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COMMUTATIVE ALGEBRAIC GROUPS AND REFINEMENTS OF THE GELFOND-FELDMAN MEASURE

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ABSTRACT. The main theorem of this paper is a measure of algebraic independence for numbers associated with a one-parameter subgroup of a commutative algebraic group defined over a number field. Qualitative results in this setting have been given by M. Waldschmidt, R. Tubbs and M. Ably, who provided measures as well. We refine Ably's quantitative results, separating the degree and the height in the limit case when the group contains a copy of the additive group of complex numbers, i.e., $\mathbf{G}_a$. This new results provides several interesting corollaries, in particular, a generalization of G. Diaz's refined Gelfond-Feldman measure to higher dimensions and an improvement of Tubbs' elliptic Gelfond-Feldman measure.

1. Introduction and statement of result. We begin with a review of the standard objects in this general setting. Although our presentation is slightly different, this is essentially the setting of [**1**] or [**43**]. Let G be a commutative algebraic group of dimension $d \geq 1$ defined over a number field K. Let $\mathbf{G}_a$ denote the additive group of complex numbers and $\mathbf{G}_m$ the multiplicative group of complex numbers. We assume that G decomposes as

$$G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2$$

with $d_0 \in \{0, 1\}$, $d_1 \geq 0$ and G_2 a commutative algebraic group of dimension $d_2 = d - d_0 - d_1$, defined over K, and with no linear factor.

We let $\phi : \mathbf{C} \to G(\mathbf{C})$ be a one-parameter subgroup, i.e., an analytic homomorphism whose image is Zariski dense in $G(\mathbf{C})$. Given complex

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numbers $y_1, \dots, y_m$ which are linearly independent over $\mathbf{Q}$, we define $Y = \mathbf{Z}y_1 + \cdots + \mathbf{Z}y_m$ and $\Gamma = \phi(Y)$. We let $l = \mathrm{rank}_{\mathbf{Z}}(Y \cap \ker \phi)$, we suppose that $l < m$, and we assume, without loss of generality, that $y_{m-l+1}, \dots, y_m \in \ker \phi$.

To provide an algebraic independence result, we'll need an embedding, say χ, of G into multiprojective space. We'll specify χ more carefully below. Having fixed an embedding, we will give a measure of algebraic independence for a point ω, specified below, whose coordinates are simply coordinates of the points $\chi \circ \phi(y_j)$ with $1 \leq j \leq m-l$.

We also need to impose conditions which insure that we have "enough points" to generate a transcendence result. To simplify these conditions, we introduce quantities $\mu^\sharp$ and κ as in [**47**, p. 388] and [**1**]. We let π_0 and π_1 be projections of G onto $\mathbf{G}_a^{d_0}$ and $\mathbf{G}_m^{d_1}$, respectively. For every algebraic subgroup $G' \subsetneq G$ defined over K, we let $r = \dim(G/G')$, $r_0 = \dim(\mathbf{G}_a^{d_0}/\pi_0(G'))$, $r_1 = \dim(\mathbf{G}_m^{d_1}/\pi_1(G'))$, and $r_2 = r - r_0 - r_1$. With this in mind, we let

$$\mu^\sharp = \min_{G' \subsetneq G} \left\{ \frac{\mathrm{rank}_{\mathbf{Z}} \Gamma/(\Gamma \cap G') + r_1 + 2r_2}{r} \right\};$$

and by taking $G' = 0$, we see that

$$\mu^\sharp \leq \frac{m - l + d_1 + 2d_2}{d}.$$

We also define

$$\kappa = \frac{\mu^\sharp d - d_1 - 2d_2}{(1 - l/m)\mu^\sharp}.$$

We must make further assumptions on the points $y_1, \dots, y_m$, namely assumptions regarding the distribution of the points of $\Gamma = \phi(Y)$ among certain algebraic subgroups of G. This technical hypothesis, referred to as (H), will be carefully outlined below.

Lastly, we will use $\mathrm{Deg}\,(J)$, $\mathrm{Ht}\,(J)$, $T(J)$ and $||J||_\omega$, respectively, to denote the degree, height, size and absolute value at ω for an ideal $J \subseteq K[\mathcal{X}_1, \dots, \mathcal{X}_n]$ and a point $\omega \in \mathbf{C}^n$. Precise meanings for these notions, discussed more carefully below, have been provided by Yu. Nesterenko and P. Philippon.

We're now in a position to state the main result.

Theorem 1. *Suppose K, G, ϕ and $y_1, \ldots, y_m$ as above are given, that the corresponding κ satisfies $\kappa > 1$, that $\mu^\sharp > 2$ if G is nonlinear, and that hypothesis* (H) *is satisfied. Suppose G is embedded into projective space and ω is specified as below. Then there exists a constant $C > 0$, depending on K, G, ϕ, $y_1, \ldots, y_m$, $n, \kappa, \mu^\sharp$, and the embedding χ, such that for all ideals $J \subseteq K[\mathcal{X}_1, \ldots, \mathcal{X}_n]$ of codimension $n + 1 - \kappa$, degree* $\mathrm{Deg}\,(J)$*, height* $\mathrm{Ht}\,(J)$ *and size $T(J)$, we have*

$$\log ||J||_\omega \geq -\exp(C\mathrm{Deg}\,(J)^{d_0\kappa/(d-\kappa)}T(J)).$$

Letting $l = 0$ and assuming $\mu^\sharp$ is maximal yields the qualitative result of [**43**, Main Theorem]. Using the criteria of algebraic independence of Jabbouri and Philippon [**16**] and ideas of [**34**], M. Ably was able to quantify that result and weaken the technical hypothesis. We state his complete result [**1**, Théorème Principal] as Theorem 2 below. Note that in the case where $d_0 = 1$ we have separated the degree and height, thereby providing a refinement of his measure.

As Ably, we use the standard construction and the Philippon-Jabbouri criteria for algebraic independence. In order to take full advantage of this criteria, however, we choose parameters which *explicitly* depend on the degree and height of the ideal J. This idea was used by Diaz [**12**] to separate the degree and height in the classical Gelfond-Feldman measure.

You may note that Ably provides a measure in the nonlimit case as well. In our result, we have stated only the limit case, as our method does not allow a separation of degree and height in the nonlimit case. The proof and construction given here suggest that it is impossible to make a similar refinement in this case; we address this in a closing remark.

Theorem 2 [**1**]. *Suppose that hypothesis* (H) *is satisfied and that* $\kappa > 1$; *if G is nonlinear, we also suppose that $\mu^\sharp > 2$. Let k be an integer ≥ 0 such that $\kappa \geq k + 1$. Then there exists a real number*

$$C_3 = C_3(G, \chi, \phi, [K : \mathbf{Q}], x_1, \ldots, x_{d_1}, y_1, \ldots, y_m, k) > 0$$

such that

(1) *if* $\kappa = k+1$, *the function*

$$\Phi_1(T) = \exp(C_3 T^{d/(d-\kappa d_0)})$$

is a measure of algebraic independence of ω *in dimension* k,

(2) *if* $\kappa > k+1$, *the function*

$$\Phi_2(T) = C_3(T/(\log T)^{(d-\kappa d_0)/\kappa d})^{\kappa/(\kappa-k-1)}$$

is a measure of algebraic independence of ω *in dimension* k.

Before continuing with the preliminary details, we offer brief historical remarks. It is imperative to note the significant contributions of Masser, Wüstholz and Philippon on zero estimates, as well as Nesterenko and Philippon for the introduction of commutative algebra to transcendence theory and the development of criteria for algebraic independence. Their contributions were monumental as well as fundamental. As a side remark, the first quantitative results when not all of the numbers are algebraically independent were given by Nesterenko [**27**] and Philippon [**31**] as results of their independent work on criteria for algebraic independence.

Before Theorem 2 was provided by Ably, the Gelfond-Schneider problem and further generalizations had already been studied, independently, in the commutative algebraic group setting, by R. Tubbs and M. Waldschmidt. For instance, [**39**] contains a general result (Theorem 3) which includes Gelfond's algebraic independence of α^{β}, α^{β^2} when $\alpha \in \overline{\mathbf{Q}}$ with $\alpha \log \alpha \neq 0$ and β cubic over $\mathbf{Q}$ as well as the elliptic analogue which was provided by Masser and Wüstholz. More generally, Tubbs' work on algebraic groups provides a variety of corollaries establishing small transcendence degree for values of exponential and elliptic functions. In particular, he requires that G be defined only over an arbitrary subfield K of $\mathbf{C}$, not necessarily a number field. And he exploits general hypotheses such as periodicity or that some or all of the points under consideration have coordinates which are algebraic or which correspond to torsion points on the algebraic group. See [**39, 41, 42**] for such results.

The study of transcendence results in the commutative algebraic group setting was initiated by Waldschmidt after much work by himself and others in the exponential and elliptic settings. In particular, large transcendence degree for the Gelfond-Schneider problem in the commutative algebraic group setting was first established in [**45, 46**] under a strong technical hypothesis. Strengthened results are provided in [**43**] under a similar hypothesis. (A very general result, the theorem of the algebraic subgroup, is given in [**47**].)

2. Further preliminaries.

Specifying the embedding and the point ω. First we consider the map $\psi : \mathbf{C} \to G_2(\mathbf{C})$ given by the projection of $\phi(\mathbf{C})$ onto the G_2 component of $G(\mathbf{C})$. We make the following observation. If ψ is nontrivial and $T_{G_2}(\mathbf{C})$ is the tangent space to G_2 at the origin, then we have $\psi = \exp_{G_2} \circ \mathrm{Lie}\,\psi$, where $\mathrm{Lie}\,\psi : \mathbf{C} \to T_{G_2}(\mathbf{C})$ is the tangent map of ψ at the origin.

We let $\chi_2 : G_2(\mathbf{C}) \to \mathbf{P}_N(\mathbf{C})$ be a K-embedding of $G_2(\mathbf{C})$ into projective N-space as defined by J.-P. Serre in [**36**]. Then $\chi_2 \circ \exp_{G_2} : T_{G_2}(\mathbf{C}) \to \mathbf{P}_N(\mathbf{C})$ is given by analytic functions with order of growth at most two. We will note these functions by $\Theta_0, \Theta_1, \dots, \Theta_N$.

Given χ_2, we let χ be the natural K-embedding of $G(\mathbf{C})$ into $\mathbf{P}_{d_0}(\mathbf{C}) \times \mathbf{P}_{d_1}(\mathbf{C}) \times \mathbf{P}_N(\mathbf{C})$. Then $\chi \circ \phi$ can be represented as

$$(1,z;1,e^{x_1 z},\dots,e^{x_{d_1} z};\Theta_0(\mathrm{Lie}\,\psi(z)),\dots,\Theta_N(\mathrm{Lie}\,\psi(z))). \tag{1}$$

Here the coordinates corresponding to $\mathbf{P}_{d_0}, \mathbf{P}_{d_1}$ or $\mathbf{P}_N$ do not appear when d_0, d_1 or d_2, respectively, is zero. We note here that $x_1, \dots, x_{d_1}$ are $\mathbf{Q}$-linearly independent since $\phi(\mathbf{C})$ is Zariski dense in $G(\mathbf{C})$. Our technical hypothesis (H) will include a quantification of this linear independence as well.

With this embedding in mind, we would like to consider the algebraic independence of the coordinates of $\chi \circ \phi(y_j)$ for $1 \le j \le m-l$. We may assume, by virtue of a linear transformation, that $\Theta_0(\mathrm{Lie}\,\psi(y_j)) \ne 0$ for $1 \le j \le l-m$. Then we let

$$\omega = \Bigg(1, y_j; 1, \exp(x_1 y_j), \dots, \exp(x_{d_1} y_j); \\ 1, \frac{\Theta_1(\mathrm{Lie}\,\psi(y_j))}{\Theta_0(\mathrm{Lie}\,\psi(y_j))}, \cdots, \frac{\Theta_N(\mathrm{Lie}\,\psi(y_j))}{\Theta_0(\mathrm{Lie}\,\psi(y_j))} : 1 \le j \le m-l\Bigg), \tag{2}$$

where here, as above, we omit the corresponding irrelevant coordinates if $d_0 = 0$, $d_1 = 0$ or $d_2 = 0$. We also note that $\mathbf{P}_{d_0} \times \mathbf{P}_{d_1} \times \mathbf{P}_N \hookrightarrow \mathbf{P}_n$ via the Segre embedding.

Defining the degree, height, size and "absolute value at a point" for an ideal. As mentioned above, we need notions of the degree and height of an ideal as well as "the absolute value of an ideal evaluated at a point." Such ideas were provided by Yu. V. Nesterenko, via the introduction of resultant ideals (otherwise known as U-resultants, Chow forms, or "U-éliminante forms") and were extended by P. Philippon. In [**24**], we find the notion of the degree of a homogeneous prime ideal over a Noetherian ring, and in [**26**], the height of such an ideal over $\mathbf{Z}$. In [**25, 27**], Nesterenko defines the absolute value at a point for any unmixed homogeneous ideal $\mathfrak{h}$ of rank $n-d \le n$ in the polynomial ring $\mathbf{Z}[\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_n]$, provided $\mathfrak{h} \cap \mathbf{Z} = \{0\}$. These ideas were extended by Philippon in [**32**] to ideals over an arbitrary number field, using the Mahler measure, and thereby providing an invariant height.

Following the latter, we let v be a place of our number field K and n_v the local degree (i.e., the degree of K_v over $\mathbf{Q}_v$). We also let $\mathbf{C}_v$ be the completion of the algebraic closure of K_v and σ_v an embedding of K into $\mathbf{C}_v$ extending the canonical embedding of K into K_v. For a polynomial P with coefficients in K, we let $M_v(P)$ be the maximum absolute value of the coefficients of $\sigma_v(P)$ if v is finite and the Mahler measure of $\sigma_v(P)$ if v is infinite. Then the height and invariant height, respectively, of P are defined by

$$\bar{\underline{h}}(P) := \frac{1}{[K:\mathbf{Q}]} \sum_v n_v \max\{0, \log M_v(P)\}$$
$$\underline{h}(P) := \frac{1}{[K:\mathbf{Q}]} \sum_v n_v \log M_v(P).$$

More generally, consider a homogeneous ideal $I \subseteq K[\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_n]$ of codimension $n+1-r$ and an index $\underline{d} \in \mathbf{N}^r$. Again, following [**32**, Definition 1.14], we define

$$\operatorname{Ht}_{\underline{d}}(I) := \underline{h}(f_I)$$

and

$$\operatorname{Deg}_{\underline{d}}(I) := d^0(f_I)$$

where f_I is a "U-éliminante form" of I of index $\underline{d}$ and d^0 is the total degree. We omit the subscript $\underline{d}$ when the index is $\underline{1} = (1, \dots, 1) \in \mathbf{N}^r$. Further, for $\underline{x} \in \mathbf{P}_n(\mathbf{C}_v)$, we give [**32**, Définition 1.15] morphisms $\tilde{\mathfrak{d}}_{\underline{x},\underline{d}}$ (also noted as $\tilde{\mathfrak{d}}_{\underline{x}}$ or $\tilde{\mathfrak{d}}$) and then define the "absolute value of index $\underline{d}$ of I at $\underline{x}$" by

$$||I||_{\underline{x},v,\underline{d}} := M_v(\tilde{\mathfrak{d}}_{\underline{x},\underline{d}}(f_I))/M_v(f_I).$$

For an ideal J of $K[\mathcal{X}_1, \dots, \mathcal{X}_n]$ of codimension $n+1-r$ and $\underline{\theta} \in \mathbf{C}_v^n$, we let

$$||J||_{\underline{\theta}} := ||^h J||_{\underline{\underline{\theta}},v,\underline{d}},$$

where

$$\underline{\underline{\theta}} = (1 : \theta_1 : \cdots : \theta_n)$$

if $\underline{\theta} = (\theta_1, \dots, \theta_n)$ and

$$\mathrm{Deg}\,(J) := \mathrm{Deg}\,({}^h J) \quad \text{and} \quad T(J) := \mathrm{Ht}\,({}^h J) + \mathrm{Deg}\,({}^h J),$$

where ${}^h J$ is the homogenization of J in $K[\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_n]$ generated by the homogenizations of the elements of J.

