there are U-numbers and also T or S-numbers in the set of real numbers with unbounded partial quotients. Moreover, techniques using continued fractions were used by E. Burger and T. Struppeck to prove the existence of U_2-numbers whose partial quotients lie in a given set and obey a given distribution (see [**6**]). For the field of formal Laurent series a similar technique is used in the present paper and in [**5**].

Let $K((1/z))$ be the field of formal Laurent series (in $1/z$) over an arbitrary field K. The sets $\mathcal{Z} = K[z]$, $\mathcal{Q} = K(z)$, $\mathcal{R} = K((1/z))$ play the analogous roles of $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, respectively. We define the 0-adic absolute value $|\cdot|_0$ on $\mathcal{Z}$ by

$$|f(z)|_0 = e^\delta \quad \text{if } f(z) = z^\delta f_1(z)$$
$$\text{with} \quad f_1(z) \in \mathcal{Z},\ f_1(0) \neq 0; \quad |0|_0 = 0.$$

This absolute value is then extended to $\mathcal{Q}$ and to its completion $\mathcal{R} = K((1/z))$.

In this context Roth's theorem was generalized by S. Uchiyama [**17**] and Schmidt's theorem by E. Dubois [**7**] and M. Ratliff [**14**] when $\operatorname{char}(K) = 0$. These theorems do not hold when $\operatorname{char}(K) \neq 0$. Osgood [**13**] and de Mathan [**11**] gave some counterexamples.

P. Bundschuh [**4**] extended Mahler's classification to the field of formal Laurent series over a finite field K and gave some explicit examples of U-numbers. The main result of this paper was announced in the note [**5**], which contained a sketch of the proof when $\operatorname{char}(K) \neq 2$. In the present paper we give the complete proof for any infinite field K or any K with $\operatorname{char}(K) \neq 2$.

1. Mahler's classification. Let us consider polynomials

$$P(X) = c_n X^n + \cdots + c_0 \quad \text{with} \quad c_j = c_j(z) \in K[z].$$

For convenience we denote the degree of P with respect to X by $d(P)$ and the degree of c_j with respect to z by $\delta(c_j)$ (thus $|c_j|_0 = e^{\delta(c_j)}$). The logarithmic height of P is defined as

$$h(P) = \operatorname{Max}\{\delta(c_j) : 0 \leq j \leq n\}.$$

For integers $n \geq 1$, $h \geq 0$, we consider

$$\mathcal{P}_{n,h} = \{P \in \mathcal{Z}[X], d(P) \leq n, h(P) \leq h\}. \tag{4}$$

So when K is finite, one may easily define $w(n,h,\alpha)$ by

$$\operatorname{Inf}\{|P(\alpha)|_0 : P \in \mathcal{P}_{n,h}\} = e^{-nhw(n,h,\alpha)}. \tag{5}$$

In this formula e^{-h} plays the role H did in the introduction. Then formulas (2) and (3) determine a similar classification for S, T and U.

When K is infinite we use the same definition, but only after proving that the infimum in (5) is positive.

Lemma 1. *Let α be a transcendental number in $\mathcal{R}$ over $\mathcal{Q}$. Then, for any integers $n \geq 1$, $h \geq 0$, we have*

$$\mathit{Inf}\{|P(\alpha)|_0 : P \in \mathcal{P}_{n,h}\} > 0. \tag{6}$$

If α is algebraic of degree d over $\mathcal{Q}$ the same result holds for $h \geq 0$ and $1 \leq n < d$.

Proof. If $|\alpha|_0 > 1$ we consider $\beta = \alpha z^{-\delta(\alpha)}$ (so $|\beta|_0 = 1$) and the correspondence

$$P = c_n X^n + \cdots + c_0 \longleftrightarrow \tilde{P} = c_n z^{n\delta(\alpha)} X^n + \cdots + c_0$$

so that $P(\alpha) = \tilde{P}(\beta)$ and $h(\tilde{P}) \leq h(P) + n\delta(\alpha)$. As we have

$$\begin{aligned}\mathrm{Inf}\,\{|P(\alpha)|_0, P \in \mathcal{P}_{n,h}\} &= \mathrm{Inf}\,\{|\tilde{P}(\beta)|_0, P \in \mathcal{P}_{n,h}\}\\ &\geq \mathrm{Inf}\,\{|Q(\beta)|_0, Q \in \mathcal{P}_{n,h_1}\}\end{aligned}$$

where $h_1 = h + n\delta(\alpha)$, the lemma for β, n, h_1 gives the lemma for α, n, h. So it is enough to prove (6) when $|\alpha|_0 \leq 1$.

