ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 3, Summer 1996

SUBMODULES OF PRODUCTS OF QUASI-PERIODIC MODULES

W. DALE BROWNAWELL

To Wolfgang M. Schmidt in Honor of his 60th Birthday

ABSTRACT. Previous work of J. Yu, A. Thiery and the author determines the connected components of (reduced) algebraic subgroups of products of various Drinfeld A-modules. Here we extend that work to products of the associated higher dimensional modules whose A-action is given by the corresponding derivations.

The first step is to further investigate the $\mathbf{F}_q$-linear relations holding on a subgroup of products of Drinfeld modules ϕ_i. Then submodules of the products of higher dimensional modules correspond to certain subspaces of derivations. One natural way these subspaces arise is as pullbacks of relations on submodules of products of Drinfeld modules.

Notation. $\mathbf{F}_q$ is a finite field of $q = p^s$ elements, $\mathcal{C}$ is a smooth projective geometrically irreducible curve over $\mathbf{F}_q$, ∞ is a closed point on $\mathcal{C}$, k is the function field of $\mathcal{C}$ over $\mathbf{F}_q$, A is the ring of functions in k regular on $C\backslash\{\infty\}$, k_∞ is the completion of k at ∞, $\bar{k}_\infty$ is the algebraic closure of k_∞, $\mathcal{T}$ is the ring $\bar{k}_\infty\{F\}$ of "twisted polynomials" in the q-power Frobenius element F.

0. Introduction. For applications in transcendence, it has become useful to understand the algebraic subgroups of products of interesting commutative algebraic groups. In [**10**], algebraic subgroups of products of simple algebraic groups were shown to be subproducts whose factors have coordinates satisfying linear relations over the respective endomorphism rings.

In particular, it was shown in [**10**] and in [**2**] (see also [**4**]) that, for Weierstrass elliptic functions $\wp_i$ corresponding to non-isogenous elliptic curves $\mathcal{E}_i$, the functions $\wp_i(u_{ij}z)$ are algebraically independent

Received by the editors on June 9, 1995, and in revised form on December 14, 1995.
Research supported in part by NSF.

exactly when all the numbers u_{ij} are linearly independent over the multiplication ring of each curve $\mathcal{E}_i$. In fact, the main independence result for functions in [**2**] included various $\zeta_i(u_{ij}z)$ as well, where the $\zeta_i(z)$ are Weierstrass quasi-periodic functions.

In [**6**], V.G. Drinfeld introduced the very fruitful notion of elliptic modules, now usually called Drinfeld modules, which are rich analogues over function fields over finite fields of elliptic curves. See [**9**] for a nice exposition. In a remarkable series of papers [**14–17**], J. Yu has developed transcendence theory in the setting of Drinfeld modules. For an overview of this and related work, see the surveys [**18, 13**]. In particular, Yu has begun (and A. Thiery has continued [**12**]) the investigation of transcendence properties of quasi-periodic functions associated to Drinfeld modules, which are components of the exponential function of a related t-module. (See [**1**] for t-modules.)

In [**12**] and [**3**], we investigated the algebraic subgroups of products of such modules which are closed under the A-action. In this note we extend these considerations to related quasi-periodic modules. The consequences for independence of values must await some quantitative investigations, as in the usual situation over $\mathbf{C}$. For increased accessibility, we will sometimes indicate arguments which have been given elsewhere before.

Background: Drinfeld modules and Drinfeld exponential functions. Let $F = F^1$ be the qth power function (the sth iterate of the Frobenius map $X \mapsto X^p$). Let $\mathcal{T} = \bar{k}_\infty\{F\}$ denote the *twisted* (noncommutative) *polynomial ring* of operators generated by F over $\bar{k}_\infty$. This ring of *twisted polynomials* $T = \sum a_h F^h$ is evidently isomorphic to the ring of $\mathbf{F}_q$*-linear polynomials* $P(X) = \sum a_h X^{q^h}$, whose multiplication is composition of polynomials. Still for clarity, we maintain the distinction between the $\mathbf{F}_q$-linear polynomials and the operations (twisted polynomials) given by substituting into them. The ring $\bar{k}_\infty\{F\}$ of twisted polynomials has no zero divisors and in fact is a right (and a left) division ring. In particular, every left ideal in $\bar{k}_\infty\{F\}$ is principal. If $T \in \mathcal{T}$, we define $\operatorname{subdeg}_F T = \max n$ such that $T \in \bar{k}_\infty\{F\}F^n$, with the understanding that $\operatorname{subdeg}_F 0 = \infty$.

One definition of a Drinfeld A-module, with the additional specification of a rank $r > 0$ is an $\mathbf{F}_q$-linear homomorphism $\phi : A \to \bar{k}_\infty\{F\}$

with the following property: for each $a \in A$, $a \neq 0$,

$$\phi(a) = aF^0 + a_1F^1 + \cdots + a_mF^m \tag{1}$$

for $a_1, \ldots, a_m$ in $\bar{k}_\infty$, $m = rd(a)$, where d is the valuation associated with the point ∞, and $a_m \neq 0$. When L is a subfield of $\bar{k}_\infty$ and $\phi(A) \subset L\{F\}$, we say that ϕ is *defined over* L.

There is also a more analytic way of looking at Drinfeld modules, which will be useful to us later on. An *A-lattice* Λ *of rank* r is a finitely generated discrete A-module of $\bar{k}_\infty$ of projective rank r. Given such a Λ, Drinfeld defines the following "exponential" function:

$$e_\Lambda(z) = z \prod_{\omega \in \Lambda}{}' \left(1 - \frac{z}{\omega}\right),$$

where $\prod'$ means that no term corresponding to $0 = \omega \in \Lambda$ appears in the product. This is an $\mathbf{F}_q$-linear power series, since its partial sums are $\mathbf{F}_q$-linear. Drinfeld showed that $e_\Lambda(z)$ is an $\mathbf{F}_q$-linear surjective homomorphism onto $\bar{k}_\infty$. It is not difficult to show that, for each $a \in A$, there is a unique twisted polynomial $\phi_\Lambda(a)$ such that

$$e_\Lambda(az) = \phi_\Lambda(a)e_\Lambda(z). \tag{2}$$

The map $a \mapsto \phi_\Lambda(a) \in \bar{k}_\infty\{F\}$ gives a Drinfeld module of rank r, and Drinfeld showed [**6**] that all Drinfeld modules can be accounted for in this way. For more details on the basic material of this section and surveys of Drinfeld modules, see [**9**].