When we're considering polynomials $P(X)$ instead of ideals, we fall back on the more familiar notation $d(P)$ to denote the maximum partial degree and $\deg(P)$ to denote the total degree. When P is also in $\mathbf{Z}[\mathcal{X}_1, \dots, \mathcal{X}_n]\backslash\{0\}$, we use $H(P)$ to denote the usual height, i.e., the maximum absolute value of the coefficients of P, and we use $t(P) = \max\{1 + d(P), \log H(P)\}$.

Specifying the technical hypothesis. Before specifying the technical hypothesis, we need an additional definition. We recall our K-embedding χ of G into $\mathbf{P}_{d_0} \times \mathbf{P}_{d_1} \times \mathbf{P}_N$. As in [**33**, p. 358], for positive real numbers $\overline{D}_0, \overline{D}_1, \overline{D}_2$, we say that a subvariety V of G is incompletely defined in G by equations of multidegree $\leq (\overline{D}_0, \overline{D}_1, \overline{D}_2)$ if V is an irreducible component of $G \cap Z(I)$, where $Z(I) \subset \mathbf{P}_{d_0} \times \mathbf{P}_{d_1} \times \mathbf{P}_N$ is the set of common zeros of an ideal I generated by polynomials over K of multidegree at most $(\overline{D}_0, \overline{D}_1, \overline{D}_2)$ with respect to the coordinates of $\mathbf{P}_{d_0} \times \mathbf{P}_{d_1} \times \mathbf{P}_N$.

For $h = (h_1, \dots, h_m) \in \mathbf{Z}^m$, we define $h \cdot y = h_1 y_1 + \cdots + h_m y_m$ and for positive real numbers S, we let $\mathbf{Z}^m(S) = \{h \in \mathbf{Z}^m : |h_j| < S, 1 \leq j \leq m\}$. We also let $T_G(\mathbf{C})$ be the tangent space of G at the origin with associated tangent map $\mathrm{Lie}\,\phi$, and we fix a norm $||\cdot||$ on $T_G(\mathbf{C})$.

We specify the following technical hypothesis which is very similar to (H) of [**1**].

(H) There exist positive constants c_0 and S_0 such that for all $S \geq S_0$ and for all algebraic connected subgroups $G' \subsetneq G \subset \mathbf{P}_{d_0} \times \mathbf{P}_{d_1} \times \mathbf{P}_N$, incompletely defined in G by equations of multi-degree at most $(S^{\kappa\mu^\sharp}(\log S), S^{\mu^\sharp - 1}(\log S)^{d_0/d}, 2S^{\mu^\sharp - 2}(\log S)^{d_0/d})$, and all $h \in \mathbf{Z}^m(S)$, we have:

(1) if $h \neq 0$, then $|h \cdot y| \geq \exp(-c_0 S \log S)$ and

(2) either $\phi(h \cdot y) \in G'(\mathbf{C})$ or for all $u \in T_G(\mathbf{C})$ such that $\exp_G(u) \in G'(\mathbf{C})$, we have

$$||u - \mathrm{Lie}\,\phi(h \cdot y)|| \geq \exp(-S^{\kappa\mu^\sharp} \log S).$$

Remark . For the proof, it is not necessary that hypothesis (H) be satisfied for all the subgroups of G mentioned above. It suffices, in fact, if (H) hold for certain such subgroups indicated more precisely below.

3. Special cases: new corollaries and previous results. In the pure exponential and elliptic cases, Theorem 1 has some interesting corollaries which refine the present measures of algebraic independence. We discuss these below, along with a review of previous quantitative results and some discussion of the progress on qualitative results, so as to provide an historical perspective.

3.1. *Exponential setting.* One popular example corresponds to the group $G = \mathbf{G}_a \times \mathbf{G}_m^{d_1}$ (so $d = d_1 + 1$) and the one-parameter subgroup $\phi(z) : \mathbf{C} \to G(\mathbf{C})$ given by $(z, e^{x_1 z}, \dots, e^{x_{d_1} z})$. In this case, the exponential map $\exp_G : \mathbf{C}^d \to G(\mathbf{C})$ is given by $\exp_G(z_0, z_1, \dots, z_{d_1}) = (z_0, e^{z_1}, \dots, e^{z_{d_1}})$ and we have $\phi = \exp_G \circ \mathrm{Lie}\,\phi$ where clearly $\mathrm{Lie}\,\phi : z \mapsto (z, x_1 z, \dots, x_{d_1} z)$.

If $x_1, \dots, x_{d_1}$ are $\mathbf{Q}$-linearly independent complex numbers, then $\phi(\mathbf{C})$ is Zariski-dense in $G(\mathbf{C})$. Suppose we also have a second set of $\mathbf{Q}$-linearly independent complex numbers $y_1, \dots, y_m$. With this in mind, we will consider the following technical hypothesis (H_1).

(H_1) Suppose there exist $C_4 > 0$ and $S_4 > 0$ such that for all $S \geq S_4$, for all $\lambda \in \mathbf{Z}^{d_1}(S)$ and all $h \in \mathbf{Z}^m(S)$, we have

$$|\lambda \cdot x| \geq \exp(-S^{md/(2m+d-2)})$$

and

$$|h \cdot y| \geq \max\{\exp(-C_4 S \log S), \exp(-S^{md/(m+2d-1)})\}.$$

This technical hypothesis insures that (H) is satisfied for certain subgroups of $\mathbf{G}_a \times \mathbf{G}_m^{d_1}$ as noted.

Here we consider the algebraic independence of the coordinates of $\omega = (y_1, \ldots, y_m, e^{x_1 y_1}, \ldots, e^{x_1 y_m}, \ldots, e^{x_{d_1} y_1}, \ldots, e^{x_{d_1} y_m})$. We have $\kappa = md/(m+d-1)$ and $\mu^\sharp = (m+d-1)/d$, and the statement of the theorem is as follows.

Corollary 3. *Suppose $\kappa = md/(m+d-1) > 1$ and (H_1) holds. Then there exists a positive C_5, depending only on G, x and y such that for $\omega = (y_1, \ldots, y_m, e^{x_1 y_1}, \ldots, e^{x_1 y_m}, \ldots, e^{x_{d_1} y_1}, \ldots, e^{x_{d_1} y_m})$ and for all ideals J of $K[\mathcal{X}_1, \ldots, \mathcal{X}_{md}]$ of codimension $md + 1 - \kappa$, degree* $\mathrm{Deg}\,(J)$*, and size $T(J)$, we have*

$$\log ||J||_\omega \geq -\exp(C_5 (\mathrm{Deg}\, J)^{m/d_1} T(J)).$$

From Theorem 2, we see that the previous measure was

$$\exp(C_3 T(J)^{(m+d_1)/d_1}).$$

In particular, consider $\alpha \in \mathbf{C}\backslash\{0\}$ with $\log\alpha \neq 0$ and $\beta \in \overline{\mathbf{Q}}$ of degree $d_1 \geq 2$. Letting $x_1, \ldots, x_{d_1}$ be $\log\alpha$, $\beta\log\alpha, \ldots, \beta^{d_1-1}\log\alpha$ and $y_1, \ldots, y_m$ be $1, \beta, \ldots, \beta^{d_1-1}$, we can choose $\omega = (\alpha, \alpha^\beta, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d_1-1}})$.[1] The technical hypothesis easily follows from a Liouville estimate, and we have the following corollary which is a generalization of the well-known Gelfond-Feldman measure.

Corollary 4. *Let α be a complex number such that $\alpha\log\alpha \neq 0$, and let β be an algebraic number of degree d_1 at least two. Then there exists a positive constant C_6, depending only on α and β such that for* *all ideals J of $K[\mathcal{X}_1, \dots, \mathcal{X}_{d_1}]$ of codimension $(d_1+1)/2$, degree* $\operatorname{Deg}(J)$ *and size $T(J)$, we have*

$$\log ||J||_{(\alpha, \alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d_1-1}})} \geq -\exp(C_6(\operatorname{Deg} J)T(J)).$$

This is a refinement of the measure $\exp(C_3 T(J)^2)$ given in [**1**].

In the case where α is also algebraic, the Gelfond-Feldman measure is essentially as in Corollary 4, with the exception that we may also consider ideals of dimension at most $(d_1-1)/2$ evaluated now at the (d_1-1)-tuple $(\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d_1-1}})$.

These results bring us to the limit of the current method. We take a few pages to provide a history of the exponential Gelfond-Schneider problem and improvements of the original Gelfond-Feldman measure. Unless otherwise noted, we assume that $\alpha \in \overline{\mathbf{Q}}$ with $\alpha \log \alpha \neq 0$ and $\beta \in \overline{\mathbf{Q}}$ of degree $d_1 \geq 2$.

Having established the transcendence of α^β, A.O. Gelfond [**14**] went on to provide a transcendence measure as well. He also considered the algebraic independence of the $d_1 - 1$ numbers $\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d_1-1}}$ with α and β as above, now known as the Gelfond-Schneider problem, and thereby established the well-known algebraic independence of α^β and α^{β^2} over $\mathbf{Q}$ when β is a cubic irrational [**13**] and [**14**, Theorem 1, pp. 132–133]. In 1950, Gelfond and N.I. Feldman [**15**] used his transcendence measure for α^β to provide a measure of this new algebraic independence. They gave the following result: *For every $\varepsilon > 0$, there exists a $t(\varepsilon) > 0$ such that for every nonzero polynomial $P(X,Y) \in \mathbf{Z}[X,Y]$ with $t(P) > t(\varepsilon)$, we have* $\log |P(\alpha^\beta, \alpha^{\beta^2})| > -\exp(t(P)^{4+\varepsilon})$.

In 1977, D. Brownawell [**3**] improved the original measure, separating the degree and the height to get something slightly better than $-\exp(C_7 d(P)^3 t(P))$ for some positive constant C_7. Actually, this strengthened measure followed from the proof of a more general result, namely a lower bound for pairs of relatively prime integral polynomials evaluated at $(\alpha, \alpha^\beta, \alpha^{\beta^2})$ when $\alpha \in \mathbf{C}$ (not necessarily algebraic) and β a cubic irrational number, improving similar lower bounds given in [**2**]. The main ingredients of Brownawell's proof were the semi-resultants of G.V. Chudnovsky and a simultaneous approximation result for a and a^b when $b \in \overline{\mathbf{Q}} \backslash \mathbf{Q}$ and $a \in \mathbf{C}$ with $a \log a \neq 0$. The second of these ingredients was provided by M. Mignotte and M. Waldschmidt in [**23**]; Brownawell [**2**] and [**44**] refer to other such results.

In 1987, using a very general theorem of P. Philippon [**31**, Théorème 2] on measures of algebraic independence, G. Diaz [**10**] was able to replace the $t(P)^{4+\varepsilon}$ in the original Gelfond-Feldman measure with $t(P)^{2+\varepsilon}$. Furthermore, by appealing to an improved criteria due to Philippon and E.M. Jabbouri [**16**], he [**12**] was also able to separate the degree and height to produce the lower bound $-C_8 \exp(C_9 d(P) t(P))$ for some positive constants C_8 and C_9.

Meanwhile, studies continued on the algebraic independence of $\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d_1-1}}$ when $\alpha \in \overline{\mathbf{Q}}$ with $\alpha \log \alpha \neq 0$ and β algebraic of degree $d_1 \geq 2$. More generally, authors began to consider the independence of sets of the form $\{x_i, y_j, e^{x_i y_j} : 1 \leq i \leq n, 1 \leq j \leq m\}$, $\{x_i, e^{x_i y_j} : 1 \leq i \leq n, 1 \leq j \leq m\}$, $\{e^{x_i y_j} : 1 \leq i \leq n, 1 \leq j \leq m\}$, as in Corollary 3 above. The best qualitative result to date for this more general Gelfond-Feldman problem is given, independently, by G. Diaz [**11**] and Nesterenko [**29**]. As corollaries, Diaz stated that $\mathrm{trdeg}_{\mathbf{Q}} \mathbf{Q}(\alpha, \alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d_1-1}})$ is at least $[(d_1+1)/2]$ when $\alpha \in \mathbf{C}$ and $\mathrm{trdeg}_{\mathbf{Q}} \mathbf{Q}(\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta_{d_1-1}})$ is at least $[(d_1+1)/2]$ when $\alpha \in \overline{\mathbf{Q}}$. He had introduced a trick attributed to Chudnovsky [**7**, pp. 375–376] to improve the lower bound of $[d_1/2]$ given independently by Nesterenko [**28**] and Philippon [**32**]. Other important ingredients of his proof were the criteria of algebraic independence established in [**32**], a combination of analytic estimates from [**8**] and [**35**], and a zero lemma established in [**33**].[2] Nesterenko [**29**] strengthened Diaz's general result by establishing the same lower bound on the transcendence degree under a looser technical hypothesis.