Let α be such that $|\alpha|_0 \leq 1$ and suppose that there exist integers n, h such that

$$\mathrm{Inf}\,\{|P(\alpha)|_0 : P \in \mathcal{P}_{n,h}\} = 0. \tag{7}$$

We will now prove that there exists a polynomial P in $\mathcal{P}_{n,h}$ such that $P(\alpha) = 0$ and $P \neq 0$. We write

$$\begin{aligned}\alpha^i &= \sum_{k\geq 0} a_{i,k} z^{-k} \quad \text{with} \quad a_{i,k} \in K \quad \text{for } k \geq 0,\ i \geq 0,\\ P(X) &= \sum_{0\leq i\leq n} c_i X^i, \qquad c_i = \sum_{0\leq j\leq h} c_{ij} z^j \quad \text{with} \quad c_{ij} \in K.\end{aligned} \tag{8}$$

Then

$$P(\alpha) = \sum_{0\leq i\leq n} \sum_{0\leq j\leq h} c_{ij} z^j \sum_{k\geq 0} a_{i,k} z^{-k} = \sum_{s\geq -h} l_s(\underline{c}) z^{-s},$$

where $\underline{c} = (c_{i,j} : 0 \le i \le n, 0 \le j \le h)$ and

$$l_s(\underline{c}) = \sum_{0 \le i \le n} \sum_{0 \le j \le h} a_{i,s+j} c_{i,j}, \qquad s \ge -h \tag{9}$$

are linear forms in the $(n+1)(h+1)$ unknowns $c_{i,j}$ with coefficients $a_{i,s+j}$ in K.

From (7) there exists, for any $r \ge -h$, a polynomial P in $\mathcal{P}_{n,h}$ such that $|P(\alpha)|_0 \le e^{-r}$. This means that the system

$$l_s(\underline{c}) = 0 \quad \text{for} \quad -h \le s < r$$

has a nonzero solution $\underline{c}$ in $K^{(n+1)(h+1)}$. Denote by M_r the matrix $(a_{i,s+j})$ with $r+h$ rows indexed by $-h \le s < r$ and $(n+1)(h+1)$ columns indexed by $0 \le i \le n$, $0 \le j \le h$. Then the rank of M_r is less than $(n+1)(h+1)$. The nondecreasing and stationary sequence rank$\,(M_r)$ has a maximum m less than $(n+1)(h+1)$. From a basis $l_{s(1)}, \dots, l_{s(m)}$ we get a nonzero solution $\underline{c}$ such that $l_{s(t)}(\underline{c}) = 0$ for $1 \le t \le m$ and then, $l_s(\underline{c}) = 0$ for any s. The corresponding polynomial P (from (8)) satisfies $P(\alpha) = 0$. This contradicts the fact that α is transcendental or algebraic of degree $d > n$. □

2. ***U*-numbers.** We say that α is *Liouville* if for any $\Omega > 0$ there exist q, p in $\mathcal{Z}$ such that $|q\alpha - p|_0 < |q|_0^{-\Omega}$. Clearly α is Liouville if and only if α is a U_1-number. Similarly, α is a U_2-number if and only if

$$\begin{cases} \forall\, \Omega > 0, \exists\, q, p_1, p_2 \text{ in } \mathcal{Z} \text{ such that } |q + p_1\alpha + p_2\alpha^2|_0 < |q|_0^{-\Omega} \\ \exists\, \Omega_1 > 0, \forall\, q, p \text{ in } \mathcal{Z} \text{ we have } |q\alpha - p|_0 > |q|_0^{-\Omega_1}. \end{cases} \tag{10}$$

Using the continued fraction algorithm, E. Burger and T. Struppeck [**6**] in the real case and [**5**] in the formal Laurent series case announced the existence of U_2-numbers α, with the property that for $b_1, b_2, \dots$ in $\mathbf{Z}$ or $\mathcal{Z}$ (with Char$\,(K) \ne 2$), $(\alpha + b_1)^2, (\alpha + b_2)^2, \dots$ are Liouville. We will now write the complete proof, replacing $(\alpha + b)^2$ by $\alpha + b - 1/(\alpha + b)$ and this is valid for any base field K.