As suggested by P. Philippon, we consider the case of $\mathbf{G}_a$ with the ordinary A-action to be a *trivial* A-Drinfeld module with $r = 0$, $\phi(a) = aF^0$ for all a, and $e_\phi(z) = z$.

Isogenies, endomorphisms, and multiplications. If $\phi : A \to L\{F\}$ and $\psi : A \to L\{F\}$ are Drinfeld modules defined over L, then a *morphism* from ϕ to ψ defined over L is an element T of $L\{F\}$ for which

$$T\phi(a) = \psi(a)T, \qquad \forall\, a \in A,$$

and we write that $T \in \operatorname{Hom}(\phi, \psi)$. If such $T = uF^i +$ higher order terms, $u \neq 0$, then, recalling equation (1), we see that the lowest order terms in the preceding displayed equation are $ua^{q^i}F^i = auF^i$. Thus, on taking $a \in A$ transcendental over $\mathbf{F}_q$, we see that $i = 0$ and $T = uF^0+$ higher terms, with $u \neq 0$. We also write $\nabla T = u$. A nonzero morphism is called an *isogeny*.

In Theorem 2.2 of [**3**], the following result was shown:

Theorem 0.1. *Let ϕ and ψ be Drinfeld modules. Let $P(X) = uX+$ higher terms be an $\mathbf{F}_q$-linear polynomial over $\bar{k}_\infty$ with $u \neq 0$, and let $T \in \mathcal{T}$ be the corresponding twisted polynomial. The following conditions are equivalent*:

(1) *T is an isogeny from ϕ to ψ, i.e.,*

$$T\phi(a) = \psi(a)T, \qquad \forall\, a \in A.$$

(2) $P(e_\phi(z)) = e_\psi(uz)$.

(3) $u\mathrm{Ker}\,(P(e_\phi(z))) = \mathrm{Ker}\, e_\phi$.

We call $R_\phi := \{\nabla T : T \in \mathrm{Hom}\,(\phi, \phi)\}$ the *ring of multiplications of ϕ.* Thus $A \subset R_\phi$ and the nonzero elements of R_ϕ correspond to the isogenies which are endomorphisms of ϕ. It is also clear that ϕ extends uniquely to R_ϕ via $\phi(\nabla T) = T$. Drinfeld also showed that $R_\phi = \{c \in k_\infty : c\mathrm{Ker}\, e_\phi \subset \mathrm{Ker}\, e_\phi\}$. Compare, e.g., Corollary 2.4 of [**3**]. Thus, ϕ extends to an R_ϕ-Drinfeld module, also denoted ϕ, and $\phi(R_\phi) \subset L'\{F\}$ with L' a finite extension of L. (In the case of a trivial Drinfeld module, $\bar{k}_\infty$ may be considered the ring of endomorphisms.)

1. Algebraic A-submodules of products of Drinfeld modules. The basic result of this section is the following:

Classification Theorem 1.1. *Let $\mathcal{D}_1, \dots, \mathcal{D}_s$ be the additive group with A-action furnished by the mutually nonisogenous A-Drinfeld modules $\phi_1, \dots, \phi_s$. For $i = 1, \dots, s$, let e_i denote the exponential map associated to ϕ_i, select $f_i \in \mathbf{N}$, let $\mathbf{e}_i$ denote the map $\bar{k}_\infty^{f_i} \to \bar{k}_\infty^{f_i}$ obtained by applying e_i to each coordinate of $\mathbf{z}_i \in \bar{k}_\infty^{f_i}$, and let $M_i \subset \bar{k}_\infty$ denote the field of quotients of R_{ϕ_i}.*

Then the connected, reduced algebraic subgroups H of

$$G = \mathcal{D}_1^{f_1} \times \cdots \times \mathcal{D}_s^{f_s}$$

which are closed under the action of A are classified by the points $V = (V_1, \ldots, V_s)$ of

$$\mathrm{Grass}_{f_1}(M_1) \times \cdots \times \mathrm{Grass}_{f_s}(M_s).$$

More precisely, H and V correspond to each other in the following way: for all

$$\mathbf{z} = (\mathbf{z}_1, \ldots, \mathbf{z}_s) \in \bar{k}_\infty^{f_1+\cdots+f_s},$$

$$\exp(\mathbf{z}) \in H(\bar{k}_\infty) \Longleftrightarrow \mathbf{z}_i \perp V_i, \qquad i = 1, \ldots, s,$$

with respect to the ordinary inner product on each $\bar{k}_\infty^{f_i}$, where $\exp(\mathbf{z}) = (\mathbf{e}_1(\mathbf{z}_1), \ldots, \mathbf{e}_s(\mathbf{z}_s))$.

By $\mathrm{Grass}_{f_i}(M_i)$ we designate the set of all M_i-subspaces of $K_i^{f_i}$.

Proof. This follows from the investigations in [**3**] and [**12**], where the connected component of the identity is determined set-theoretically, which is equivalent to treating the reduced case, as we are doing here.

In particular, Theorem A of [**3**] shows that the reduced connected component H_0 of H is equal to the direct product

$$H_0(\bar{k}_\infty) = H_1(\bar{k}_\infty) \times \cdots \times H_s(\bar{k}_\infty)$$

of the reduced connected components H_i of the various $H \cap \mathcal{D}_i^{f_i}$ (and that the direction $\Longrightarrow$ holds). So we are reduced to the case that $G = \mathcal{D}_i^{f_i}$, which is well-known when rank $\phi_i = 0$. When rank $\phi_i > 0$, the theorem of the appendix of Chapter III of [**12**] shows both directions $\Longleftrightarrow$:

$$H_i(\bar{k}_\infty) = \{\mathbf{e}_i(\mathbf{z}_i) : \mathbf{z}_i \in \bar{k}_\infty^{f_i}, \mathbf{z}_i \perp V_i\}, \tag{3}$$

for a uniquely determined $V_i \in \mathrm{Grass}_{f_i}(M_i)$, and conversely, any choice of $V_i \in \mathrm{Grass}_{f_i}(M_i)$ determines a unique reduced connected A-closed subgroup H_i of $\mathcal{D}_i^{f_i}$ via (3). $\square$

This result gives a good understanding of the connected, reduced A-closed algebraic subgroups of $\prod \mathcal{D}_j^{f_j}$. The main goal of the present note is to extend this result to products of quasi-periodic modules.