In his work, Nesterenko uses a somewhat different algebraic approach and sometimes provides quantitative results as well. For instance, the following corollary [**28**, Theorem 5] provides a measure of algebraic independence in the Gelfond-Schneider setting. *Let α and β be algebraic numbers with $\alpha \log \alpha \neq 0$, with β of degree $d_1 \geq 2$, and let $\tau \in \mathbf{R}$ with $0 < \tau < (d_1+1)/2$. Then there exists a constant $C_{10} = C_{10}(\alpha, \beta, \tau) > 0$ such that for any set of polynomials $P_j \in \mathbf{Z}[\mathcal{X}_1, \dots, \mathcal{X}_{d_1-1}]$, $j = 1, \dots, N$, which generate an ideal of height $d_1 - r$ in $\mathbf{Z}[\mathcal{X}_1, \dots, \mathcal{X}_{d_1-1}]$, where $0 \leq r < \tau$ and $T \geq t(P_j)$,*

$j = 1, \dots, N$, *we have*

$$\max_{1 \le j \le N} \{|P_j(\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d_1-1}})|\} \ge \exp(C_{10} T^{(d_1-r)\tau/(\tau-r)}).$$

In the case where the P_j's generate a prime ideal $\mathfrak{p} \subseteq \mathbf{Z}[\mathcal{X}_1, \dots, \mathcal{X}_{d_1-1}]$ of height $d_1 - r$, the notion of height (i.e., rank) used by Nesterenko [**26**, for instance, p. 9] provides that $\mathbf{Z}[\mathcal{X}_1, \dots, \mathcal{X}_{d_1-1}]/\mathfrak{p}$ has dimension r. Nesterenko's results can be compared to Theorem 2 above, particularly in the nonlimit case, provided one pays particular attention to the codimension as Ably's result applies to ideals in d_1 variables while Nesterenko works in $\mathbf{Z}[\mathcal{X}_1, \dots, \mathcal{X}_{d_1-1}]$.

And, finally, Theorem 2 [**1**, Corollaire 4] brings us to the present, providing the measures $\exp(C_{11} T(J)^2)$ of algebraic independence for $(\alpha, \alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d_1-1}})$ in codimension $(d_1+1)/2$ and

$$C_{11}(T(J)/(\log T(J))^{1/(d_1+1)})^{(d_1+1)/(d_1-2k-1)}$$

in codimension $n - k > (d_1+1)/2$ for α and β as in Corollary 4.

3.2. *Elliptic setting.* Here we let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3, associated elliptic curve E, and lattice of periods Ω. We use $\mathcal{O}$ to denote the ring of endomorphisms of E and k the associated field of multiplications.

Letting σ be the corresponding Weierstrass σ function, we have a parametrization

$$p(z) = (\sigma^3(z), \sigma^3(z)\wp(z), \sigma^3(z)\wp'(z))$$

of the complex points on E. Here we might consider the group $G = \mathbf{G}_a \times E^{d_2}$ (so $d = d_2 + 1$), complex numbers $x_1, \dots, x_{d_2}$, and the one-parameter subgroup $\phi(z) : \mathbf{C} \to G(\mathbf{C})$ given by $(1, z, p(x_1 z), \dots, p(x_{d_2} z))$. If the numbers $x_1, \dots, x_{d_2}$ are k-linearly independent, then $z, \wp(x_1 z), \dots, \wp(x_{d_2} z)$ are algebraically independent, and thus we've guaranteed that $\phi(\mathbf{C})$ is Zariski dense in $G(\mathbf{C})$.

As in the exponential case, we let $y_1, \dots, y_m$ be a second set of complex numbers which are linearly independent over $\mathbf{Q}$. Here we may consider the algebraic independences of the coordinates of

$\omega = (y_1, \ldots, y_m, p(x_1 y_1), \ldots, p(x_1 y_m), \ldots, p(x_{d_2} y_1), \ldots, p(x_{d_2} y_m))$. We have

$$\kappa = [k : \mathbf{Q}]m(d_2 + 1)/([k : \mathbf{Q}]m + d - 1)$$

and

$$\mu^\sharp = ([k : \mathbf{Q}]m + d - 1)/(d_2 + 1).$$

For $\lambda = (\lambda_1, \ldots, \lambda_{d_2}) \in \mathcal{O}^{d_2}$, we define $|\lambda| := \max_{1 \le i \le d_2} |\lambda_i|$ and for $S \in \mathbf{R}^+$ we let $\mathcal{O}^{d_2}(S) = \{\lambda \in \mathcal{O}^{d_2} : 0 < |\lambda| \le S\}$ and similarly for $h \in \mathcal{O}^m$ and $\mathcal{O}^m(S)$. We also define $\lambda \cdot x = \sum \lambda_i x_i$ where the sum is over $i = 1, \ldots, d_2$, and we let $h \cdot y = \sum h_j y_j$ where this sum ranges over $j = 1, \ldots, m$. In this setting, we consider the following technical hypothesis.

(H_2) Suppose there exist $C_{12} > 0$ and $S_{12} > 0$ such that for all $S \ge S_{12}$, all $\lambda \in \mathcal{O}^{d_2}(S)$, and all $h \in \mathcal{O}^m(S)$, we have

$$|\lambda \cdot x| \ge \exp(-S^{[k:\mathbf{Q}]m/(6(\mu^\sharp - 2)+1)})$$

and

$$|h \cdot y| \ge \max\{\exp(-C_{12} S \log S), \exp(-S^{[k:\mathbf{Q}]m/(4(\mu^\sharp - 2)+3)})\}.$$

In this setting we have elliptic analogues of Corollaries 3 and 4 which will be easily anticipated by the reader.

Corollary 5. *Suppose* $\kappa = [k : \mathbf{Q}]m(d_2+1)/([k : \mathbf{Q}]m+d-1) > 1$ *and* (H_2) *holds. Then there exists a positive* C_{13}*, depending only on* G, x *and* y *such that for* ω *as above and for all ideals* J *of* $K[\mathcal{X}_1, \ldots, \mathcal{X}_{md}]$ *of codimension* $md + 1 - \kappa$*, degree* $\operatorname{Deg}(J)$ *and size* $T(J)$*, we have*

$$\log ||J||_\omega \ge -\exp(C_{13} (\operatorname{Deg} J)^{[k:\mathbf{Q}]m/d_2} T(J)).$$

From Theorem 2, we see that the previous measure was

$$\exp(C_{14} T(J)^{([k:\mathbf{Q}]m + d_2)/d_2}).$$

We have a refinement in the special case of the original elliptic Gelfond-Schneider problem, too.

Corollary 6. *Let u be a complex number and β an algebraic number of degree d_2 over k such that $\wp(u), \wp(\beta u), \dots, \wp(\beta^{d_2-1}u)$ are defined, and suppose that $d_2 > 2/[k:\mathbf{Q}]$. Then there exists a positive constant C_{15}, depending only on u and β such that for all ideals J of $K[\mathcal{X}_1, \dots, \mathcal{X}_{d_2}]$ of codimension $(d_2+1)/3$ (if $k = \mathbf{Q}$) or $(d_2+1)/2$ (if $[k:\mathbf{Q}] = 2$), degree $\operatorname{Deg}(J)$, and size $T(J)$, we have*

$$\log ||J||_{(\wp(u), \wp(\beta u), \dots, \wp(\beta^{d_2-1}u))} \geq -\exp(C_{15}(\operatorname{Deg} J)^{[k:\mathbf{Q}]/2} T(J)).$$

Again, a brief history is in order. According to [**37**], Schneider provided the following result: *Let $\wp(z)$ be the Weierstrass elliptic function with algebraic invariants. If $\wp(u)$ and β are both algebraic and $\beta \notin k$, then $\wp(\beta u)$ is transcendental.* The next step, an elliptic analogue of Gelfond's algebraic independence result for the usual exponential function was announced by D.W. Masser and G. Wüstholz in [**19**] and established in [**22**]. They proved a variety of results for a Weierstrass elliptic function $\wp(z)$ with algebraic invariants g_2 and g_3. In the case where $\wp$ also has complex multiplication, an interesting corollary was the Gelfond analogue which can be stated as follows. *Suppose $\wp$ has complex multiplication over $k \neq \mathbf{Q}$. Then, if u is a complex number such that $\wp(u)$ is defined and is algebraic over $\mathbf{Q}$ and if β is cubic over k, then the numbers $\wp(\beta u)$ and $\wp(\beta^2 u)$ are defined and are algebraically independent over $\mathbf{Q}$.* A fundamental ingredient of their proof was provided by their work concerning zero estimates on algebraic groups [**21**]. According to [**38**], "this approach was initiated by Nesterenko [**24**], developed in a fruitful manner by Brownawell and Masser [**5**], and then extended by Masser and Wüstholz in their papers [**19, 18, 21**]."

Tubbs provided a variety of quantitative results in this setting, beginning with a transcendence measure for $\wp(\beta u)$ when u is a nontorsion algebraic[3] point of $\wp(z)$, $\beta \notin k$ is an algebraic number, and $\wp$ has algebraic invariants. Using lower bounds for linear forms in elliptic logarithms, provided in [**17**] and [**9**], Tubbs [**38**] was able to extend his transcendence measure to provide a quantitative version of the Masser-Wüstholz algebraic independence result. He proved: *Let $\wp(z)$ denote a Weierstrass elliptic function with complex multiplication and algebraic invariants. Suppose u is a nontorsion algebraic point for $\wp(z)$ and that β is cubic over the field of multiplications of $\wp(z)$. Then for every $\varepsilon > 0$, there exists a real number $t(\varepsilon) > 0$ such that for every nonzero* *integral polynomial $P(X,Y)$ with $t(P) > t(\varepsilon)$, we have*

$$\log|P(\wp(\beta u), \wp(\beta^2 u))| > -\exp(t(P)^{4+\varepsilon}).$$

In the case where $\wp$ has complex multiplication, β is cubic over k and $\zeta = (\wp(u), \wp(\beta u), \wp(\beta^2 u))$ is defined, Tubbs [**40**] also provides a lower bound for pairs of relatively prime integral polynomials evaluated at ζ, providing an elliptic analogue to [**2**].

The first result establishing many algebraic independent values of elliptic functions in the Gelfond-Schneider setting was given by Masser and Wüstholz [**20**]; there they considered elliptic curves with algebraic invariants and without complex multiplication. Philippon [**30**], in his thesis, provided a strong refinement, dealing with both the non-c.m. and c.m. cases, and establishing transcendence degrees close to the present limit of the method. Other large transcendence degree results for the general elliptic Gelfond-Schneider problem were provided in [**4, 6**] along with quantitative results, and in [**45, 46**] as consequences of transcendence results for commutative algebraic groups. In the first case, Waldschmidt removes the hypothesis that g_2, g_3 be algebraic but imposes an additional condition on either the j-invariant $j(E)$ of the elliptic curve or the ratio τ of a pair of fundamental periods of $\wp$.

The best qualitative result to date for the general elliptic Gelfond-Schneider problem was provided, independently, by Ably [**1**, Corollaire 7] and Tubbs [**43**, Corollary of theorem 2], as consequences of their results for commutative algebraic groups. With regards to the elliptic analogue of the original problem, we have the following corollary: *Suppose β is algebraic of degree $d_2 > 2/[k : \mathbf{Q}]$ over k, and let u be a complex number such that $\wp(z)$ is defined at $u, \beta u, \dots, \beta^{d_2-1}u$ and $\wp(u)$ is algebraic. Put*

$$t = \begin{cases} \left[\frac{d_2+1}{3}\right] & \textit{if } k = \mathbf{Q} \\ \left[\frac{d_2+1}{2}\right] & \textit{if } k \neq \mathbf{Q}. \end{cases}$$

Then at least t of the values $\wp(\beta u), \dots, \wp(\beta^{d_2-1}u)$ are algebraically independent. The best quantitative result to date follows from [**1**, Corollaire 5], the elliptic corollary of Theorem 2 above.[4]

4. Proof of main result. As mentioned above, our proof follows the typical outline for algebraic independence results. In particular, we construct an auxiliary function, alter the auxiliary function as in [**11**], find a zero-free region via the zero estimates of [**33**], and then bound the auxiliary function by the typical interpolation lemma as in [**11**]. For much of this, we follow the details of Ably [**1**]. Lastly, as in [**12**], we appeal to E.M. Jabbouri's [**16**] version of P. Philippon's [**32**] criteria for algebraic independence, choosing our parameters to depend explicitly on the degree and height of the ideal J.

We let K, G, ϕ and $y_1, \dots, y_m$ be as given above. We may define quantities $\mu^\sharp$ and κ, specify a particular point ω, and prescribe the technical hypothesis (H). All of this is done as above.

We choose a transcendence basis $\theta_1, \dots, \theta_t$ and an integer θ_{t+1} over $\mathbf{Z}[\theta_1, \dots, \theta_t]$ such that $K(\omega) = \mathbf{Q}(\theta_1, \dots, \theta_t, \theta_{t+1})$. For simplicity of notation, we let $\theta = (\theta_1, \dots, \theta_t)$, $\theta' = (\theta_1, \dots, \theta_{t+1})$, and similarly $X = (X_1, \dots, X_t)$, $X' = (X_1, \dots, X_{t+1})$. For $z_0 \in \mathbf{C}^n$ and positive real numbers R, we define $B^n(z_0, R)$ to be the open ball in $\mathbf{C}^n$ with center z_0 and radius R. As usual, we have the following proposition.

Proposition 7. *Suppose hypothesis* (H) *is satisfied,* $\kappa > 1$*, and furthermore,* $\mu^\sharp > 2$ *if* G *is nonlinear. Then there exist positive constants* S_0, a_1, a_2, a_3 *and choices of the parameters* D_0, D_1, D_2, M *and* r *such that*:

For all $S \geq S_0$*, there exists an ideal* $\mathcal{J}_S = \langle Q_{S,1}, \dots, Q_{S,m(S)} \rangle$ *in* $K[X_1, \dots, X_t]$ *such that*

(a) *the set of zeros of* $\mathcal{J}_S$ *in* $B^t(\theta, \exp(-r(S)))$ *is empty,*

(b) $\max_{1 \leq j \leq m(S)} |Q_{S,j}(\theta)| \leq \exp(-a_1 r(S)) + \exp(-a_2 M^m \log S)$,

(c) $\max_{1 \leq j \leq m(S)} d(Q_{S,j}) \leq a_3((D_0 - 1) + (D_1 - 1)S + (D_2 - 1)S^2)$,

and

(d) $\max_{1 \leq j \leq m(S)} \mathrm{Ht}\,(Q_{S,j}) \leq a_3((D_0-1)\log S + (D_1-1)S + (D_2-1)S^2)$.

Proof. We run through the highlights of the proof to illustrate what conditions arise on the parameters D_0, D_1, D_2, M, r and S. As these parameters depend on S, we may sometimes write $r(S)$, for example, to emphasize this dependence. In the final stage, when we choose the parameters, we will actually specify $(\overline{D}_0, \overline{D}_1, \overline{D}_2)$, instead of (D_0, D_1, D_2), and we will define D_i by $D_i = \max\{1, [\overline{D}_i]\}$ where $[\cdot]$

denotes the integer part. Some of the more tedious details of the proof have been omitted here but can be found in both [**43**] and [**1**].