Suppose $\mathcal{A}$ is a subset of $\mathcal{Z} \backslash K$ and $\mathcal{P} : \mathcal{A} \to [0,1]$ is a probability measure on $\mathcal{A}$. For α in $\mathcal{R} \backslash \mathcal{Q}$ with continued fraction $\alpha =$

$[a_0, a_1, \dots, a_n \dots]$ we say that the *asymptotic density of partial quotients of* α *agrees with* $\mathcal{P}$ if for each $a \in \mathcal{A}$,

$$\mathcal{P}(a) = \lim_{N\to\infty} \frac{1}{N} \#\{n, 1 \le n \le N, a_n = a\}.$$

Theorem. *Let* $a_0 \in \mathcal{Z}$, $\mathcal{B}$ *and* $\mathcal{A}$ *be subsets of* $\mathcal{Z}\backslash K$. *Further assume that* $\#\mathcal{A} \ge 2$ *and* $a_0 + \mathcal{B} \subset \mathcal{A}$. *Let* $\mathcal{P}$ *be a probability measure on* $\mathcal{A}$. *Then there exist uncountably many* U_2*-numbers,* $\alpha = [a_0, a_1, \dots] \in K((1/z))$ *such that*

(i) *any partial quotient* a_n, $n \ge 1$, *is in* $\mathcal{A}$;

(ii) *the asymptotic density of the partial quotients of* α *agrees with* $\mathcal{P}$;

(iii) *for any* n *large enough,* $|a_n|_0 < e^n$;

(iv) *for each* b *in* $\mathcal{B}$, $\alpha + b - 1/(\alpha + b)$ *is a Liouville number.*

When the 0-adic absolute values of the elements in $\mathcal{A}$ are bounded, the theorem asserts that there exist U_2-numbers α which are badly approximable and for which $\alpha+b-1/(\alpha+b)$ is Liouville (and therefore very approximable) for each b in $\mathcal{B}$.

To prove the theorem when $\mathcal{B} = \{b\}$, we construct a sequence $(\alpha_j)_{j\ge1}$ of quadratic irrationals over $\mathcal{Q}$ with partial quotients in $\mathcal{A}$ such that the asymptotic density is close to $\mathcal{P}$ and such that $\alpha_j + b - 1/(\alpha_j + b)$ is in $\mathcal{Q}$ and is a very good approximation of $\alpha + b - 1/(\alpha + b)$ where $\alpha = \lim_{j\to\infty} \alpha_j$.

2.1. Properties of continued fractions. In 1924, E. Artin [**1**] introduced the continued fraction algorithm in a field of Laurent series (see also [**11**]). For $\alpha \in \mathcal{R}$ we denote the partial quotients, the complete quotients and the convergents of α, respectively, by a_n, $\alpha^{(n)}$ and p_n/q_n. We have the standard formulas

$$\alpha = [a_0, a_1, \cdots a_n \cdots], \quad a_n \in \mathcal{Z}, \quad n \ge 0,$$
$$|a_n|_0 > 1, \quad n \ge 1,$$
$$\alpha^{(n)} = [a_n, a_{n+1}, \dots], \quad n \ge 0,$$
$$\alpha^{(n)} = a_n + 1/\alpha^{(n+1)},$$

$$p_{-2}=0, \qquad p_{-1}=1, \qquad p_{n+1}=a_{n+1}p_n+p_{n-1}, \quad n\geq -1,$$
$$q_{-2}=1, \qquad q_{-1}=0, \qquad q_{n+1}=a_{n+1}q_n+q_{n-1}, \quad n\geq -1.$$

Many standard properties of continued fractions remain valid in this context but there are differences. For example, if the sequence of partial quotients of α is periodic, then α is quadratic over $\mathcal{Q}$. (For a purely periodic α with length l we have $\alpha=(p_l\alpha+p_{l-1})/(q_l\alpha+q_{l-1})$). But the converse does not hold when K is infinite. It is only true when K is finite. (For example, $\alpha=[z,z,z/2,2z,\dots,z/2^n,2^nz,\dots]$ satisfies $z\alpha^2+(1-z^2)\alpha-2z=0$).