Remark. We take this opportunity to note the omission of the "return isogeny" R_{ji} from ψ_j to ϕ_i (cf. Corollary 2.3 of [**3**]) in the proof of Theorem 3.2 of [**3**, p. 352]. The proof there will be complete if R_{ji} is applied to both $T_{i\tau}^{i\sigma}$ and $P_{i\tau}^{i\sigma}$ on lines -3 and -1.

Before we turn to our main task, we need some background.

Biderivations. When rank $\phi > 1$, one obtains related functions which are said to be *quasi-periodic* (with respect to the A-lattice Λ_ϕ). To describe them, we consider $\bar{k}_\infty\{F\}$ as an A-bimodule, where right multiplication is given by $\phi(a)$ and left multiplication by a itself. Then a *ϕ-biderivation* δ of A is an $\mathbf{F}_q$-linear map $\delta : A \mapsto \bar{k}_\infty\{F\}F$ such that

$$\delta(ab) = a\delta(b) + \delta(a)\phi(b), \qquad \forall\, a, b \in A.$$

There are three different fundamental types of ϕ-biderviations.

Differentials of the first kind. One ϕ-biderivation is obtained directly from ϕ as

$$\delta_\phi(a) = \phi(a) - aF^0, \qquad \forall\, a \in A.$$

Strictly inner biderivations. Let $D_{si}(\phi)$ denote the $\bar{k}_\infty$-vector space of *exact* or *strictly inner* ϕ-biderivations $\delta_\phi^{(T)}$, which are those obtained from the Drinfeld module ϕ and any twisted polynomial $T \in \bar{k}_\infty\{F\}F$ via

$$\delta_\phi^{(T)}(a) := T\phi(a) - aT, \qquad \forall\, a \in A.$$

Notice that, for nonzero such T,

$$\deg_F \delta_\phi^{(T)} = \deg_F T + \deg_F \delta_\phi(a) > \deg_F \delta_\phi(a).$$

Set $D_{in}(\phi) := \bar{k}_\infty \delta_\phi \oplus D_{si}(\phi)$.

Strictly reduced biderivations. When $r > 1$, those biderivations δ for which $\deg_F \delta(a) < \deg_F \delta_\phi(a)$ are said to be *strictly reduced*. Let

$D_{sr}(\phi)$ denote the $\bar{k}_\infty$-vector space of strictly reduced biderivations and set $D_{\rm red}(\phi) = \bar{k}_\infty\delta_\phi \oplus D_{sr}(\phi)$.

Then the "Hodge decomposition," equation (2.14) in [**7**], is that $D(\phi)$, the full $\bar{k}_\infty$-vector space of ϕ-biderivations, decomposes as

$$D(\phi) = D_{sr}(\phi) \oplus \bar{k}_\infty\delta_\phi \oplus D_{si}(\phi).$$

If ϕ is defined over $\bar{k}$, one obtains [**15**] the analogous decomposition for the modules $D_*(\phi)_{\bar{k}}$ of ϕ-biderivations δ defined over $\bar{k}$, i.e., with $\delta(A) \subset \bar{k}\{F\}F$:

$$D(\phi)_{\bar{k}} = D_{sr}(\phi)_{\bar{k}} \oplus \bar{k}\delta_\phi \oplus D_{si}(\phi)_{\bar{k}},$$

where $*$ stands for either si or sr or for the lack of a subscript, and moreover each

$$D_*(\phi) = D_*(\phi)_{\bar{k}} \otimes_{\bar{k}} \bar{k}_\infty.$$

Thus the de Rham cohomology satisfies

$$H^*_{DR}(\phi) := D(\phi)/D_{si}(\phi) \cong D_{sr}(\phi) \oplus \bar{k}_\infty\delta_\phi = D_{\rm red}(\phi).$$

P. Deligne and G. Anderson (see also [**7**, Section 5]) noticed that

$$\dim_{\bar{k}_\infty} H^*_{DR}(\phi) = \operatorname{rank}\phi = r.$$

E.-U. Gekeler proves the de Rham isomorphism in [**7**]

$$H^*_{DR}(\phi) \cong \operatorname{Hom}_A(\Lambda_\phi, \bar{k}_\infty),$$

by constructing the quasi-periodic functions inducing given elements of the module $\operatorname{Hom}_A(\Lambda_\phi, \bar{k}_\infty)$, while Anderson had a "shtuka" proof (still unpublished, but see [**8**] for related material).

Quasi-periodic functions and the quasi-periodic module. Given a biderivation δ, there is a unique (up to a nonzero scalar) entire $\mathbf{F}_q$-linear solution (e.g., [**7**]) $Y = F_\delta(z)$ of the functional equation

$$Y(az) - aY(z) = \delta(a)e_\phi(z), \qquad \forall\, a \in A, \tag{4}$$

satisfying the "initial condition" $Y(z) \equiv 0 \bmod z^q$. The function F_δ is said to be *quasi-periodic* with respect to δ and L, since

$$F_\delta(z+\omega) = F_\delta(z) + F_\delta(\omega), \qquad \forall\, z \in \bar{k}_\infty,\ \omega \in \Lambda, \tag{4.1}$$

$$F_\delta(z) \quad \text{is } A\text{-linear on } \Lambda. \tag{4.2}$$

It follows directly from (4) that F_δ is $\bar{k}_\infty$-linear in the index δ:

$$F_{c\delta+c\gamma}(z) = cF_\delta(a) + dF_\gamma(z). \tag{4.3}$$

We can readily account for the quasi-periodic functions arising from the "inner derivations," i.e., those of the form $\delta_\phi^{(T)}$, with $T \in \bar{k}_\infty\{F\}$, possibly with nonzero coefficient of F^0. The related quasi-periodic functions are easily verified to be of the form

$$F_{\delta_\phi^{(T)}}(z) := Te_\phi(z) - uz, \tag{5}$$

where $T = uF^0+$ higher terms. In particular, $F_{\delta_\phi}(z) = e_\phi(z) - z$.