We recall that $K(\omega) = \mathbf{Q}(\theta_1, \ldots, \theta_{t+1})$ and θ_{t+1} is integral over $\mathbf{Z}[\theta_1, \ldots, \theta_t]$. We let $R(\theta_1, \ldots, \theta_t, X_{t+1})$ be the minimal polynomial for θ_{t+1} over $\mathbf{Z}[\theta_1, \ldots, \theta_t]$. We also recall that $\gamma_j := \phi(y_j) \in G(K(\omega)) \subseteq \mathbf{P}_{d_0} \times \mathbf{P}_{d_1} \times \mathbf{P}_N$. Thus, we have multiprojective coordinates for γ_j, $1 \leq j \leq m$, given by

$$(3) \quad (Q(\theta), A_{d_0,j}(\theta'); Q(\theta), B_{1,j}(\theta'), \ldots, B_{d_1,j}(\theta'); \\ Q(\theta), C_{1,j}(\theta'), \ldots, C_{N,j}(\theta'))$$

where $Q(X)$, $A_{d_0,j}(X')$, $B_{i,j}(X')$ and $C_{s,j}(X')$ are integral polynomials over K (for $i = 1, \ldots, d_1$, $s = 1, \ldots, N$ and $j = 1, \ldots, m$) of size at most c_1. (Here c_1 is a positive constant which depends only on $G, \phi, \chi, y_1, \ldots, y_m$ and $\theta_1, \ldots, \theta_{t+1}$.)[5] Furthermore, $Q(\theta)$ is nonzero and we may suppose that

$$|Q(\theta)| \geq c_2$$

where c_2 is a positive constant depending only on $G, \phi, \chi, y_1, \ldots, y_m$ and $\theta_1, \ldots, \theta_{t+1}$.

In establishing our zero-free region, we will want to consider $\tilde{\theta}$ in the ball $B^t(\theta, \rho(S))$ of $\mathbf{C}^t$ with center θ and radius $\rho(S) = \exp(-r(S))$. For any $\tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_t)$ in this ball, we know from [**46**, p. 263] that there exists a simple root $\tilde{\theta}_{t+1}$ of $R(\tilde{\theta}_1, \ldots, \tilde{\theta}_t, X_{t+1})$ which satisfies

$$|\tilde{\theta}_{t+1} - \theta_{t+1}| \leq \exp(-r(S)/2).$$

For $j = 1, \ldots, m$, we let $\tilde{\gamma}_j$ be the point with multiprojective coordinates given by evaluating the coordinate polynomials of (3) at $\tilde{\theta}$ and $\tilde{\theta}' := (\tilde{\theta}_1, \ldots, \tilde{\theta}_t, \tilde{\theta}_{t+1})$. Waldschmidt notes that $\tilde{\gamma}_j \in G(\mathbf{C})$. Furthermore, for $j = 1, \ldots, m$, there exists $\tilde{y}_j \in T_G(\mathbf{C})$ such that $\tilde{\gamma}_j = \exp_G \tilde{y}_j$ and

$$|\tilde{y}_j - y_j| \leq \exp(-r(S)/2).$$

We will have replaced θ' with $\tilde{\theta}'$ in each coordinate of γ_j when expressed as in (3). Since $|Q(\theta)| \geq c_2$, we know from a difference lemma (see, for example, [**11**, p. 7]) that $|Q(\tilde{\theta})| \neq 0$ provided $r(S) > c_3$, where

c_3 is a positive constant depending only on c_2, $\max_{1\le i\le t}|\theta_i|$, t and $t(Q)$. Thus, we have multiprojective coordinates

$$(Q(\tilde{\theta}), A_{d_0,j}(\tilde{\theta}'); Q(\tilde{\theta}), B_{1,j}(\tilde{\theta}'), \dots, B_{d_1,j}(\tilde{\theta}'); \\ Q(\tilde{\theta}), C_{1,j}(\tilde{\theta}'), \dots, C_{N,j}(\tilde{\theta}'))$$

for $\tilde{\gamma}_j$.

In order to carry out our construction, we will also use multiprojective coordinates for certain points of $\Gamma = \phi(Y)$ in terms of θ'. For $h = (h_1, \dots, h_m) \in \mathbf{N}^m$, we let $||h|| = h_1 + \dots + h_m$ and (as above) $h \cdot y = h_1 y_1 + \dots + h_m y_m$. Multiprojective coordinates for $h \cdot \gamma := h_1\gamma_1 + \dots + h_m\gamma_m$ were given by Tubbs in [**41, 39**] and [**43**] and can also be found in [**1**] where the G_2-coordinates are selected even more carefully.

We consider the additive coordinates first. From the group law on $\mathbf{G}_a$, we see that $\mathbf{P}_{d_0}$-coordinates for the additive part of $h \cdot \gamma$ can be given by

$$\left(Q(\theta), \sum_{j=1}^{m} h_j A_{d_0,j}(\theta')\right). \tag{4a}$$

From the group law on $\mathbf{G}_m^{d_1}$, we see that

$$\left(Q(\theta)^{||h||}, \prod_{j=1}^{m} B_{1,j}(\theta')^{h_j}, \dots, \prod_{j=1}^{m} B_{d_1,j}(\theta')^{h_j}\right) \tag{4b}$$

are $\mathbf{P}_{d_1}$ coordinates for the multiplicative part of $h \cdot \gamma$. Projective coordinates corresponding to the G_2 part of $h \cdot \gamma$ cannot be given so explicitly in terms of the G_2-coordinates for $\gamma_1, \dots, \gamma_m$. However, we have the following lemma which, except for notation, is taken directly from the work of Ably. (In particular, the polynomials $D^{\beta}_{s,h}$ with $\beta \in \mathcal{B}_h$ and $s = 0, 1, \dots, N$ correspond to u_i^{β} for $\beta \in \mathcal{B}_h$ and $i = 0, 1, \dots, N$ of [**1**, p. 212].)

Lemma 8. *For each $h \in \mathbf{N}^m(S)$, there exists a finite set $\mathcal{B}_h$ and polynomials $D^{\beta}_{s,h}(X')$ (for $\beta \in \mathcal{B}_h$ and $s = 0, 1, \dots, N$) which are integral over K and such that:*

(a) $\max\{d(D^{\beta}_{s,h}), \mathrm{Ht}\,(D^{\beta}_{s,h}) : \beta \in \mathcal{B}_h, s = 0, 1, \dots, N\} \leq c_4 S^2$ *and*

(b) *for every* $\tilde{\theta} \in B^t(\theta, \rho(S))$, *there exists* $\beta \in \mathcal{B}_h$ *such that*

$$(D^{\beta}_{0,h}(\tilde{\theta}'), \dots, D^{\beta}_{N,h}(\tilde{\theta}')) \tag{4c}$$

is a set of $\mathbf{P}_N$*-coordinates for the* G_2 *portion of* $h \cdot \tilde{\gamma} := h_1\tilde{\gamma}_1 + \cdots + h_m\tilde{\gamma}_m$.

There is one technicality to overcome at the start. Namely, we would like lower bounds for the multiprojective coordinates of $h \cdot \gamma$. When $d_2 > 0$, we have to use another approach, as no such bound holds, a priori, for the G_2-coordinates. We act as if we have already specified $D_2(S)$ which is at least 1, and we consider cases based on whether or not we have a suitable lower bound. This idea was used by Tubbs in [**40**]. In the more general algebraic groups setting, the same idea appears in [**43**] and, independently, in [**1**] and Ably's thesis. Tubbs uses a single set of G_2-coordinates for each $h \cdot \gamma$; this allows him to conclude that $h \cdot \tilde{\gamma}$ is in $\mathbf{P}_N$ for all but finitely many $\tilde{\theta}$ "near" θ. As noted in Lemma 8, Ably uses a family which contains a set of projective coordinates for each $\tilde{\theta}$ "near" θ, i.e., a complete set of nonvanishing $\mathbf{P}_N$-coordinates for $\pi_2(h \cdot \gamma)$. This idea was instrumental in loosening the technical hypothesis of [**43**] to (H) in [**1**].

In the first case, we suppose that $d_2 > 0$ and there exists an $h \in \mathbf{Z}^m(S)$ such that

$$\max_{\substack{\beta \in \mathcal{B}_h \\ 0 \leq s \leq N}} |D^{\beta}_{s,h}(\theta')| < \exp(-r(S)/5(D_2(S) - 1)). \tag{5}$$

We let $P^{\beta}_{s,h}(\theta) = r(D^{\beta}_{s,h}(\theta, X_{t+1}), R(\theta, X_{t+1}))$ for $s = 0, 1, \dots, N$ and $\beta \in \mathcal{B}_h$, where r denotes Chudnovsky's semi-resultant. Then we let

$$\mathcal{J}_S = \langle P^{\beta}_{s,h}(X) : s = 0, 1, \dots, N; \beta \in \mathcal{B}_h \rangle.$$

For each $\tilde{\theta} \in B^t(\theta, \rho(S))$, using Lemma 8(b) and the nonvanishing property of semi-resultants, we know that $P^{\beta}_{s,h}(\tilde{\theta}) \neq 0$ for some $s \in \{0, 1, \dots, N\}$ and some $\beta \in \mathcal{B}_h$, so (a) of Proposition 7 is established. Using other properties of semi-resultants, (b) follows from (5) provided

$c_5(D_2(S)-1)S^2 \leq r(S)$, while (c) and (d) of Proposition 7 follow from condition (a) of Lemma 8.

In the second case, we have $d_2 = 0$ or we may assume that

$$\max_{\substack{\beta \in \mathcal{B}_h \\ 0 \leq s \leq N}} |D^{\beta}_{s,h}(\theta')| \geq \exp(-r(S)/5(D_2(S)-1)) \tag{6}$$

for every $h \in \mathbf{N}^m(S)$. For each such h, we fix $\beta(h) \in \mathcal{B}_h$ and $s(h) \in \{0, 1, \ldots, N\}$ such that

$$|D^{\beta(h)}_{s(h),h}(\theta')| \geq \exp(-r(S)/5(D_2(S)-1)).$$

Then, for each $\tilde{\theta} \in B^t(\theta, \rho(S))$, the differences lemma shows that

$$\max_{0 \leq s \leq N} |D^{\beta(h)}_{s,h}(\tilde{\theta}')| \neq 0$$

provided $r(S) \geq c_6 S^2$ where c_6 depends only on c_4, $\max_{1 \leq i \leq t+1} |\theta_i|$ and t. Then we know that

$$(D^{\beta(h)}_{0,h}(\tilde{\theta}'), \ldots, D^{\beta(h)}_{N,h}(\tilde{\theta}'))$$

is a set of $\mathbf{P}_N$-coordinates for $\pi_2(h \cdot \tilde{\gamma})$.

Constructing the auxiliary function. In this second case, we follow the standard construction of an auxiliary function with many zeros. We let $\mathcal{O}_K$ denote the ring of integers in K. We create a homogeneous polynomial P in $\mathcal{O}_K[W, Y, Z]$ of multidegree at most $(D_0 - 1, D_1 - 1, D_2 - 1)$ in $W = (W_0, W_{d_0})$, $Y = (Y_0, \ldots, Y_{d_1})$ and $Z = (Z_0, \ldots, Z_N)$. We construct P so that the associated function

$$F(z) = P \circ \chi \circ \phi(z),$$

that is,

$$P(1, z; 1, e^{x_1 z}, \ldots, e^{x_{d_1} z}; \Theta_0(\operatorname{Lie}\psi(z)), \ldots, \Theta_N(\operatorname{Lie}\psi(z))),$$

satisfies

$$F(h \cdot y) = 0$$

for all $h \in \mathbf{N}^m(M)$.

We let $\{M_\mu : \mu \in \mathcal{M}\}$ be a set of multihomogeneous monic monomials of multidegree (D_0-1, D_1-1, D_2-1) in (W, Y, Z) which is maximal with respect to the property that the elements M_μ are linearly independent modulo the ideal defining G in $\mathbf{P}_{d_0} \times \mathbf{P}_{d_1} \times \mathbf{P}_N$. We may choose these monomials so that

$$\operatorname{card}(\mathcal{M}) = c_7 D_0^{d_0} D_1^{d_1} D_2^{d_2}$$

where c_7 depends only on G, d_0, d_1 and d_2.

Then we seek an auxiliary polynomial $P(W, Y, Z)$ of the form

$$\sum_{\mu \in \mathcal{M}} P_\mu(\theta) M_\mu(W, Y, Z),$$

where the P_μ are polynomials over $\mathcal{O}_K$. We recall that $R(\theta_1, \ldots, \theta_t, X_{t+1}) \in \mathbf{Z}[\theta_1, \ldots, \theta_t, X_{t+1}]$ is the minimal polynomial for θ_{t+1} over $\mathbf{Q}(\theta_1, \ldots, \theta_t)$. Then for each $\mu \in \mathcal{M}$ and $h \in \mathbf{N}^m(S)$, we define $H_{\mu,h}(X') \in \mathcal{O}_K[X']$ by the equation

$$H_{\mu,h}(\theta') = M_\mu(\chi \circ \phi(h \cdot y))$$

where the projective $\mathbf{G}_a^{d_0}$- ($\mathbf{G}_m^{d_1}$- and G_2-) coordinates of $\phi(h \cdot y)$ are represented as in (4a), respectively (4b) and (4c), and where we've used the polynomial R to insure that

$$\deg_{X_{t+1}} H_{\mu,h} \leq \deg_{X_{t+1}} R.$$

Then the system of equations

$$\{F(h \cdot y) = 0 : h \in \mathbf{N}^m(M)\} \tag{7}$$

gives rise to an equivalent system

$$\{H_h(X') = 0 : h \in \mathbf{N}^m(M)\}$$

where $H_h(X')$ is defined by

$$H_h(X') := \sum_{\mu \in \mathcal{M}} P_\mu(X) H_{\mu,h}(X')$$

for $h \in \mathbf{N}^m(S)$.

Before solving this system, we first express $H_h(X')$ as a polynomial in X_{t+1} of (partial) degree (with respect to X_{t+1}) at most $\deg_{X_{t+1}} R$ and with coefficients in $\mathcal{O}_K[X]$. Setting the coefficients of these polynomials (in X_{t+1}) to zero, each equation of (7) yields $\deg_{X_{t+1}} R$ equations involving X.

This third system of equations will have a solution provided we allow $\deg_{X_i} P_\mu$ to be at least as large as $\max\{\deg_{X_i}(H_{\mu,h})\}$ for $i = 1, \ldots, t$ where the maximum is over all $\mu \in \mathcal{M}$ and $h \in \mathbf{N}^m(M)$. We solve this third system formally, letting the P_μ be polynomials whose coefficients are the unknowns of our system of linear equations and whose partial degrees equal $\max\{\deg_{X_i}(H_{\mu,h})\}$ for $i = 1, \ldots, t$. From our multiprojective coordinates for $\phi(h \cdot y)$ in (4a)–(4c), we see then that

$$\max_{\mu \in \mathcal{M}} d(P_\mu) \leq c_8((D_0 - 1) + (D_1 - 1)M + (D_2 - 1)M^2).$$

Siegel's lemma over number fields provides a nontrivial integral solution over K and hence a set of coefficient polynomials $\{P_\mu : \mu \in \mathcal{M}\}$ in $\mathcal{O}_K[X]$ satisfying

$$\max_{\mu \in \mathcal{M}} \operatorname{Ht}(P_\mu) \leq c_9((D_0 - 1)\log M + (D_1 - 1)M + (D_2 - 1)M^2).$$

Of course, this yields a set of nonzero auxiliary polynomials

$$\{H_h(X') : h \in \mathbf{N}^m(S)\}$$

whose coefficients are also integers in K. We know that this application of Siegel's lemma is legitimate, provided that $[K : \mathbf{Q}]$ times the rank of this system is strictly less than the number of unknowns.