Since the absolute value is an ultrametric we get [**11**] the approximation

$$|q_n(q_n\alpha-p_n)|_0=1/|a_{n+1}|_0, \tag{11}$$

$$|q_n\alpha-p_n|_0=\mathrm{Min}\,\{|q\alpha-p|_0:(p,q)\in\mathcal{Z}^2,|q|_0<|q_{n+1}|_0\}. \tag{12}$$

From (11) and (12) we can say that α is badly approximable if and only if the absolute value of its partial quotients is bounded.

Lemma 2. *Let a_0 be in $\mathcal{Z}\backslash K$ and $(a_1,\dots,a_t)$ be a symmetric sequence in $(\mathcal{Z}\backslash K)^t$. Then the quadratic number α defined by the purely periodic continued fraction*

$$[\overline{a_0,a_1,\dots,a_t,a_0}]$$

is such that $\alpha-1/\alpha\in\mathcal{Q}$.

Proof. Using $\alpha^{(t+2)}=\alpha$ and the conjugate of the relation $\alpha^{(n)}=a_n+1/\alpha^{(n+1)}$ for $0\leq n\leq t+2$, we easily get that the continued fraction of $-1/\alpha'$ is that of α with the period reversed (here α' is the algebraic conjugate of α). Using symmetry we have $-1/\alpha'=\alpha$. Thus, $\mathrm{Tr}\,(\alpha)=\alpha+\alpha'=\alpha-1/\alpha\in\mathcal{Q}$. □

2.2 Proof of the theorem. First we suppose that $\mathcal{B}=\{b\}$.

Let $(\varepsilon_j)_{j\geq 1}$, $(\Omega_j)_{j\geq 1}$ be two monotonic sequences of positive real numbers with $\lim_{j\to\infty}\varepsilon_j=0$ and $\lim_{j\to\infty}\Omega_j=\infty$. Denote by $\mathcal{A}^*$ the set of words composed with the alphabet $\mathcal{A}$. If $W = (w_1, \dots, w_N) \in \mathcal{A}^*$ with length N we use the notation $\overrightarrow{W} = W$, $\overleftarrow{W} = (w_N, \dots, w_1)$, $W^{M+1} = W^M W$ for M a natural number. The density of $s \in \mathcal{A}$ with respect to W is

$$\delta(s, W) = \#\{n : 1 \leq n \leq N, w_n = s\}/N.$$

We now recursively define an infinite sequence $(\alpha^{(j)})_{j \geq 1}$ of quadratic irrationals. We select a word W_1 in $\mathcal{A}^*$ which satisfies $\#W_1 \geq b/\varepsilon_1$ and

$$|\delta(a, W_1) - \mathcal{P}(a)| < \varepsilon_1/3 \quad \text{for all } a \in \mathcal{A}.$$

Let α_1 be the quadratic $[a_0, \overline{\overrightarrow{W}_1, \overleftarrow{W}_1, a_0 + b, a_0 + b}]$, and then $\beta_1 = \alpha_1 + b$ satisfies (from Lemma 2)

$$\beta_1 - \frac{1}{\beta_1} = \frac{r_1}{s_1} \in \mathcal{Q}, \qquad \gcd(r_1, s_1) = 1.$$