Given $T \in \mathrm{Hom}\,(\phi,\psi)$, it is easy to check that pre-multiplication by T gives a $\bar{k}_\infty$-linear map $T^* : D(\psi) \to D(\phi)$:

$$(T^*\delta)(a) = \delta(a)T, \qquad \forall\, a \in A,$$

which we call the *pull-back* of δ by T. Thus, when $T \in \mathrm{End}\,(\phi)$, say $T = \tau F^0+$ higher terms, and $\delta \in D(\phi)_L$, then

$$T^*\delta = \delta_\phi^{(S)} + \sum_{i=2}^{r} c_i\delta_i,$$

where $S \in L'\{F\}F$, $c_i \in L'$, and $\delta_2, \ldots, \delta_r$ form a basis for $H_{sr}(\phi)_{L'}$. However, as noted in [**3**, pp. 361, 362], one sees directly from the functional equation for $F_\delta(z)$ that

$$F_{T^*\delta}(z) = F_\delta(\tau z). \tag{6}$$

Thus, we obtain the fundamental relation:

$$F_\delta(\tau z) = Se_\phi(z) - \sigma z + \sum_{i=2}^{r} c_iF_i(z), \tag{7}$$

where $S = \sigma F^0 +$ higher terms and each F_i is the quasi-periodic function corresponding to δ_i. Combining this remark with Theorem 5.1′ in [**3**], we obtain the following more general form of the classification theorem:

Theorem 1.2. *Let Φ denote a set of nonisogenous A-Drinfeld modules. For each $\phi \in \Phi$, let $r_\phi = \operatorname{rank}_A R_\phi$, $U(\phi) \subset k_\infty$ and $u_\phi = \operatorname{rank}_{R_\phi}(\sum_{u\in U(\phi)} R_\phi u)$. Then*

$$\operatorname{tr\,deg}_{\bar{k}_\infty} \bar{k}_\infty(z, \dots, e_\phi(uz), \dots, F_\delta(uz), \dots)_{\substack{\phi\in\Phi \\ u\in U(\phi) \\ \delta\in D(\phi)}} = 1 + \sum r_\phi u_\phi.$$

For later use, we now record that pull-backs respect the Hodge decomposition.

Proposition 1.3. *If T is an isogeny from ϕ to ψ, then T^* induces isomorphisms*

(1) $T^*_{DR} : H^*_{DR}(\psi) \to H^*_{DR}(\phi)$ *and*
(2) $T^*_{sr} : D_{sr}(\psi) \to D_{sr}(\phi)$.

Proof. First we notice that $T^*(D_{in}(\psi)) \subset D_{in}(\phi)$: For $S \in \bar{k}_\infty\{F\}$ and $a \in A$,

$$\begin{aligned} T^*\delta_\psi^{(S)}(a) &= (S\psi(a) - aS)T = S(\psi(a)T) - aST \\ &= ST(\phi(a)) - aST = \delta_\phi^{(ST)}(a). \end{aligned}$$

Therefore, as $D_{sr}(\psi) \simeq D(\psi)/D_{in}(\psi)$ (and similarly for ϕ), T^* induces a homomorphism from $D_{sr}(\psi)$ to $D_{sr}(\phi)$. Moreover, when S has no F^0 term, neither does ST. Therefore, T^* similarly induces a homomorphism from $H^*_{DR}(\psi)$ to $H^*_{DR}(\phi)$, as $H^*_{DR}(\psi) \simeq D(\psi)/D_{si}(\psi)$.

It is now sufficient to verify injectivity of T^*_{DR} and T^*_{sr}, since isogenous Drinfeld modules have the same rank, which is equal to the dimension of the de Rham cohomology by the above-cited results. For injectivity, it is enough to verify that the only reduced (and then strictly reduced) ψ-biderivation δ with $T^*\delta$ strictly inner (inner) is zero.

As before, let δ_i, $i = 2, \dots, r$ form a basis for $D_{sr}(\psi)$, and let $\delta_1 = \delta_\psi$. Write $\delta = \sum a_i \delta_i$ with $a_i \in \bar{k}_\infty$. Then, when $T^*\delta$ is strictly inner (alternatively inner), for some $S \in \mathcal{T}F$ (alternatively $S \in \mathcal{T}$) and all $a \in A$,

$$T^*\delta(a) = \delta_\phi^{(S)}(a) = S\phi(a) - aS.$$

We translate this into an identity on quasi-periodic functions. By the $\bar{k}_\infty$-linearity (4.3) of the F_δ in δ,

$$F_\delta = F_{\sum a_i \delta_i} = \sum a_i F_{\delta_i}.$$

Therefore we see by (5) and (6) that writing $T = \tau F^0 +$ higher terms gives

$$\sum_{i=1}^{r} a_i F_{\delta_i}(\tau z) = F_{T^*\delta}(z) = F_{\delta_\phi^{(S)}}(z) = Se_\phi(z) - \sigma z, \tag{8}$$

where $S = \sigma F^0 +$ higher terms (and $\sigma = 0$ if $T^*\delta$ is strictly inner). By Theorem 0.1 above, $e_\psi(\tau z) = Te_\phi(z)$, which makes $e_\phi(\tau z)$ and $e_\phi(z)$ algebraically dependent. On the other hand, by Theorem 5.1 of [**3**], we know that

$$z, e_\psi(z), F_{\delta_2}(z), \dots, F_{\delta_r}(z)$$

are algebraically independent. Therefore, $F_{\delta_2}(z), \dots, F_{\delta_r}(z)$ do not actually occur in (8) and by Theorem 0.1, $a_1 T = S$, i.e., $a_i = 0$, $i = 2, \dots, r$ and δ is inner.