In order to provide an upper bound for the rank of this system, we use the work of [**34**, Lemme 6.7] as in [**1**]. For this, we introduce further notation. For subvarieties V of $\mathbf{P}_{d_0} \times \mathbf{P}_{d_1} \times \mathbf{P}_N$ and real numbers $D_0, D_1, D_2 > 0$, we define $H(V; D_0, D_1, D_2)$ as in [**33**, p. 358] to be the homogeneous polynomial equal to $(\dim V)!$ times the homogeneous part of (maximal) degree $\dim V$ of the Hilbert-Samuel polynomial of

V evaluated at D_0, D_1, D_2. We also define $\Gamma(M) = \{\phi(h \cdot y) : h \in \mathbf{N}^m(M)\}$. They showed, then, that the system of linear equations in (7) has rank at most

$$8^{\dim G'} \operatorname{card}\left(\frac{\Gamma(M)+G'}{G'}\right) H(G'; D_0, D_1, D_2)$$

for *all* connected algebraic subgroups $G' \subsetneq G$.

Counting the number of unknowns is straightforward. Then, thanks to the above mentioned work of Philippon and Waldschmidt (and the details of [**1**]), we know that Siegel's lemma applies provided

$$(\mathcal{C}1)\quad 2^{t+1}[K:\mathbf{Q}](\deg_{X_{t+1}} R) 8^{\dim G'} \operatorname{card}\left(\frac{\Gamma(M)+G'}{G'}\right) H(G'; D_0, D_1, D_2) \le \operatorname{card}(\mathcal{M}) = c_7 D_0^{d_0} D_1^{d_1} D_2^{d_2}$$

for *at least one* connected, algebraic subgroup $G' \subsetneq G$.

We note that the same inequality, essentially, had been used in [**43**]. There, Tubbs imposes a technical hypothesis (referred to as Condition 1) which allows him (roughly speaking) to replace the factor

$$\operatorname{card}\left(\frac{\Gamma(M)+G'}{G'}\right) H(G'; D_0, D_1, D_2)$$

with $H(G'; D_0, D_1, 2D_2)$. To handle the case where $\mu^\sharp$ is not necessarily maximal, Ably imposes the general condition $(\mathcal{C}1)$ as stated above.

Altering the auxiliary polynomials. Eventually, we will consider polynomials associated with $H_h(X')$ where $h \in \mathbf{N}^m(S)$ and $S > M$. This will be necessary to get the desired zero-free region. First, though, we employ a trick introduced by G.V. Chudnovsky and formalized by Diaz [**11**] to alter these auxiliary polynomials. This is necessary as H_h may vanish even at points in a small neighborhood of θ'. The trick used here is presented in [**43, 29**] and [**1**] as well.

For $i = (i_1, \ldots, i_t) \in \mathbf{N}^t$, we let $||i|| = i_1 + \cdots + i_t$ and define a differential operator

$$D^i = \left(\frac{1}{i_1! \cdots i_t!}\left(\frac{\partial}{\partial X_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial X_t}\right)^{i_t}\right).$$

For each $\tilde{\theta} \in B^t(\theta, \rho(S))$, we pick $i(\tilde{\theta})$ such that

$$||i(\tilde{\theta})|| = \min\{||i|| : D^i P_\mu(\tilde{\theta}) \neq 0 \text{ for some } \mu \in \mathcal{M}\}.$$

This choice of $i(\tilde{\theta})$ insures that for every $i \in \mathbf{N}^t$ with $||i|| < ||i(\tilde{\theta})||$, we have

$$D^i P_\mu(\tilde{\theta}) = 0 \quad \text{for every } \mu \in \mathcal{M}.$$

This will be crucial later. This "trick" for altering the auxiliary polynomials does not affect the upper bound for $\mathrm{Ht}\,(P_\mu)$ by more than a constant factor and does not impose any additional conditions on the parameters.

Now we let $I_S = \{i(\tilde{\theta}) : \tilde{\theta} \in B^t(\theta, \rho(S))\}$ and for each $i \in I_S$, we define a new auxiliary polynomial

$$H_{h,i}(X') = \sum_{\mu \in \mathcal{M}} D^i P_\mu(X) H_{\mu,h}(X').$$

For every $h \in \mathbf{N}^m(S)$ and $i \in I_S$, the polynomial $H_{h,i}$ is nonzero and has coefficients which are integers in K.

Now we consider the set of polynomials

$$\{H_{h,i} : h \in \mathbf{N}^m(S), i \in I_S\}$$

in $K[X_1, \dots, X_{t+1}]$. In the next stage, we will impose conditions on S to insure that the stated zero-free region will exist. First, we note that from the bounds on the degree and height of P_μ and the definitions of H_h and $H_{h,i}$ above, we have

$$\max_{\substack{h \in \mathbf{N}^m(S) \\ i \in I_S}} d(H_{h,i}) \leq c_{10}((D_0 - 1) + (D_1 - 1)S + (D_2 - 1)S^2)$$

for a suitable choice of c_{10}. Since

$$\mathrm{Ht}\,(D^i P_\mu) \leq d(P_\mu) \log 2 + \mathrm{Ht}\,(P_\mu)$$

we also have

$$\max_{\substack{h \in \mathbf{N}^m(S) \\ i \in I_S}} \mathrm{Ht}\,(H_{h,i}) \leq c_{11}((D_0 - 1)\log S + (D_1 - 1)S + (D_2 - 1)S^2)$$

as well.

Establishing the zero-free region. Recall that for each $\tilde{\theta} \in B^t(\theta, \rho(S))$, we may choose $\tilde{\theta}_{t+1}$ as above and we define $\tilde{\theta}' = (\tilde{\theta}_1, \dots, \tilde{\theta}_t, \tilde{\theta}_{t+1})$. We have also defined $i(\tilde{\theta})$ for each $\tilde{\theta} \in B^t(\theta, \rho(S))$. For any such $\tilde{\theta}$ (and corresponding $\tilde{\theta}'$ and $i(\tilde{\theta}) \in I_S$), we will impose additional conditions on S, D_0, D_1 and D_2 which will insure that

$$H_{h,i(\tilde{\theta})}(\tilde{\theta}') \neq 0$$

for some $h \in \mathbf{N}^m(S)$. Clearly, we will need

$$(\mathcal{C}2) \qquad M < S.$$

We arrive at the other new condition by seeking a contradiction. Suppose the opposite is true, say there exists $\tilde{\theta} \in B^t(\theta, \rho(S))$ such that $H_{h,i(\tilde{\theta})}(\tilde{\theta}') = 0$ for all $h \in \mathbf{N}^m(S)$.

We define a new polynomial $\tilde{P}$ by

$$(8) \qquad \tilde{P}(W, Y, Z) = \sum_{\mu \in \mathcal{M}} D^{i(\tilde{\theta})} P_\mu(\tilde{\theta}) M_\mu(W, Y, Z).$$

We evaluate $\tilde{P}$ at $h \cdot \tilde{\gamma}$, using the multiprojective coordinates

$$\Bigg(Q(\tilde{\theta}), \sum_{j=1}^m h_j A_{d_0,j}(\tilde{\theta}'); Q(\tilde{\theta})^{||h||}, \prod_{j=1}^m B_{1,j}(\tilde{\theta}')^{h_j}, \cdots, \\ \prod_{j=1}^m B_{d_1,j}(\tilde{\theta}')^{h_j}; D_{0,h}^\beta(\tilde{\theta}'), \dots, D_{N,h}^\beta(\tilde{\theta}')\Bigg)$$

where $\beta \in \mathcal{B}_h$ is chosen so that we have $\mathbf{P}_N$ coordinates for the G_2 portion of $h \cdot \tilde{\gamma}$. We note that the righthand side of (8) is

$$\sum_{\mu \in \mathcal{M}} D^{i(\tilde{\theta})} P_\mu(\tilde{\theta}) H_{\mu,h}(\tilde{\theta}').$$

That is, the righthand side is simply $H_{h,i(\tilde{\theta})}(\tilde{\theta}')$, which equals zero since we solved formally. Thus, $\tilde{P}$ vanishes on $\tilde{\Gamma}(S) = \{h \cdot \tilde{\gamma} : h \in \mathbf{N}^m(S)\} =$

$\{\phi(h\cdot\tilde{y}) : h\in \mathbf{N}^m(S)\}$. However, by construction, i.e., alteration, of the auxiliary polynomial P, we know that $D^{i(\tilde{\theta})}P_\mu(\tilde{\theta})\neq 0$ and hence $\tilde{P}$ is a nonzero polynomial. Then Philippon's zero lemma [**33**, Théorème 2.1] shows that there exists a connected subgroup G' of G, distinct from G, and incompletely defined in G by multihomogeneous equations of multidegree at most $(\overline{D}_0,\overline{D}_1,2\overline{D}_2)$ such that

$$\operatorname{card}\left(\frac{\tilde{\Gamma}(S)+G'}{G'}\right)H(G';\overline{D}_0,\overline{D}_1,\overline{D}_2)\leq H(G;\overline{D}_0,\overline{D}_1,2\overline{D}_2). \tag{9}$$

In order to apply our technical hypothesis (H) and get a contradiction, it is only necessary that (H) hold for these "obstruction subgroups" G', or for all proper connected subgroups G', incompletely defined in G by equations of multidegree at most $(\overline{D}_0,\overline{D}_1,2\overline{D}_2)$ and satisfying (9) for some perturbation $\tilde{\Gamma}(S)$ of $\Gamma(S)$. Then, by this weakened technical hypothesis (H) and the fact that we have a homomorphism taking $\tilde{\gamma}$ to γ, we have a well-defined map from $((\tilde{\Gamma}(S)+G')/G')$ into $(\Gamma(S)+G')/G'$ given by $h\cdot\tilde{\gamma}+G'\mapsto\phi(h\cdot y)+G'$ provided

$$r(S)\geq 3S^{\kappa\mu^\sharp}\log S \tag{$\mathcal{C}3$}$$

and

$$\begin{cases}\overline{D}_0\leq S^{\kappa\mu^\sharp}\log S\\ \overline{D}_1\leq S^{\mu^\sharp-1}(\log S)^{d_0/d}\\ \overline{D}_2\leq S^{\mu^\sharp-2}(\log S)^{d_0/d}.\end{cases}$$

This map is surjective, so

$$\operatorname{card}\left(\frac{\Gamma(S)+G'}{G'}\right)\leq\operatorname{card}\left(\frac{\tilde{\Gamma}(S)+G'}{G'}\right).$$

Combining this with inequality (9), we have

$$\begin{aligned}\operatorname{card}\left(\frac{\Gamma(S)+G'}{G'}\right)H(G';\overline{D}_0,\overline{D}_1,\overline{D}_2)&\leq H(G;\overline{D}_0,\overline{D}_1,2\overline{D}_2)\\ &\leq 2^{d_2}H(G;\overline{D}_0,\overline{D}_1,\overline{D}_2)\end{aligned}$$

where the last inequality follows from the homogeneity of $H(G;\overline{D}_0,\overline{D}_1,\cdot)$.

We get the desired contradiction and thus zero-free region provided

$$(\mathcal{C}5) \qquad \operatorname{card}\left(\frac{\Gamma(S)+G'}{G'}\right)\frac{H(G';\overline{D}_0,\overline{D}_1,\overline{D}_2)}{H(G;\overline{D}_0,\overline{D}_1,\overline{D}_2)} > 2^{d_2}$$

for every proper connected algebraic subgroup G' of G.

Bounding the auxiliary function. Just as in previous work, we use an extrapolation formula from the work of [**8**] and [**35**] to provide the upper bound for $|H_{h,i}(\theta')|$.

First we estimate $|H_{h,i}(\theta')|$ when $h \in \mathbf{N}^m(M)$. We fix $i \in I_S$ and choose $\tilde{\theta} \in B^t(\theta, \rho(S))$ such that $i = i(\tilde{\theta})$. We write

$$\begin{aligned}
&\text{(A)} \qquad H_{h,i}(\theta') = \sum_{\mu\in\mathcal{M}} (D^i P_\mu(\theta) - D^i P_\mu(\tilde{\theta}))H_{\mu,h}(\theta') \\
&\text{(B)} \qquad\qquad + \sum_{\mu\in\mathcal{M}} D^i P_\mu(\tilde{\theta})(H_{\mu,h}(\theta') - H_{\mu,h}(\tilde{\theta}')) \\
&\qquad\qquad\qquad + \sum_{\mu\in\mathcal{M}} D^i P_\mu(\tilde{\theta})H_{\mu,h}(\tilde{\theta}').
\end{aligned}$$

By our construction, $H_h(X') \equiv 0$ for $h \in \mathbf{N}^m(M)$. And, by the minimality of $i = i(\tilde{\theta})$, we have $D^i H_h(\tilde{\theta}') = \sum_{\mu\in\mathcal{M}} D^i P_\mu(\tilde{\theta})H_{\mu,h}(\tilde{\theta}')$. So the last sum on the righthand side vanishes.

To estimate the "differences" in (A) and (B), we apply a difference lemma again. We have

$$|D^i P_\mu(\theta) - D^i P_\mu(\tilde{\theta})| \le \exp(-r(S))c_{12}^{\deg(P_\mu)}\exp(\operatorname{Ht}(D^i P_\mu))$$

where c_{12} is a positive constant depending only on t and $|\theta|$. The righthand side of this equation is bounded by $\exp(-r(S)/2)$ provided

$$c_{12}^{\deg(P_\mu)}\exp(\operatorname{Ht}(D^i P_\mu)) \le \exp(r(S)/2).$$

It suffices if

$$c_{14}((D_0-1)\log S + (D_1-1)S + (D_2-1)S^2) \le r(S),$$

where c_{14} depends only on t, $|\theta|$, and previous constants. The same upper bound of $\exp(-r(S)/2)$ holds for $|H_{\mu,h}(\theta') - H_{\mu,h}(\tilde{\theta}')|$ under the same condition (up to a possibly new constant $c_{15} \geq c_{14}$). Furthermore, this condition follows from $(\mathcal{C}3)$ and

$$(\mathcal{C}4) \qquad \begin{cases} c_{15}\overline{D}_0 \leq S^{\kappa\mu^\sharp} \\ c_{15}\overline{D}_1 \leq S^{\mu^\sharp-1}(\log S)^{d_0/d} \\ c_{15}\overline{D}_2 \leq S^{\mu^\sharp-2}(\log S)^{d_0/d}. \end{cases}$$

Now we estimate the sums (A) and (B) above. Each of these sums is at most $\exp(-r(S)/3)$ provided $(\mathcal{C}3)$ and $(\mathcal{C}4)$ hold with a possibly different constant, say $c_{16} \geq c_{15}$, depending on c_{14} and c_7.