We choose a natural number M_1 such that the convergent

$$\rho(W_1, M_1) = [a_0, (\overrightarrow{W}_1, \overleftarrow{W}_1, a_0 + b, a_0 + b)^{M_1 - 1}, \overrightarrow{W}_1, \overleftarrow{W}_1, a_0 + b]$$

satisfies

$$|\alpha_1 - \rho(W_1, M_1)|_0 < |b^2 r_1|_0^{-\Omega_1}.$$

We suppose that W_j, $\alpha^{(j)}$, M_j and $\rho(W_j, M_j)$ have already been described and now we describe how to generate $W_{j+1}, \alpha^{(j+1)}$, M_{j+1} and $\rho(W_{j+1}, M_{j+1})$. We select $W'_{j+1} \in \mathcal{A}^*$ such that

$$\#W'_{j+1} \geq 6M_j(1 + \#W_j)/\varepsilon_{j+1}, \tag{13}$$

$$|\delta(a, W'_{j+1}) - \mathcal{P}(a)| < \varepsilon_{j+1}/3 \quad \text{for all } a \in \mathcal{A}, \tag{14}$$

and

(15) W'_{j+1} contains a sequence which does not occur in the sequence of partial quotients of α_j.

If l denotes the period length of α_j, then there are only l sequences of length l in the sequence of partial quotients of α_j. As $\#\mathcal{A} \geq 2$, (13), (15) and (iii) are easy to get. For (14) we further require that the density in any truncation of W'_{j+1} does not stray too far apart from $\mathcal{P}$. Next we define

$$W_{j+1} = ((\overrightarrow{W}_j, \overleftarrow{W}_j, a_0+b, a_0+b)^{M_j}, W'_{j+1}), \tag{16}$$

$$\begin{aligned}\alpha_{j+1} &= [a_0, \overline{\overrightarrow{W}_{j+1}, \overleftarrow{W}_{j+1}, a_0+b, a_0+b}],\\ \beta_{j+1} &= \alpha_{j+1} + b.\end{aligned} \tag{17}$$

By Lemma 2, we find

$$\beta_{j+1} - \frac{1}{\beta_{j+1}} = \frac{r_{j+1}}{s_{j+1}} \in \mathcal{Q}, \qquad \gcd(r_{j+1}, s_{j+1}) = 1. \tag{18}$$

We choose a natural integer M_{j+1} such that the convergent

$$\begin{aligned}\rho(W_{j+1}, M_{j+1}) = [a_0, (\overrightarrow{W}_{j+1}, \overleftarrow{W}_{j+1}, a_0+b, a_0+b)^{M_{j+1}-1},\\ \overrightarrow{W}_{j+1}, \overleftarrow{W}_{j+1}, a_0+b]\end{aligned} \tag{19}$$

satisfies

$$|\alpha_{j+1} - \rho(W_{j+1}, M_{j+1})|_0 < |b^2 r_{j+1}|_0^{-\Omega_{j+1}}. \tag{20}$$

Then the first $\#W_j$ partial quotients of α_j and α_{j+1} are equal. Hence, $\alpha = \lim_{j\to\infty} \alpha_j$ exists. The continued fraction of α satisfies (i) and (iii) and by (15) is not periodic.

To prove (ii) it is enough, with (14), to show that for any $a \in \mathcal{A}$ we have

$$\begin{aligned}\operatorname{Max}(|\delta(a, m(W_{j+1}, M_{j+1})) - \delta(a, W_{j+1})|,\\ |\delta(a, W_{j+1}) - \delta(a, W'_{j+1})|) < \varepsilon_{j+1}/3,\end{aligned} \tag{21}$$

where $m(W_{j+1}, M_{j+1})$ is the word associated to $\rho(W_{j+1}, M_{j+1})$ without the first a_0. From (19), we have

$$\delta(a, m(W_{j+1}, M_{j+1})) = \frac{(2\delta(a, W_{j+1})\#W_{j+1} + 2\eta)M_{j+1} - \eta}{(2\#W_{j+1} + 2)M_{j+1} - 1}$$

where $\eta = 1$ if $a = a_0 + b$ and $\eta = 0$ otherwise. Then, using (16) and (13) we have

$$\begin{aligned}|\delta(a, m(W_{j+1}, M_{j+1})) - \delta(a, W_{j+1})| \\ &\le \frac{\delta(a, W_{j+1})(2-2\eta)M_{j+1} + |\delta(a, W_{j+1}) - \eta|}{(2\#W_{j+1}+2)M_{j+1} - 1} \\ &\le \frac{2M_{j+1}+1}{(2\#W_{j+1}+2)M_{j+1} - 1} \\ &\le \frac{2}{\#W_{j+1}} \le \frac{\varepsilon_{j+1}}{3}.\end{aligned}$$