If δ is strictly reduced, then $a_1 = 0$ and clearly $\delta = 0$. If $T^*\delta$ is strictly inner, then $\sigma = 0$ and thus $a_1\tau = 0$. But, as T is an isogeny, $\tau \neq 0$; hence $a_1 = 0$ and thus again $\delta = 0$. This completes the proof. □

Quasi-periodic A-modules. For fixed strictly reduced biderivations $\delta_2, \dots, \delta_r$ with linearly independent images in the de Rham cohomology $H^*_{DR}(\phi)$, let $F_2(z), \dots, F_r(z)$ be the associated quasi-periodic functions. The *quasi-periodic* A-module Q is what we call $\mathbf{G}_a^r$ with A-action given by left multiplication with matrices

$$\Phi(a) = \begin{pmatrix} a & & 0 & \delta_r(a) \\ & \ddots & 0 & \vdots \\ & & a & \delta_2(a) \\ & & 0 & \phi(a) \end{pmatrix} = aIF^0 + \begin{pmatrix} 0 & & 0 & \delta_r(a) \\ & \ddots & 0 & \vdots \\ & & 0 & \delta_2(a) \\ & & 0 & \delta_\phi(a) \end{pmatrix},$$

where the entries not indicated by at least a dot are zero, and where we have written a for aF^0 in the first matrix. If $r = 0$ or 1, then $\mathcal{Q} = \mathcal{D}$. We call the mapping

$$\mathrm{Exp}\ : \begin{pmatrix} z_r \\ \vdots \\ z_2 \\ z_1 \end{pmatrix} \longmapsto \begin{pmatrix} F_r(z_1) + z_r \\ \vdots \\ F_2(z_1) + z_2 \\ e(z_1) \end{pmatrix}$$

the *exponential map* of the quasi-periodic A-module $\mathcal{Q}$ associated to ϕ. Then $\Phi(a)\mathrm{Exp}\,(\mathbf{z}) = \mathrm{Exp}\,(a\mathbf{z})$, where $a\mathbf{z} = (az_1, \cdots, az_r)^t$.

Morphisms of quasi-periodic modules. A *morphism* of $\mathcal{Q}$ to another quasi-periodic A-module $\mathcal{Q}_\Psi$, with A-action given by

$$\Psi(a) = \begin{pmatrix} a & & 0 & \partial_r(a) \\ & \ddots & 0 & \vdots \\ & & a & \partial_2(a) \\ & & 0 & \psi(a) \end{pmatrix},$$

is an $\mathbf{F}_q$-linear map P on $\mathbf{G}_a^r$ which is compatible with the two A-module structures. In other words, P is given by a matrix of twisted polynomials (t_{ij}) such that for each $a \in A$, first of all,

$$\begin{aligned} \psi(a)(t_{11}X + t_{12}Y_2 + \cdots + t_{1r}Y_r) = t_{11}\phi(a)X + t_{12}(aY_2 + \delta_2(a)X) + \cdots \\ + t_{1r}(aY_r + \delta_r(a)X) \end{aligned}$$

and for each $a \in A$ and each $i = 2, \ldots, r$,

$$\begin{aligned} a\Big(t_{i1}X + \sum_{i\geq 2} t_{ij}Y_j + \partial_i(a)\Big)\Big(\sum_{j\geq 2} t_{ij}Y_j + t_{11}X\Big) \\ = \sum_{j\geq 2} t_{ij}(aY_j + \delta_j(a)X) + t_{i1}\phi(a)X, \end{aligned}$$

where we have maintained the previous nonstandard indexing.

Looking at the coefficients in the first of these expressions, we find that $\phi(a)t_{1j} = t_{1j}a$, for all a and for all $j \geq 2$. Since $\deg_F \psi(a) > 0$, we see that $t_{12} = \cdots = t_{1r} = 0$ and

$$t_{11}\phi(a) = \psi(a)t_{11},$$

i.e., t_{11} is either zero or an isogeny from ϕ to ψ.

Since we can choose $a \in A$ transcendental over $\mathbf{F}_q$, we see from the second type of expression above that, when $i \geq 2$, each

$$at_{ij} = t_{ij}a,$$

and thus $t_{ij} \in \bar{k}_\infty F^0$. From the coefficient of X we see that

$$at_{i1} + \partial_i(a)t_{11} = \sum_{j\geq 2} t_{ij}\delta_j(a) + t_{i1}\phi(a),$$

or

$$\partial_i(a)t_{11} = \delta_\phi^{(t_{i1})} + \sum_{j\geq 2} t_{ij}\delta_j(a).$$

2. Algebraic A-submodules of products of quasi-periodic A-modules. In this section we want to determine the connected, reduced A-stable subgroups H of $\mathbf{G}_a^{f_0} \times \mathcal{Q}_1^{f_1} \times \cdots \times \mathcal{Q}_s^{f_s}$ with given projection $\pi(H)$ in $\mathcal{D}_1^{f_1} \times \cdots \times \mathcal{D}_s^{f_s}$.

In particular, for a fixed basis of V in $\mathrm{Grass}_{f_0}\bar{k}_\infty \times \mathrm{Grass}_{f_1}M_1 \times \cdots \times \mathrm{Grass}_{f_s}M_s$ corresponding to $\pi(H)$, we establish a correspondence between

(1) subgroups of products of quasi-periodic A-modules with projection $\pi(H)$ and

(2) subspaces of the appropriate κ-fold product of the de Rham cohomology of the associated Drinfeld modules.

Characterization of algebraic A-submodules of products of quasi-periodic modules.

Quasi-periodic notation. *Let $G = \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_s$, where the $\mathcal{Q}_i$ are the quasi-periodic modules attached to the Drinfeld A-modules ϕ_i, and let $\mathcal{D}_i$ denote $\mathbf{G}_a$ with A-action given by ϕ_i, $i = 1, \ldots, s$. Let $\pi : G \to \mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_s$ denote the projection map. For each i, let M_i denote the field of quotients of the ring R_i of multiplications of ϕ_i and r_i the A-rank of R_i. For each i, let X_i designate the coordinate* *corresponding to elements of* $\mathcal{D}_i$*, and* $Y_{i2},\dots,Y_{ir_i}$ *the coordinates corresponding to the linearly independent strictly reduced* ϕ_i*-biderivations* $\delta_{i2},\dots,\delta_{ir_i}$*. For each subspace* V *of* $D(\phi_1)\times\cdots\times D(\phi_s)$*, we can use the Hodge decomposition to project onto the strictly reduced components and obtain a short exact sequence*

$$0\longrightarrow V_{in}\longrightarrow V\longrightarrow \pi_{sr}(V)\longrightarrow 0.$$

For any A*-closed reduced algebraic subgroup* K *of* $\mathcal{D}$*, let*

$$\boldsymbol{\delta}_K=\Big\{(\delta_{\phi_1}^{(T_1)},\dots,\delta_{\phi_s}^{(T_s)}):\sum T_iX_i \text{ vanishes on } K\Big\}$$

and

$$\boldsymbol{\delta}^K=\Big\{(\delta_1,\dots,\delta_s):\sum \delta_i(a)X_i \text{ vanishes on } K,\forall\, a\in A\Big\}.$$

Since the vectors of derivations lying in $\boldsymbol{\delta}_K$ arise from $\mathbf{F}_q$-linear relations on K, we say that $\boldsymbol{\delta}_K$ comprises the K-*based* biderivations. Since evaluating the biderivations of $\boldsymbol{\delta}^K$ gives relations on K, we say that $\boldsymbol{\delta}^K$ comprises the K-*consistent* biderivations.