We have then

$$|H_{h,i}(\theta')| \leq \exp(-r(S)/4)$$

for $h \in \mathbf{N}^m(M)$ provided we also have $r(S) \geq 12\log 2$.

We extend this bound to the larger set $\mathbf{N}^m(S)$ in the usual way, via an interpolation lemma as in [**11, 43, 1**]. We choose real numbers $R_1(S)$ and $R(S)$ such that $2 < R_1 < R/4$ and $\max|h \cdot y| \leq R_1$ where the maximum is over all $h \in \mathbf{N}^m(S)$. We choose $R_1 = c_{17}S$ where $c_{17} = \max\{2, m|y_j| : 1 \leq j \leq m\}$ and $R = S^{1+\varepsilon}$ with $0 < \varepsilon < \max\{1, (\kappa-1)\mu^\sharp/2\}$. (We know that such an ε can be chosen since $\kappa > 1$.) Since $\varepsilon > 0$, we have $R > 4R_1$ provided $S > \max\{1, (4c_{17})^{1/\varepsilon}\}$.

We also note that the technical hypothesis (H) insures that

$$|h \cdot y| \geq \exp(-c_0 M \log M)$$

for $h \in \mathbf{N}^m(M)$ provided $M \geq S_0$.

Then we apply the upper bound of [**11**, p. 10] to the analytic function

$$\tilde{F} = \tilde{P} \circ \chi \circ \phi(z)$$

where $\chi \circ \phi(z)$ is represented as in (1) and $\tilde{P}$ is defined as in (8). We have, then,

$$(10) \qquad \begin{aligned} |\tilde{F}|_{R_1} \leq 2|\tilde{F}|_R \left(\frac{4R_1}{R}\right)^{M^m} &+ \left(\frac{18R_1}{Mc_{18}}\right)^{M^m} \\ &\cdot \left(\frac{c_{18}}{2c_0 M \log M}\right)^{M^{m-1}} \sum_{h \in \mathbf{N}^m(M)} |\tilde{F}(h \cdot y)| \end{aligned}$$

where $c_{18} = \max\{|y_j| : 1 \le j \le m\}$.

We bound $|\tilde{F}|_R$ by hand, noting that

$$\begin{aligned}|\tilde{F}|_R &\le \operatorname{card}(\mathcal{M}) \max_{\mu\in\mathcal{M}}\{|D^i P_\mu(\tilde{\theta})|\} \max_{\mu\in\mathcal{M}}\{|M_\mu(\chi\circ\phi)|_R\}\\ &\le \exp(c_{19}((D_0-1)\log R + (D_1-1)R + (D_2-1)R^2)).\end{aligned}$$

We bound $|\tilde{F}(h\cdot y)|$ for $h \in \mathbf{N}^m(M)$ by appealing to our bound for $|H_{h,i}(\theta')|$ where $h \in \mathbf{N}^m(M)$. We know that

$$\begin{aligned}|\tilde{F}(h\cdot y)| &= \left|\sum_{\mu\in\mathcal{M}} D^i P_\mu(\tilde{\theta}) M_\mu(\chi\circ\phi(h\cdot y))\right|\\ &\le \left|\sum_{\mu\in\mathcal{M}} (D^i P_\mu(\tilde{\theta}) - D^i P_\mu(\theta)) M_\mu(\chi\circ\phi(h\cdot y))\right| \quad \text{(C)}\\ &\quad + \left|\sum_{\mu\in\mathcal{M}} D^i P_\mu(\theta) M_\mu(\chi\circ\phi(h\cdot y))\right|. \quad \text{(D)}\end{aligned}$$

To estimate (C), we recall that

$$|D^i P_\mu(\tilde{\theta}) - D^i P_\mu(\theta)| \le \exp(-r(S)/2)$$

provided $(\mathcal{C}3)$ and $(\mathcal{C}4)$ hold as above. Then, since we know the orders of growth of the coordinates of $\chi\circ\phi(z)$, we have

$$\begin{aligned}c_7 D_0^{d_0} D_1^{d_1} D_2^{d_2} \exp(-r(S)/2) \exp(c_{20}((D_0-1)\log M&\\ + (D_1-1)M + (D_2-1)M^2))&\end{aligned}$$

as an upper bound for the sum in (C). This is at most $\exp(-r(S)/3)$ provided $(\mathcal{C}2)$, $(\mathcal{C}3)$ and $(\mathcal{C}4)$ hold with a (possibly) different constant $c_{21} \ge c_{16}$ in $(\mathcal{C}4)$.

To bound the expression in (D), we note that this sum is

$$\frac{\Theta_{s(h)}(\operatorname{Lie}\psi(h\cdot y))^{D_2-1}}{Q(\theta)^{D_0+||h||D_1}(D^{\beta(h)}_{s(h),h}(\theta'))^{D_2-1}} H_{h,i}(\theta')$$

where $|D^{\beta(h)}_{s(h),h}(\theta')| \ge \exp(-r(S)/5(D_2(S)-1))$. From the properties of theta functions, we also have

$$|\Theta_{s(h)}(\operatorname{Lie}\psi(h\cdot y))| \le \exp(c_{22}S^2).$$

Combining this with $|Q(\theta)| \geq c_2$ and our lower bound for $|D^{\beta(h)}_{s(h),h}(\theta')|$, we have

$$\left|\frac{\Theta_{s(h)}(\mathrm{Lie}\,\psi(h\cdot y))^{D_2-1}}{Q(\theta)^{D_0+||h||D_1}(D^{\beta(h)}_{s(h),h}(\theta'))^{D_2-1}}\right| \leq \exp(c_{22}(D_2-1)S^2 - (D_0+mMD_1)\log c_2 + r(S)/5).$$

Then the sum in (D) is at most $\exp(-r(S)/20)$ and

$$|\tilde{F}(h\cdot y)| \leq \exp(-r(S)/21)$$

provided $(\mathcal{C}3)$ and $(\mathcal{C}4)$ hold and $r(S) \geq 420\log 2$.

Finally, from (10), we have

$$\begin{aligned}|\tilde{F}|_{R_1} \leq 2\exp(c_{19}((D_0-1)\log R + (D_1-1)R \\ + (D_2-1)R^2))(4c_{17}S^{-\varepsilon})^{M^m} \\ + \left(\frac{c_{23}S}{M}\right)^{M^m}\left(\frac{c_{24}}{M\log M}\right)^{M^{m-1}} M^m\exp(-r(S)/21)\end{aligned}$$

where $c_{23} = 18c_{17}/c_{18}$ and $c_{24} = c_{18}/2c_0$. The first term here is bounded by

$$\exp(-M^m\varepsilon(\log S)/2)$$

provided $S \geq (4c_{17})^{2/\varepsilon}$ and provided

$$(\mathcal{C}6) \qquad M^m\varepsilon\log S \geq 4(\log 2 + c_{19}((D_0-1)\log R + (D_1-1)R + (D_2-1)R^2))$$

is satisfied. The second term here is bounded by

$$\exp(-r(S)/22)$$

provided

$$(\mathcal{C}7) \qquad r(S) \geq 462(M^m\log(c_{23}S))$$

and

$$(\mathcal{C}8) \qquad M \geq \max\{S_0, c_{24}, m^{1/m}, e\}.$$

So we have

$$|\tilde{F}(h \cdot y)| \leq \exp(-M^m \varepsilon(\log S)/2) + \exp(-r(S)/22).$$

Then, for $h \in \mathbf{N}^m(S)$, we note that

$$\begin{aligned} &\Bigl|\sum_{\mu\in\mathcal{M}} D^i P_\mu(\tilde{\theta}) H_{\mu,h}(\theta')\Bigr| \\ &\qquad = \left|\frac{Q(\theta)^{D_0+||h||D_1}(D^{\beta(h)}_{s(h),h}(\theta'))^{D_2-1}}{\Theta_{s(h)}(\operatorname{Lie}\psi(h\cdot y))^{D_2-1}}\right| |\tilde{F}(h\cdot y)|. \end{aligned}$$

By the theory of theta functions, we have $|\Theta_{s(h)}(\operatorname{Lie}\psi(h \cdot y))| \geq \exp(-c_{25}S^2)$, so

$$\begin{aligned} \Bigl|\sum_{\mu\in\mathcal{M}} D^i P_\mu(\tilde{\theta}) H_{\mu,h}(\theta')\Bigr| &\leq \exp(c_{26}((D_0-1) + m(D_1-1)S \\ &\qquad + (D_2-1)S^2) - r(S)/22) \\ &\quad + \exp(c_{26}((D_0-1) + m(D_1-1)S \\ &\qquad + (D_2-1)S^2) - M^m\varepsilon(\log S)/2) \\ &\leq \exp(-r(S)/23) + \exp(-M^m\varepsilon(\log S)/3) \end{aligned}$$

provided $(\mathcal{C}3)$, $(\mathcal{C}4)$ and $(\mathcal{C}6)$ hold with (possibly) larger constants c_{27} and c_{28} in $(\mathcal{C}4)$ and $(\mathcal{C}6)$, respectively.

Lastly, we note that

$$\begin{aligned} |H_{h,i}(\theta')| &\leq \Bigl|\sum_{\mu\in\mathcal{M}} (D^i P_\mu(\theta) - D^i P_\mu(\tilde{\theta})) H_{\mu,h}(\theta')\Bigr| \\ &\quad + \Bigl|\sum_{\mu\in\mathcal{M}} D^i P_\mu(\tilde{\theta}) H_{\mu,h}(\theta')\Bigr| \\ &\leq \exp(-r(S)/24) + \exp(-\varepsilon M^m(\log S)/3), \end{aligned}$$

again provided $(\mathcal{C}3)$, $(\mathcal{C}4)$ and $(\mathcal{C}5)$ hold with a possibly larger constant c_{29} in $(\mathcal{C}4)$.

Defining the ideals. For $h \in \mathbf{N}^m(S)$ and $i \in I_S$, we define $H^*_{h,i}(X)$ by

$$H^*_{h,i}(\theta) = r(H_{h,i}(\theta, X_{t+1}), R(\theta, X_{t+1})),$$

where $R(\theta, X_{t+1})$ is the minimal polynomial for θ_{t+1} over $\mathbf{Z}[\theta_1, \dots, \theta_t]$ and r denotes Chudnovsky's semi-resultant. Then we define the ideal $\mathcal{J}_S$ by

$$\mathcal{J}_S = \langle H^*_{h,i} : h \in \mathbf{N}^m(S), i \in I_S \rangle.$$

From our previous work, the properties of semi-resultants, and the conditions we've imposed, we see that Proposition 7 is established.

It remains, however, to show that we can choose suitable parameters D_0, D_1, D_2, M and r which, together with S, satisfy $(\mathcal{C}0)$–$(\mathcal{C}8)$.

Choice of the parameters. Our objective now is to choose the parameters $D_0(S)$, $D_1(S)$, $D_2(S)$, $M(S)$ and $r(S)$ to satisfy the conditions:

$$(\mathcal{C}0) \qquad S \text{ is sufficiently large,}$$

$$(\mathcal{C}1) \quad \begin{aligned} 2^{t+1}[K:\mathbf{Q}](\deg{}_{X_{t+1}} R) 8^{\dim G'} & \\ \cdot \operatorname{card}\left(\frac{\Gamma(M)+G'}{G'}\right) & H(G'; D_0, D_1, D_2) \\ \leq c_7 D_0^{d_0} D_1^{d_1} D_2^{d_2} & \end{aligned}$$

for some connected algebraic subgroup G' of G,

$$(\mathcal{C}2) \qquad M < S,$$

$$(\mathcal{C}3) \qquad r \geq 3 S^{\kappa \mu^\sharp} \log S,$$

$$(\mathcal{C}4) \qquad \begin{cases} c_{29}\overline{D}_0 \leq S^{\kappa\mu^\sharp} \\ c_{29}\overline{D}_1 \leq S^{\mu^\sharp - 1} (\log S)^{d_0/d} \\ c_{29}\overline{D}_2 \leq S^{\mu^\sharp - 2} (\log S)^{d_0/d}, \end{cases}$$

$$(\mathcal{C}5) \qquad \operatorname{card}\left(\frac{\Gamma(S)+G'}{G'}\right) \frac{H(G'; \overline{D}_0, \overline{D}_1, \overline{D}_2)}{H(G; \overline{D}_0, \overline{D}_1, \overline{D}_2)} > 2^{d_2}$$

for every proper connected algebraic subgroup G' of G,

$$(\mathcal{C}6) \quad c_{28}((D_0-1)\log R + (D_1-1)R + (D_2-1)R^2) \leq M^m \varepsilon \log S,$$

$$(\mathcal{C}7) \qquad c_{30}(M^m \log S) \leq r,$$

and

$$(\mathcal{C}8) \qquad M \geq \max\{S_0, c_{24}, m^{1/m}, e\}.$$

Given an ideal $J \subseteq K[\mathcal{X}_1, \ldots, \mathcal{X}_n]$, we will define τ below in terms of $\operatorname{Deg}(J)$, $\operatorname{Ht}(J)$ and $T(J)$. Given τ, we define D_0, D_1 and D_2 as follows. If $d_i = 0$, then we choose $\tilde{D}_i = 2^{d_2+1}(\log S)^{-d_0/d}$. Otherwise, for some sufficiently large constant ν, we let

$$\begin{aligned}
\tilde{D}_0 &= 2\frac{1}{\nu^{m-l}}\tau(\log S)^{-1-(d_0/d)} \\
\tilde{D}_1 &= 2\frac{1}{\nu^{m-l}}\frac{S^{\mu^\sharp-1+\mu^\sharp d_0/(d-1)}(\log S)^{d_0/d(d-1)}}{\tau^{d_0/(d-1)}} \\
\tilde{D}_2 &= 2\frac{1}{\nu^{m-l}}\frac{S^{\mu^\sharp-2+\mu^\sharp d_0/(d-1)}(\log S)^{d_0/d(d-1)}}{\tau^{d_0/(d-1)}}.
\end{aligned}$$

Following the ideas of [**34**], much as in [**1**], we let

$$A(S, G') = \left(\frac{1}{2^{d_2+1}}\operatorname{card}\left(\frac{\Gamma(S) + G'}{G'}\right)\frac{H(G'; \tilde{D}_0, \tilde{D}_1, \tilde{D}_2)}{H(G; \tilde{D}_0, \tilde{D}_1, \tilde{D}_2)}\right)^{1/\dim(G/G')}$$

and define

$$A(S) = \min_{G' \subset G} A(S, G')$$

where G' ranges over all proper connected algebraic subgroups of G. We also define

$$B(S) = \min\{A(S), 2^{-(d_2+1)}(\log S)^{d_0/d}\}$$

and for $i = 0, 1, 2$, we let

$$\overline{D}_i(S) = \tilde{D}_i(S)B(S),$$

and

$$D_i = \max\{1, [\overline{D}_i]\}$$

as mentioned previously.