Similarly, from (16),

$$\delta(a, W_{j+1}) = \frac{2M_j(\delta(a, W_j)\#W_j + \eta) + \delta(a, W'_{j+1})\#W'_{j+1}}{2M_j(\#W_j + 1) + \#W'_{j+1}}$$

$$\begin{aligned}|\delta(a, W_{j+1}) - \delta(a, W'_{j+1})| \\ &= \frac{2M_j[\#W_j(\delta(a, W_j) - \delta(a, W'_{j+1})) + \eta - \delta(a, W'_{j+1})]}{2M_j(\#W_j + 1) + \#W'_{j+1}}\end{aligned}$$

and with (13) we have

$$\begin{aligned}|\delta(a, W_{j+1}) - \delta(a, W'_{j+1})| &\le \frac{2M_j(\#W_j + 1)}{2M_j(\#W_j + 1) + \#W'_{j+1}} \\ &\le \frac{\varepsilon_{j+1}}{3}.\end{aligned}$$

This proves (21) and condition (ii).

Now we prove that $\alpha + b - 1/(\alpha + b)$ is Liouville and that α is a U_2-number. By (19) and (17) we remark that the continued fractions of $\rho(W_j, M_j)$, α_j and α have the same beginning. Since the number of partial quotients which are the same for α and α_j is greater than the length of $\rho(W_j, M_j)$, we get

$$|\alpha - \alpha_j|_0 < |\alpha_j - \rho(W_j, M_j)|_0.$$

Using (18) and (20), we have

$$\left|\alpha + b - \frac{1}{\alpha + b} - \frac{r_j}{s_j}\right|_0 = \left|(\alpha - \alpha_j)\left(1 + \frac{1}{(\alpha + b)(\alpha_j + b)}\right)\right|_0$$
$$= |\alpha - \alpha_j|_0 < |b^2 r_j|_0^{-\Omega_j}.$$

Hence, $\alpha + b - 1/(\alpha + b)$ is Liouville.

To show that $w_2(\alpha) = \infty$ (and then $w(\alpha) = \infty$) we consider:

$$Q(\alpha) = s_j\alpha^2 + (2bs_j - r_j)\alpha + (b^2 - 1)s_j - br_j \quad \text{and} \quad h = h(Q).$$

We have $e^h = \operatorname{Max}(|s_j|_0, |2bs_j - r_j|_0, |(b^2 - 1)s_j - br_j|_0) < |b^2 r_j|_0$, and $|Q(\alpha)|_0 < |s_j(\alpha + b)|_0 . |b^2 r_j|_0^{-\Omega_j} < |b^2 r_j|_0^{1-\Omega_j} < (e^h)^{(1-\Omega_j)}$. From $e^{-2hw(2,h,\alpha)} = \operatorname{Inf}\{|P(\alpha)|_0, P \in \mathcal{P}_{2,h}\} < (e^h)^{(1-\Omega_j)}$, we easily get $\omega(2, h, \alpha) \geq (\Omega_j - 1)/2$. Since $\lim_{j\to\infty} \Omega_j = \infty$, we have $w_2(\alpha) = +\infty$. So α is U_2 if α is not Liouville.

We denote the convergent of α by p_n/q_n. For any $q \in \mathcal{Z}$, there exists an n such that

$$|q_n|_0 \leq |q| < |q_{n+1}|_0.$$

We have, for any n, $|q_n|_0 = |a_n q_{n-1}|_0$ so using (12) and the condition (iii) we have $|q_n|_0 \geq e^n \geq |a_{n+1}|_0$ for n large enough. So for any $p \in \mathcal{Z}$ we get

$$|q\alpha - p|_0 \geq |q_n\alpha - p_n|_0 \geq |a_{n+1}q_n|_0^{-1} \geq |q_n|_0^{-2} \geq |q|_0^{-2},$$
$$e^{-hw_1(1,h,\alpha)} = \operatorname{Inf}(|P(\alpha)|_0, P \in \mathcal{P}_{1,h}) \geq |\alpha|_0^2 e^{-2h}.$$

Then $w_1(\alpha) = \limsup_{h\to\infty} w(1, h, \alpha) \leq 2$. So α isn't Liouville and is U_2.