Quasi-periodic classification Theorem 2.1. *Let* H *be a reduced, connected algebraic subgroup of* G *which is invariant under the action of* A *and which has codimension* $\kappa+\mu$*. Let the projection* $\pi(H)$ *have codimension* κ *in* $\mathcal{D}$*. Then there is a unique subspace* W *of* $D(\phi_1)\times\cdots\times D(\phi_s)$ *consisting of* s*-tuples* $\boldsymbol{\delta}=(\delta_1,\dots,\delta_s)$ *of biderivations* $\delta_i=\sum_{j=2}^{r_i}c_{ij}^{(\delta)}\delta_{ij}+\delta_{\phi_i}^{(T_i)}$*, such that*

(1) $W\subset\boldsymbol{\delta}^{\pi(H)}$,

(2) $W_{in}:=W\cap(D_{in}(\phi_1)\times\cdots\times D_{in}(\phi_s))=\boldsymbol{\delta}_{\pi(H)}$,

(3) $\dim\pi_{sr}(W)=\mu$*, and*

(4) $L_\delta(\mathbf{Y},\mathbf{X})=\sum_{ij}c_{ij}^{(\delta)}Y_{ij}+\sum T_iX_i$ *vanishes on* H*, for all* $\boldsymbol{\delta}\in W$*.*

Remark. Note that rank $(c_{ij}^{(\delta)})_{\delta,ij}=\dim\pi_{sr}(W)$.

Remark. Conversely, let K denote any connected, reduced A-closed algebraic subgroup of codimension κ in $\mathcal{D}$. Let $V\subset D(\phi_1)\times\cdots\times D(\phi_s)$

be K-consistent, with $V_{in} = \boldsymbol{\delta}_K$. Then since $L_\delta(\Phi(a)(Y,X)) = aL_\delta(Y,X) - \sum \delta_i(a)X_i$, the condition that $L_\delta(\mathbf{Y},\mathbf{X}) = 0$ for all $\boldsymbol{\delta} \in V$, defines an A-closed algebraic subgroup H in G of codimension equal to $\kappa + \operatorname{rank}(c_{ij}^\delta)_{\delta,ij} = \kappa + \dim \pi_{sr}(V)$ such that $\pi(H) = K$.

Therefore, we are led to consider the set $\mathcal{S}$ of pairs (K,W) where

(1) K is a reduced, connected A-closed algebraic subgroup of $\mathcal{D}_1 \times \cdots \times \mathcal{D}_s$, and

(2) W is a K-consistent subspace of $D(\phi_1) \times \cdots \times D(\phi_s)$ with $W_{in} = \boldsymbol{\delta}_K$.

Corollary 2.2. *There is a bijection between $\mathcal{S}$ and the set of reduced, connected A-closed algebraic subgroups H of G via*

$$H \longleftrightarrow (K,W)$$

if and only if

$$\pi(H) = K \quad \textit{and} \quad H = \textit{ Zeros of } (L_\delta(\mathbf{Y},\mathbf{X}))_{\delta \in W}.$$

In particular, for given K the unique maximal A-closed connected, reduced algebraic subgroup H of $\prod \mathcal{Q}_i^{f_i}$ with $\pi(H) = K$, corresponds to $W = \boldsymbol{\delta}_K$.

Remark. I do not know whether there is a unique minimal A-closed connected, reduced algebraic subgroup projecting onto K corresponding to $\boldsymbol{\delta}^K$. This occurs exactly when $(\boldsymbol{\delta}^K)_{in} = \boldsymbol{\delta}_K$.

Proof of Theorem. We define W to be the set of $\boldsymbol{\delta}$ for which conditions (1) and (4) of the theorem hold. By definition, then, whenever $\boldsymbol{\delta} \in W_{in}$, the corresponding $c_{ij}^{(\delta)} = 0$, so the $\mathbf{F}_q$-linear polynomial $\sum T_i X_i$ vanishes on H for every $a \in A$. Hence $\boldsymbol{\delta} \in \boldsymbol{\delta}_{\pi(H)}$, i.e., $W_{in} \subset \boldsymbol{\delta}_{\pi(H)}$. On the other hand, whenever $(T_1, \dots, T_s) \in \mathcal{T}^s$ such that $\sum T_i X_i$ vanishes on $\pi(H)$ and thus on H, then since H is A-stable, $\sum T_i \phi(a) X_i$ also vanishes on H. Clearly $a \sum T_i X_i$ also vanishes on H and therefore so does the difference

$$\boldsymbol{\delta}(a)\mathbf{X} = \sum \delta_{\phi_i}^{(T_i)}(a) X_i.$$

In this way we see that $\boldsymbol{\delta}_{\pi(H)} \subset W_{in}$ and establish equality (2) of the theorem.

The main part of the proof consists in showing that W is large enough. For that, we construct elements of W associated with minimal relations. We write

$$\begin{aligned}\mathbf{Y} &= (\mathbf{Y}_1; \dots ; \mathbf{Y}_s) = (Y_{12}, \dots , Y_{1r_1}; \dots , Y_{s2}, \dots , Y_{sr_s}),\\ \mathbf{X} &= (X_1, \dots , X_s).\end{aligned}$$

Among the algebraic relations

$$S(\mathbf{Y}, \mathbf{X}) = S(\mathbf{Y}_1, \dots , \mathbf{Y}_s, \mathbf{X}) = 0 \tag{9}$$

holding for the coordinates of elements of H, select one which is minimal in the following dual sense:

(1) it actually involves a nonempty minimal set of variables Y_{ij} and (with that set fixed) a minimal (possibly empty) set, of variables X_i, and, among all such relations,

(2) it is of minimal degree with respect to some lexicographic ordering in which the X_i come last.