Our objective is to choose $M(S)$ and $r(S)$ and to impose conditions on ν, τ, S and $r(S)$ which will insure that the parameters $S, M, D_0, D_1, D_2, \overline{D}_0, \overline{D}_1, \overline{D}_2, R$ and r satisfy the conditions $(\mathcal{C}1)$ through $(\mathcal{C}8)$.

We consider $(\mathcal{C}1)$ first. Here we exploit the fact that this condition does not have to be satisfied for every connected algebraic subgroup $G' \subsetneq G$ but only for *at least one* such G'. This idea allowed Ably to replace Condition 1 of [**43**] with a weaker technical hypothesis. The verification of $(\mathcal{C}1)$ is essentially as in [**1**], with one small technicality.

Ably's choices of $\overline{D}_i$ insure that $\overline{D}_i(S) \geq 1$ for $i = 0, 1, 2$ and for all S under consideration. Then he defines $D_i = [\overline{D}_i]$ and uses the inequalities $\overline{D}_i \geq D_i \geq \overline{D}_i/2$ at the point where he establishes the condition $(\mathcal{C}1)$. In our case, we don't necessarily have $\overline{D}_i(S) \geq 1$ for all values of i and S. We address this by defining $D_i = \max\{1, [\overline{D}_i]\}$ above; this insures that $D_i \geq \overline{D}_i/2$ for $i = 0, 1, 2$. We also make the following observation. When $D_0 = 1$ (alternatively $D_1 = 1$ or $D_2 = 1$), we are constructing polynomials which don't involve any $\mathbf{G}_a$-, respectively $\mathbf{G}_m$- or G_2-, coordinates, so we may disregard the corresponding entry in $H(G'; D_0, D_1, D_2)$. This observation allows us to assume, without loss of generality, that $\overline{D}_i \geq 1$. With this in mind, we proceed with the verification of $(\mathcal{C}1)$.

In the first case, we suppose that

$$B(S) = A(S).$$

From the definition of $A(S)$, we know that there exists a connected algebraic subgroup $G'_0 \subsetneq G$ such that $A(S) = A(S, G'_0)$. We then choose to satisfy $(\mathcal{C}1)$ with G'_0.

Using this choice of G'_0 with the definition of $A(S)$, we have

$$A(S)^{\dim(G/G'_0)} \geq A(S, G'_0)^{\dim(G/G'_0)}.$$

Exploiting this inequality, the definition of $A(S, G'_0)$, the equations $\tilde{D}_i = \overline{D}_i/B$ for $i = 0, 1, 2$, the homogeneity of $H(G; \cdot, \cdot, \cdot)$, and the equality $B(S) = A(S)$, we have

$$\begin{aligned} A(S)^{\dim(G/G'_0)} \geq \frac{1}{2^{d_2+1}} &\operatorname{card}\left(\frac{\Gamma(S) + G'_0}{G'_0}\right) \\ &\cdot \frac{H(G'_0; \overline{D}_0, \overline{D}_1, \overline{D}_2)}{H(G; \overline{D}_0, \overline{D}_1, \overline{D}_2)} A(S)^{\dim(G/G'_0)}. \end{aligned}$$

That is,

$$H(G;\overline{D}_0,\overline{D}_1,\overline{D}_2) \geq \frac{1}{2^{d_2+1}}\operatorname{card}\left(\frac{\Gamma(S)+G_0'}{G_0'}\right) \cdot H(G_0';\overline{D}_0,\overline{D}_1,\overline{D}_2). \tag{11}$$

On the righthand side of $(\mathcal{C}1)$, thanks to Lemma 3.4 of [**33**], we have

$$D_0^{d_0}D_1^{d_1}D_2^{d_2} = \frac{d_0!d_1!d_2!}{d!\deg G_2}H(G;D_0,D_1,D_2).$$

Furthermore, since $D_i \geq \overline{D}_i/2$, we have

$$\begin{aligned} H(G;D_0,D_1,D_2) &\geq H(G;\overline{D}_0/2,\overline{D}_1/2,\overline{D}_2/2) \\ &= \frac{1}{2^d}H(G;\overline{D}_0,\overline{D}_1,\overline{D}_2). \end{aligned}$$

Using these two relationships followed by (11), we see that the right-hand side of $(\mathcal{C}1)$ is at least

$$c_7\frac{d_0!d_1!d_2!}{d!\deg G_2}\frac{1}{2^{d+d_2+1}}\operatorname{card}\left(\frac{\Gamma(S)+G_0'}{G_0'}\right)H(G_0';\overline{D}_0,\overline{D}_1,\overline{D}_2).$$

We impose a stronger condition, namely,

$$\nu^{d+1}M \leq S. \tag{$(\mathcal{C}2)'$}$$

From this, we have

$$\operatorname{card}\left(\frac{\Gamma(S)+G_0'}{G_0'}\right) \geq \frac{\nu^{d+1}}{\operatorname{card}T(\Gamma)}\operatorname{card}\left(\frac{\Gamma(M)+G_0'}{G_0'}\right)$$

where $T(\Gamma)$ is the set of torsion points of Γ. Combined with our previous upper bound, we see that the righthand side of $(\mathcal{C}1)$ is at least

$$c_7\frac{d_0!d_1!d_2!}{d!\deg G_2}\frac{1}{2^{d+d_2+1}}\frac{\nu^{d+1}}{\operatorname{card}T(\Gamma)} \cdot \operatorname{card}\left(\frac{\Gamma(M)+G_0'}{G_0'}\right)H(G_0';\overline{D}_0,\overline{D}_1,\overline{D}_2).$$

On the lefthand side of $(\mathcal{C}1)$, we have at most

$$2^{t+1}[K:\mathbf{Q}](\deg_{X_{t+1}} R)8^d \operatorname{card}\left(\frac{\Gamma(M)+G_0'}{G_0'}\right) H(G_0'; D_0, D_1, D_2).$$

We note that $H(G_0'; \overline{D}_0, \overline{D}_1, \overline{D}_2) \geq H(G_0'; D_0, D_1, D_2)$, since without loss of generality, we may assume that $\overline{D}_i \geq 1$ for $i = 0, 1, 2$. Then we see that $(\mathcal{C}1)$ is satisfied provided

$$\nu^{d+1} \geq \frac{2^{t+1} 2^{4d+d_2+1} d! (\deg G_2)(\operatorname{card} T(\Gamma))[K:\mathbf{Q}](\deg_{X_{t+1}} R)}{c_7 d_0! d_1! d_2!}.$$

In the second case, we have $B(S) = 2^{-(d_2+1)}(\log S)^{d_0/d}$. This time, we choose to satisfy $(\mathcal{C}1)$ with the particular connected algebraic subgroup $G' = \{0\}$ of G. With this choice, $(\mathcal{C}1)$ is satisfied provided

$$2^{t+1}[K:\mathbf{Q}](\deg_{X_{t+1}} R) M^{m-l} \leq c_7 D_0^{d_0} D_1^{d_1} D_2^{d_2}.$$

Again, since $D_i \geq \overline{D}_i/2$, we have

$$\begin{aligned} D_0^{d_0} D_1^{d_1} D_2^{d_2} &\geq \frac{1}{2^d} \overline{D}_0^{d_0} \overline{D}_1^{d_1} \overline{D}_2^{d_2} \\ &\geq \frac{1}{2^{d(d_2+1)} \nu^{d(m-l)}} S^{\kappa \mu^\sharp (1-l/m)} \end{aligned}$$

from our choices of $\tilde{D}_0, \tilde{D}_1, \tilde{D}_2$ and the definitions of $\overline{D}_i$ (for $i = 0, 1, 2$), $B(S)$ and κ. So we choose

$$M = \left[\frac{1}{\nu^{d+1}} S^{\kappa\mu^\sharp/m}\right]$$

and $(\mathcal{C}1)$ is satisfied provided

$$\nu^{m-l} \geq \frac{2^{d(d_2+1)} 2^{t+1}[K:\mathbf{Q}] \deg_{X_{t+1}} R}{c_7}.$$

Of course, $(\mathcal{C}2)'$ is satisfied as well provided $S \geq 1$ since $\kappa\mu^\sharp \leq m$.

In order to insure the desired zero-free region, we required that $(\mathcal{C}5)$ hold for every proper connected algebraic subgroup $G' \subsetneq G$. This follows easily (as in [**1**]) from our definitions of $A(S,G')$, $A(S)$, $B(S)$ and the relationships between $B(S)$, $\tilde{D}_i(S)$ and $\overline{D}_i(S)$ for $i=0,1,2$.

To see this, we multiply each side of $(\mathcal{C}5)$ by $(B(S))^{\dim(G/G')}$ and use the homogeneity of $H(G';\cdot,\cdot,\cdot)$ and $H(G;\cdot,\cdot,\cdot)$ to observe that

$$\frac{H(G';\overline{D}_0,\overline{D}_1,\overline{D}_2)}{H(G;\overline{D}_0,\overline{D}_1,\overline{D}_2)}B(S)^{\dim(G/G')}=\frac{H(G';\tilde{D}_0,\tilde{D}_1,\tilde{D}_2)}{H(G;\tilde{D}_0,\tilde{D}_1,\tilde{D}_2)}.$$

Then the lefthand side of $(\mathcal{C}5)$ times $B(S)^{\dim(G/G')}$ is simply 2^{d_2+1} times $A(S,G')^{\dim(G/G')}$ which is at least $2^{d_2+1}A(S)^{\dim(G/G')}$.

On the other hand, from the definition of $B(S)$, we have $B(S)\le A(S)$. Recalling that we've multiplied both sides of $(\mathcal{C}5)$ by $B(S)^{\dim(G/G')}$, we see that the righthand side is at most $2^{d_2}A(S)^{\dim(G/G')}$ and the condition is satisfied.

We consider condition $(\mathcal{C}3)$ which suggests that

$$r(S)=3S^{\kappa\mu^\sharp}\log S.$$

At the same time, we satisfy $(\mathcal{C}7)$ provided $\nu^{(d+1)m}\ge c_{30}/3$. We work on $(\mathcal{C}4)$ now. By the definitions of $\overline{D}_0,\tilde{D}_0$ and $B(S)$, we have

$$c_{29}\overline{D}_0\log S\le\begin{cases}c_{29}\tau/2^{d_2}\nu^{m-l} & \text{if } d_0=1\\ c_{29}\log S & \text{if } d_0=0\end{cases}$$

which is bounded by $S^{\kappa\mu^\sharp}\log S$ if S is sufficiently large (when $d_0=0$) and if τ satisfies

$$\text{(I)}\qquad \tau\le\frac{2^{d_2}\nu^{m-l}}{c_{29}}S^{\kappa\mu^\sharp}(\log S)\quad\text{when } d_0=1.$$

We also have

$$c_{29}\overline{D}_1\le\max\left\{\frac{c_{29}}{2^{d_2}\nu^{m-l}}\cdot\frac{S^{\mu^\sharp-1+\mu^\sharp d_0/(d-1)}(\log S)^{d_0/d(d-1)}(\log S)^{d_0/d}}{\tau^{d_0/(d-1)}},c_{29}\right\}$$

which is bounded by $S^{\mu^\sharp-1}(\log S)^{d_0/d}$ provided $\nu^{m-l}\ge c_{29}/2^{d_2}$ when $d_0=0$ and

$$\left(\frac{c_{29}}{2^{d_2}\nu^{m-l}}\right)^{d-1}S^{\mu^\sharp}(\log S)^{1/d}\le\tau$$

when $d_0 = 1$ and $d_1 \geq 1$. A similar upper bound holds for $c_{29}\overline{D}_2$ under the same conditions; thus $(\mathcal{C}4)$ is satisfied provided (I) holds and

$$\text{(II)} \qquad \left(\frac{c_{29}}{2^{d_2}\nu^{m-l}}\right)^{d-1} S^{\mu^\sharp}(\log S)^{1/d} \leq \tau$$

when $d_0 = 1$ and $d \geq 2$.

If we insure that

$$\text{(III)} \qquad S^{\kappa\mu^\sharp/m} \geq \nu^{d+1}\max\{S_0, c_{24}, m^{1/m}, e\}$$

we see that $(\mathcal{C}8)$ is satisfied. Furthermore, we have $M \geq 1$, which is useful below.

We turn our attention to $(\mathcal{C}6)$ last, observing first that

$$\begin{aligned} c_{28}(D_0 - 1)\log R &= c_{28}(D_0 - 1)\log S^{1+\varepsilon} \\ &\leq 2c_{28}(D_0 - 1)\log S \leq \tau/2 \end{aligned}$$

as long as $\nu^{m-l} \geq 4c_{28}/2^{d_2}$. Then, since $M \geq 1$, we may bound the first term of $(\mathcal{C}6)$ by $(M^m\varepsilon\log S)/2$ provided

$$\text{(I)}' \qquad \tau \leq \min\left\{\frac{2^{d_2}\nu^{m-l}}{c_{29}}, \frac{\varepsilon}{2^m\nu^{m(d+1)}}\right\} S^{\kappa\mu^\sharp}\log S.$$

Secondly, we have

$$\text{(12)} \qquad \begin{aligned} c_{28}((D_1 - 1)R + (D_2 - 1)R^2) = c_{28}(&(D_1 - 1)S^{1+\varepsilon} \\ &+ (D_2 - 1)S^{2+2\varepsilon}). \end{aligned}$$

When $d_0 = 0$, the righthand side is bounded by

$$\text{(13)} \qquad \frac{1}{2}(S^{\mu^\sharp}(\log S)^{d_0/d})S^{2\varepsilon}$$

where the $S^{2\varepsilon}$ is replaced with S^{ε} if $d_2 = 0$. On the other hand, when $d_0 = 1$ and $d \geq 2$, the condition

$$\text{(II)}' \qquad \begin{gathered} 2^{d-1}\left(\frac{\max\{c_{28}, c_{29}\}}{2^{d_2}\nu^{m-l}}\right)^{d-1} S^{\mu^\sharp}(\log S)^{1/d} \leq \tau \\ \text{when } d_0 = 1 \text{ and } d \geq 2 \end{gathered}$$

insures that (13) is still an upper bound for the righthand side of (12). And, finally, since $2\varepsilon < (\kappa - 1)\mu^\sharp$, we see that ($\mathcal{C}6$) is satisfied for S sufficiently large provided

$$\text{(III)}' \qquad \log S \geq \max\left\{ \frac{2(d+1)m\log\nu}{\kappa\mu^\sharp}, \left(\frac{2^m\nu^{m(d+1)}}{\varepsilon}\right)^{d/(d-d_0)} \right\}$$

and $\nu \geq \max\{S_0, c_{24}, m^{1/m}, e\}$.

Thus, we have verified ($\mathcal{C}1$)–($\mathcal{C}8$) (and hence established Proposition 7) subject to the conditions (I)$'$, (II)$'$ and (III)$'$ and provided ν and S are sufficiently large.