To get infinitely many α it is enough to remark that at each step j we have many choices for W'_{j+1} such that (13), (14) and (15) hold. This comes from $\#\mathcal{A} \geq 2$. From two different choices at one step j_0, we get two different numbers α. So we get uncountably many numbers α and the theorem holds when $\mathcal{B} = \{b\}$.

When $\mathcal{B} = \{b_1, b_2, \dots\}$ we use the idea of [**6**]. We construct a sequence of quadratic numbers $\alpha_{J,i}$ for $J \geq 1$, $1 \leq i \leq J$. For convenience, we write the formulas (16) and (17) as

$$W_{j+1} = \varphi(W_j, M_j, W'_{j+1}), \qquad \alpha_{j+1} = \psi(W_{j+1}, b).$$

We choose $W_{1,1}, \alpha_{1,1}, M_{1,1}$ as before to be $W_1, \alpha_1 = \psi(W_1, b_1)$ and M_1 with $b = b_1$. Then from $W_{J-1,i}, \alpha_{J-1,i}, M_{J-1,i}$, $J \geq 2$, $1 \leq i \leq J-1$, we put

$$W_{J,0} = W_{J-1,J-1}, \qquad \alpha_{J,0} = \alpha_{J-1,J-1}, \qquad M_{J,0} = M_{J-1,J-1}.$$

Then for $i = 1, 2, \ldots, J$ we choose $W'_{J,i}$ with properties analogous to (13), (14) and (15) for $\alpha_{J,i-1}$ with

$$W_{J,i} = \varphi(W_{J,i-1}, M_{J,i-1}, W'_{J,i}), \qquad \alpha_{J,i} = \psi(W_{J,i}, b_i).$$

We get

$$\alpha_{J,i} + b_i - \frac{1}{\alpha_{J,i} + b_i} = \frac{r_{J,i}}{s_{J,i}} \in \mathcal{Q}, \tag{22}$$

and we choose $M_{J,i}$ such that

$$|\alpha_{J,i} - \rho(W_{J,i}, M_{J,i}, b_i)|_0 < |b_i^2 r_{J,i}|_0^{-\Omega_j}. \tag{23}$$

Hence $\alpha = \lim_{J\to\infty} \alpha_{J,J}$ exists and as before it follows that α is U_2. From (22), (23) and the construction we get as before that $\alpha + b_i - 1/(\alpha + b_i)$ is Liouville for any b_i in $\mathcal{B}$. □

3.3. *Remarks.* We can focus our attention on the degree (in z) of the partial quotients in the theorem.

Corollary. *Let $a_0 \in K$, $A \subset \mathbf{N}$, P be a probability measure on A and $\mathcal{B} \subset \mathcal{Z}\backslash K$ be such that the $\delta(\mathcal{B}) \subset A$. Then there exist uncountably many U_2-numbers α such that*

(i) *for any n, $\delta(a_n) \in A$ and $\delta(a_n) < n$ for n large enough;*

(ii) *the asymptotic density of $(\delta(a_n))_{n\geq 1}$ agrees with P;*

(iii) *for any b in $\mathcal{B}$, $\alpha + b - 1/(\alpha + b)$ is Liouville.*

In [**5**], a result similar to the theorem for $\mathrm{Char}(K) \neq 2$ is given as well as a sketch of the proof of the existence of U_2-numbers α such that $(\alpha + b)^2$ is Liouville. (The hypothesis $2(a_0 + \mathcal{B}) \subset \mathcal{A}$ replaces the former $a_0 + \mathcal{B} \subset \mathcal{A}$; Lemma 2 is replaced by the following property: For a symmetric sequence $(a_1, \dots, a_t)$ the quadratic number $\beta = [a_0, \overline{a_1, \dots a_t, 2a_0}]$ is such that $\beta^2 \in \mathcal{Q}$.)

The more difficult question, "Do T-numbers exist?" is still open in this context.

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