Then

$$S(\mathbf{Y} + \mathbf{Y}', \mathbf{X} + \mathbf{X}') - S(\mathbf{Y}, \mathbf{X}) - S(\mathbf{Y}', \mathbf{X}') = 0 \tag{10}$$

for all $(\mathbf{Y}', \mathbf{X}'), (\mathbf{Y}, \mathbf{X}) \in H$, although this expression has lower maximal term in the lexicographical ordering with respect to either primed or unprimed variables than in the minimal relation (9). Consequently, as in the inductive proof of Artin's theorem on minimal algebraic relations for additive functions [**11**, Theorem 12.1], $S(\mathbf{Y}, \mathbf{X})$ is an additive polynomial, i.e., the relation (10) holds identically in the variables $(\mathbf{Y}', \mathbf{X}'), (\mathbf{Y}, \mathbf{X})$. This means that (9) can be rewritten as

$$\sum_{i=1}^{s} \sum_{j=2}^{r_i} T_{ij} Y_{ij} + \sum_{i=1}^{s} T_i X_i = 0 \tag{11}$$

with twisted polynomials T_{ij}, T_i. Since $\bar{k}_\infty$ is a perfect field,

$$\min \{\mathrm{subdeg}_F T_{ij},\ \mathrm{subdeg}_F T_i\} = 0,$$

else we could simply cancel a left factor of F out of this twisted relation, as our underlying ideal defining H is prime.

Since H is closed under the action of A, it is also true that for every $a \in A$, the relation

$$(12)\quad \sum_{i=1}^{s}\sum_{j=2}^{r_i} T_{ij}(aY_{ij}) + \sum_{i=1}^{s}\sum_{j=2}^{r_i} T_{ij}(\delta_{ij}(a)X_{ij}) + \sum_{i=1}^{s} T_i(\phi_i(a)X_i) = 0$$

holds for all $(\mathbf{X},\mathbf{Y}) \in H$.

If $\max \deg_F T_{ij} = d \geq 1$, then multiplying relation (11) by a^{q^d} gives another relation

$$(13)\qquad \sum_{i=1}^{s}\sum_{j=2}^{r_i} a^{q^d} T_{ij}Y_{ij} + \sum_{i=1}^{s} a^{q^d} T_i X_i = 0$$

holding on H. However, the difference between (12) and (13) is a relation of the form of (11), but which is smaller with respect to the lexicographical ordering on the monomials in the Y_{ij}. Thus this difference does not actually involve the Y_{ij} at all, and we have the following:

Intermediate conclusions. (1) $a^{q^d}T_{ij}Y_{ij} = T_{ij}(aY_{ij})$ for all $a \in A$, for all T_{ij}, and

(2) $\sum_i \{(\sum_j T_{ij}\delta_{ij}(a)) + (T_i\phi_i(a) - a^{q^d}T_i)\}X_i = 0$ for all $\mathbf{X} \in \pi(H)$.

Since $a \in A$ can be chosen transcendental over $\mathbf{F}_q$, conclusion (1) is equivalent to saying that

$$(1')\qquad T_{ij} = \tau_{ij}F^d, \qquad \tau_{ij} \in \bar{k}_\infty,$$

with the same d for all i, j.

We wish to show that $d = 0$. Suppose otherwise for the moment. Since $\deg_F \delta_{ij}(a) > 0$, by the minimality of degree in our choice of S and the remark following equation (11), we must have $\min \operatorname{subdeg}_F T_i = 0$. However, since we are assuming that $d \geq 1$, we may select $a \in A$ such that $a^{q^d} \neq a$ and set

$$\tilde{T}_i = T_i - (a - a^{q^d})^{(-1)}\Big\{ \sum_j T_{ij}\delta_{ij}(a) + T_i\phi_i(a) - a^{q^d}T_i \Big\}.$$

Then $\mathrm{subdeg}_F \tilde{T}_i > 0$, and since we have merely added a multiple of the polynomial occurring in conclusion (2), it is still true that for all $(\mathbf{Y}, \mathbf{X}) \in H$,

$$\sum_{i=1}^{s} \sum_{j=2}^{r_i} T_{ij} Y_{ij} + \sum_{i=1}^{s} \tilde{T}_i X_i = 0, \tag{14}$$

where the T_{ij} remain the same, but each $\mathrm{subdeg} \tilde{T}_i > 0$. Therefore, since $\bar{k}_\infty$ is perfect, we can factor out a power of F (on the left of equation (14)) to give a similar relation with $\max \deg_F T_{ij} = d - 1 \geq 0$. This contradicts the minimality of equation (11). So, in fact, $d = 0$, as desired.

Now, since $d = 0$, $T_{ij} = \tau_{ij} F^0$ and conclusion (2) can be restated for $\boldsymbol{\delta}_S = ((\sum_j \tau_{ij} \delta_{ij} + \delta_\phi^{(T_i)})_i)$, as

$$\boldsymbol{\delta}_S(a) \cdot \mathbf{X} = \sum_i \Big\{ \Big(\sum_j \tau_{ij} \delta_{ij}(a) \Big) + \delta_{\phi_i}^{(T_i)}(a) \Big\} X_i = 0, \tag{15}$$

for all $\mathbf{X} = (X_i) \in \pi(H)$ and all $a \in A$, i.e.,

$$\boldsymbol{\delta}_S \in \boldsymbol{\delta}^{\pi(H)}.$$

In addition, according to (11),

$$L_{\delta_S}(\mathbf{Y}, \mathbf{X}) = \sum_{ij} \tau_{ij} Y_{ij} + \sum_i T_i X_i \tag{16}$$

vanishes on H. Therefore, we see from (15) and (16) that the minimal relation S on the coordinates of H gives rise to a one-dimensional subspace of $D(\phi_1) \times \cdots \times D(\phi_s)$ satisfying properties (1) and (4) for $\boldsymbol{\delta}_S$.