Application of the criteria for algebraic independence and proof of Theorem 1. As in Theorem 1, let G be a commutative algebraic group of dimension $d \geq 1$ defined over a number field K, and suppose that $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2$ with $d_0 \in \{0,1\}$, $d_1 \geq 0$ and G_2 a commutative algebraic group of dimension $d_2 = d - d_0 - d_1$, defined over K, and with no linear factor. Let $\phi : \mathbf{C} \to G(\mathbf{C})$ be a one-parameter subgroup and let $y_1, \ldots, y_m$ be complex numbers which are $\mathbf{Q}$-linearly independent. Define $\mu^\sharp, \kappa$ and ω and specify the embedding χ as above. Suppose that $\kappa > 1$, that $\mu^\sharp > 2$ if G is nonlinear, and that the technical hypothesis (H) holds. Lastly, suppose that $J \subseteq K[\mathcal{X}_1, \ldots, \mathcal{X}_n]$ is an ideal of codimension $n+1-\kappa$, degree $\mathrm{Deg}\,(J)$, height $\mathrm{Ht}\,(J)$ and size $T(J)$.

To establish Theorem 1, we'll apply the following criteria for algebraic independence from [**16**], using (essentially) the polynomials of Proposition 7.

Proposition 9. *Let K be a number field, $\omega = (\omega_1, \ldots, \omega_n) \in \mathbf{C}^n$, and κ an integer belonging to $\{1, \ldots, n+1\}$. Let $1 \leq \delta, \tau, \sigma$ and U be positive real numbers with $\tau^{1/\kappa} \geq \sigma \geq 1$, $\tau \geq \delta\kappa\log(n+1)$ and $U \geq 6(4[K:\mathbf{Q}] + \kappa - 1)\tau$. Suppose that, for all integers T satisfying*

(a) $\tau/\sigma^\kappa < T \leq U/\sigma^\kappa$,

there exists a finite family of polynomials $(Q^*_{S,j})_{j=1,\ldots,m(S)} \subset K[\mathcal{X}_1, \ldots, \mathcal{X}_n]$ such that

(b) $d^0(Q^*_{S,j}) \leq \delta$,

(c) $\mathrm{Ht}\,(Q^*_{S,j}) + \delta\kappa\log(n+1) \le \tau$,

(d) $\max_{1\le j\le m(S)}\{|Q^*_{S,j}(\omega)|/|\omega|^{d^0 Q^*_{S,j}}\} \le \exp(-T\sigma^\kappa)$,

(e) *the polynomials* $Q^*_{S,j}$, $j = 1,\dots,m(S)$, *have no common zero in the ball of* $\mathbf{C}^n$ *with center* ω *and radius* $\exp(-T\sigma^{\kappa+1})$.

Then for all ideals $J \subset K[\mathcal{X}_1,\dots,\mathcal{X}_n]$ *of codimension* $n+1-\kappa$, *of height* H *and of degree* D, *satisfying*

(f) $\quad (4[K:\mathbf{Q}]+\kappa+1)(27\sigma)^\kappa(H\delta^\kappa + (\tau+\delta\log(n+1))D\delta^{\kappa-1}) \le U$

we have

$$\log||J||_\omega \ge -U.$$

To apply the criteria, we let $\sigma = 6\nu^{(d+1)m+1}$, and for a sufficiently large constant η, we define

$$\delta = \frac{\exp(\eta(\mathrm{Deg}\,J)^{d_0\kappa/(d-\kappa)}T(J))}{T(J)^{1/\kappa}},$$
$$\tau = \frac{4\kappa(\log(n+1))\delta T(J)^{d_0}}{(\mathrm{Deg}\,J)^{d_0}},$$

and

$$\begin{aligned} U &= (4[K:\mathbf{Q}]+\kappa+1)(27\sigma)^\kappa(5\kappa\log(n+1))T(J)\delta^\kappa \\ &= (4[K:\mathbf{Q}]+\kappa+1)(27\sigma)^\kappa(5\kappa\log(n+1)) \\ &\quad\cdot \exp(\eta\kappa(\mathrm{Deg}\,J)^{d_0\kappa/(d-\kappa)}T(J)). \end{aligned}$$

We note that σ,δ,τ and U are at least one provided $\nu\ge 1$ and $\eta \ge 1$. Furthermore, $\tau \ge \sigma^\kappa$ as long as $\nu \ge 6$ and $\eta \ge 2\kappa((d+1)m+1)(\log\nu)$. Our choice of τ, as defined in terms of δ, also insures that $\tau \ge \delta\kappa\log(n+1)$ and our choice of U guarantees that condition (f) of the criteria is satisfied.

For each integer T in $(\tau/\sigma^\kappa, U/\sigma^\kappa]$, we define a corresponding S to be the real number satisfying

$$T = \frac{6}{\sigma^{\kappa+1}}S^{\kappa\mu^\sharp}\log S. \tag{14}$$

Now we turn our attention to the polynomials of Proposition 7. We may suppose, without loss of generality, that $\theta_1,\dots,\theta_t \in K[\omega]$.

Expressing $\theta_1, \ldots, \theta_t$ in terms of the coordinates of ω, we have new polynomials $Q^*_{S,j}$, $j = 1, \ldots, m(S)$, in $K[\mathcal{X}_1, \ldots, \mathcal{X}_n]$ such that their set of common zeros in $B^n(\omega, \exp(-2r(S)))$ is empty and they satisfy

$$\max_{1\le j\le m(S)} |Q^*_{S,j}(\omega)|/|\omega|^{d^0 Q^*_{S,j}} \le \exp(-a'_1 r(S)) + \exp(-a'_2 M^m \log S),$$

$$\max_{1\le j\le m(S)} d^0(Q^*_{S,j}) \le a'_3((D_0-1) + (D_1-1)S + (D_2-1)S^2),$$

and

$$\max_{1\le j\le m(S)} \mathrm{Ht}\,(Q^*_{S,j}) \le a'_3((D_0-1)\log S + (D_1-1)S + (D_2-1)S^2).$$

Using our choice of $r(S)$ and the relationship (14) between T and S, we see that the zero-free ball $B^n(\omega, \exp(-2r(S)))$ has radius $\exp(-\sigma^{\kappa+1}T)$. We may also insure that

$$\exp(-a'_1 r(S)) + \exp(-a'_2 M^m \log S) \le \exp\left(-\frac{S^{\kappa\mu^\sharp}\log S}{\nu^{(d+1)m+1}}\right)$$

provided, for instance, $\nu^{(d+1)m+1} \ge 2/3a'_1$, $\nu \ge 2^{m+1}/a'_2$ and $S \ge 4$. Then

$$\max_{1\le j\le m(S)} |Q^*_{S,j}(\omega)|/|\omega|^{d^0(Q^*_{S,j})} \le \exp(-\sigma^\kappa T).$$

Now we focus on the degrees and heights of the polynomials $Q^*_{S,j}$. We have

$$a'_3(D_0-1) \le \frac{\tau}{3}(\log S)^{-1}$$

provided $\nu^{m-l} \ge 3a'_3/2^{d_2}$. Since $T > \tau/\sigma^\kappa$, we know that $6S^{\kappa\mu^\sharp}\log S > \tau\sigma = 6\nu^{(d+1)m+1}\tau$ and thus

$$\log S \ge 4\kappa(\log(n+1))(\mathrm{Deg}\,J)^{d_0\kappa/(d-\kappa)}T(J)$$

provided $\eta \ge 8\kappa^2\mu^\sharp \log(n+1)$. Then we may conclude that

$$\tau \le \delta \log S$$

and thus
$$a_3'(D_0-1)\le \delta/3.$$

Considering the $\mathbf{G}_m$ and G_2 contributions to the degrees and heights of our polynomials $Q^*_{S,j}$, we note that

$$a_3'((D_1-1)S+(D_2-1)S^2)\le 2\delta/3$$

provided

$$\text{(IV)}\quad \begin{cases} (3a_3'/2^{d_2}\nu^{m-l})^{d-1}S^{\mu^\sharp d}(\log S)\le \delta^{d-1}\tau & \text{if } d_0=1 \text{ and } d\ge 2\\ (3a_3'/2^{d_2}\nu^{m-l})S^{\mu^\sharp}\le \delta & \text{if } d_0=0.\end{cases}$$

Finally, we note that

$$\max_{1\le j\le m(S)} d^0(Q^*_{S,j})\le \delta$$

and
$$\max_{1\le j\le m(S)} \mathrm{Ht}\,(Q^*_{S,j})\le \frac{\tau}{3}+\frac{2\delta}{3}\le \tau.$$

By the criteria for algebraic independence then, we conclude that

$$\log ||J||_\omega \ge -C_1\exp(C_2(\mathrm{Deg}\,J)^{d_0\kappa/(d-\kappa)}T(J))$$

provided $C_1\ge (4[K:\mathbf{Q}]+\kappa+1)(27\sigma)^\kappa(5\kappa\log(n+1))$ and $C_2\ge \eta\kappa$. Thus, Theorem 1 is established subject to verification of conditions (I)$'$, (II)$'$, (III)$'$ and (IV).

Verification of final conditions. Condition (I)$'$ is easily established since $T>\tau/\sigma^\kappa$. We simply need to insure that ν satisfies

$$\min\left\{\frac{2^{d_2}\nu^{(d+1)m+m-l+1}}{c_{29}},\frac{\varepsilon\nu}{2^m}\right\}\ge 1.$$

Similarly, (III)$'$ is satisfied provided

$$\eta\ge \kappa\mu^\sharp\left(\frac{2^m\nu^{m(d+1)}}{\varepsilon}\right)^{d/(d-d_0)}.$$

Conditions (II)$'$ and (IV) are much more sensitive. For both, we observe that $T \le U/\sigma^\kappa$ so

$$S^{\mu^\sharp}(\log S)^{1/d} \le (U\sigma/6)^{1/\kappa}(\log S)^{(1/d)-(1/\kappa)}.$$

To simplify the tedious inequalities, we let $c_{31} = ((4[K:\mathbf{Q}]+\kappa+1)27^\kappa(5\kappa\log(n+1))/6)^{1/\kappa}$ and then note that

$$S^{\mu^\sharp}(\log S)^{1/d} \le c_{31}\sigma^{(\kappa+1)/\kappa}T(J)^{1/\kappa}\delta(\log S)^{(1/d)-(1/\kappa)}. \tag{15}$$

When $d_0 = 1$, then $\ker\phi = \{0\}$, so $l = 0$; consequently, $\kappa \le d$ with strict inequality when $d \ge 2$. On the righthand side of our previous inequality, we then see that the $\log S$ term has a negative exponent when $d_0 = 1$ and $d \ge 2$. We know that

$$S^{\kappa\mu^\sharp} > \tau\sigma/6\log S$$

and thus

$$\log S > \frac{1}{2\kappa\mu^\sharp}\log\tau$$

provided $\sigma \ge 6$. In inequality (15) then, we have

$$\begin{aligned} S^{\mu^\sharp}(\log S)^{1/d} &\le c_{31}\sigma^{(\kappa+1)/\kappa}T(J)^{1/\kappa}\delta\left(\frac{\log\tau}{2\kappa\mu^\sharp}\right)^{(1/d)-(1/\kappa)} \\ &\le c_{31}\sigma^{(\kappa+1)/\kappa}T(J)^{1/\kappa}\delta \\ &\quad\cdot\left(\frac{\eta(\operatorname{Deg}J)^{d_0\kappa/(d-\kappa)}T(J)}{3\kappa\mu^\sharp}\right)^{(1/d)-(1/\kappa)}. \end{aligned}$$

Exploiting this last inequality, we see that (II)$'$ holds, provided

$$\frac{2^{d-1}c_{31}}{4\kappa\log(n+1)}\left(\frac{\max\{c_{28},c_{29}\}}{\nu^{m-l}2^{d_2}}\right)^{d-1}\sigma^{(\kappa+1)/\kappa}(3\kappa\mu^\sharp)^{(1/\kappa)-(1/d)} \le \eta^{(1/\kappa)-(1/d)}.$$

Since $\kappa < d$ when necessary, we may insure this condition by choosing η sufficiently large. Similarly, (IV) holds provided

$$\frac{c_{31}^d}{4\kappa\log(n+1)}\left(\frac{3a_3'}{2^{d_2}\nu^{m-l}}\right)^{d-1}\sigma^{d(\kappa+1)/\kappa}(3\kappa\mu^\sharp)^{(d/\kappa)-1} \le \eta^{(d/\kappa)-1} \quad \text{if } d_0 = 1 \text{ and } d \ge 2$$

and
$$\left(\frac{3a_3'}{2^{d_2}\nu^{m-l}}\right)c_{31}\sigma^{(\kappa+1)/\kappa}(3\kappa\mu^\sharp)^{1/\kappa} \le \eta^{1/\kappa} \quad \text{if } d_0 = 0.$$
Hence, our results are established.

Warning. The reader should note that these final inequalities are a bit misleading. In particular, the constant c_{31} hides many things while σ hides many powers of ν. Furthermore, the constants c_{28}, c_{29} and a_3' have not even been completely specified.

Closing remarks regarding the nonlimit case. In considering the separation of the degree and height in our measure of algebraic independence, the earlier inequality
$$\tau \le \delta \log S$$
seems crucial for it allows us to bound the degree and heights of the polynomials $Q_{S,j}^*$ separately when applying the criteria of algebraic independence. In the nonlimit case, we would like to choose τ and δ to be "essentially" powers of $T(J)$ and $\mathrm{Deg}\,(J)$. This inequality forces the powers to be equal. For simplicity then, write $U = c_{32}\sigma^\kappa T(J)\delta^\kappa$. Now, to insure inequality (IV) when $d_0 = 1$, we would need
$$\left(\frac{c_{33}}{\nu^{m-l}}\right)^{d-1} c_{34}\sigma^{d(\kappa+1)/\kappa}T(J)^{d/\kappa}(\mathrm{Deg}\,J)^{d_0}(\log S)^{1-(d/\kappa)} \le 4\kappa(\log(n+1))T(J)^{d_0}$$
which seems impossible unless τ is exponential in $T(J)$ and $\mathrm{Deg}\,J$.

ENDNOTES

1. Throughout, we let $\log\alpha$ denote a fixed determination of the logarithm of $\alpha \in \mathbf{C}\backslash\{0\}$, and for $\beta \in \mathbf{C}$, we define $\alpha^\beta \stackrel{\text{def}}{=} e^{\beta\log\alpha}$.

2. For more about the intermediate results and the essential contributions, see [**44**] and [**4**].

3. A complex number u is called an algebraic point of $\wp$ if either u is a pole of $\wp$ or $\wp(u) \in \overline{\mathbf{Q}}$.

4. Again, for further discussion of the substantial contributions which made all this work possible, see [**4, 6, 44**].

5. Throughout the remainder of the paper, c_1, c_2, c_3, etc., and $a_1, a_1', a_2, a_2', a_3, a_3'$ will denote positive constants which are independent of the parameter S and may depend, at most, on $K, G, \phi, y_1, \ldots, y_m, \chi$ and $\theta_1, \ldots, \theta_{t+1}$.

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