It remains now to show that the dimension of the subspace of W generated by all such $\boldsymbol{\delta}_S$ has strictly reduced projection of dimension at least (and therefore exactly) μ. For that, select a maximal set B of coordinates Y_{ij} which are algebraically independent modulo the ideal of H. By definition, $\mu + \mathrm{Card}\,(B) = \sum_i (r_i - 1)$, and for each $Y = Y_{ij}$ in the complement B' of B, there is a minimal algebraic relation on the variables of $B \cup \{Y_{ij}\} \cup \{X_\ell\}_{\ell=1,\dots,s}$ involving Y_{ij}, but none actually involving a nonempty proper subset of $B \cup \{Y_{ij}\}$ and, possibly, some of the X_ℓ. When we write down the matrix of τ_{kl} occurring in the equations (15) for these relations indexed by $Y_{ij} \in B'$, we find a diagonal matrix with nonzero entries. Hence these relations define a variety of codimension equal to the cardinality of $B'(= \mu)$ in the variety defined by the equations of $\pi(H)$. In other words, these equations, together with those of $\pi(H)$, define a variety containing H of codimension $\kappa + \mu$, of which H is therefore the reduced, connected component of the identity.

Uniqueness of W follows, because if W' were another such subspace, then, by the definition of W, $W' \subset W$, and therefore also $\pi_{sr}(W') \subset \pi_{sr}(W)$. However, since $\dim\pi_{sr}(W') = \dim\pi_{sr}(W)$, $\pi_{sr}(W') = \pi_{sr}(W)$. Consequently, since by (2) $W'_{in} = \boldsymbol{\delta}_K = W_{in}$, the five lemma shows that $W' = W$, as desired. □

3. Analytic characterization of algebraic A-submodules. The above identification of the A-closed submodules of products of quasi-periodic A-modules, while complete, is not as satisfactory as the classification theorem for A-submodules of products of Drinfeld modules, because the latter establishes a sort of analytic Lie correspondence for products of Drinfeld modules. Investigating such a relationship for quasi-periodic modules will be our goal in the next section. Since the somewhat stronger hypothesis that no distinct but isogenous factors occur in the product appears in the former case and since we build on that result, we will have to invoke it here as well.

The case of a power. We first consider the special case that $H < G = \mathcal{D}^f$, since all the main features appear here without so many subscripts. We denote by W_H the subspace of Theorem 2.1 corresponding to H.

Lemma 3.1. *Let* $\kappa = \operatorname{codim}_{\mathcal{D}^f}\pi(H)$. *Then* $\dim\pi_{sr}(W_H) \leq \kappa(r-1)$.

Proof. Let us reindex if necessary so that $X_{\kappa+1}, \dots, X_f$ are algebraically independent modulo the prime ideal I of polynomials vanishing on $\pi(H)$, whereas each of $X_1, \dots, X_\kappa$ is algebraic modulo $(I, X_{\kappa+1}, \dots, X_f)$. Now if $\dim\pi_{sr}(W_H) > \kappa(r-1)$, then projection of W_H on the $\kappa(r-1)$-dimensional space formed by the first κ factors in $D_{sr}(\phi)^f$ has a nontrivial kernel. Therefore, there must be a $\boldsymbol{\delta}' \in W_H$, $\boldsymbol{\delta}' \notin (W_H)_{in}$ of the form

$$\boldsymbol{\delta}' = (\delta_\phi^{(T_1)}, \dots, \delta_\phi^{(T_\kappa)}, \delta_{\kappa+1}, \dots, \delta_f).$$

But, according to Corollary 2.2 of Theorem 2.1, the L_δ with $\boldsymbol{\delta} \in \boldsymbol{\delta}_{\pi(H)} + \bar{k}_\infty \boldsymbol{\delta}'$ define a proper algebraic A-closed subgroup of G, for which the projection onto the final $f-\kappa$ components of $\mathcal{Q}^f$ is an A-closed algebraic subgroup of dimension less than $(f-\kappa)r$, but with surjective projection to $\mathcal{D}_{\kappa+1} \times \cdots \times \mathcal{D}_f$. (For each choice of coordinates $X_{\kappa+1}, \dots, X_f$, the relations on $\pi(H)$ allow only finitely many choices of coordinates $X_1, \dots, X_\kappa$ such that $(X_1, \dots, X_f) \in \pi(H)$. Thus if, say,

$$\boldsymbol{\delta}' = \left(\sum_{j=2}^{r} c_{ij}\delta_j + \delta_\phi^{(T_i)} \right)_{i=1}^{f} \in W \subset \boldsymbol{\delta}^{\pi(H)}$$

with, say, each $c_{ij} = 0$, $i \le \kappa$, but $c_{\kappa+1,2} \neq 0$, then for each choice of X_j, $j > \kappa$ and $Y_{\kappa+1,3}, \dots, Y_{\kappa+1,r}$, and Y_{jl}, $j > \kappa$, $l = 2, \dots, r$, there are only finitely many $Y_{\kappa+1,2}$ such that (Y_{jl}, X_j) lies in the projection of H into the last $f-\kappa$ components $\mathcal{Q}^{f-\kappa} \subset \mathcal{Q}^f$. Therefore this projection of H is not surjective.)

However, according to Corollary 2.2 above, such a subgroup cannot exist. Therefore, the upper bound on the dimension of A-closed algebraic subgroups of $\mathcal{Q}_1 \times \cdots \times \mathcal{Q}_f$ with given projection $\pi(H)$ must hold as claimed. □

Our next main task will be to construct families of A-closed connected, reduced subgroups H with given projection $K < \mathcal{D}^f$. We start by finding a minimal A-closed algebraic subgroup H_K of G with $\pi(H_K) = K$. It corresponds to the unique pair $(K, W) \in S$ with W maximal.

Since $\phi(R)$ is commutative, the map

$$\phi(R)^f \times D(\phi) \longrightarrow D(\phi)^f,$$

defined by

$$(\mathbf{m}, \partial) \longmapsto ((m_1^*\partial), \dots, (m_f^*\partial)),$$

where $m_j^*\partial(a) := \partial(a)m_j$, for all $a \in A$, is $\phi(R)$-bilinear and balanced. Thus, there is a natural induced homomorphism

$$\begin{aligned}\mu : \phi(R)^f \otimes_{\phi(R)} D(\phi) &\longrightarrow D(\phi)^f \\ (m_i) \otimes \partial &\longmapsto (m_i * \partial).\end{aligned}$$

Let $\mathbf{m} * \partial$ denote the image of $\mathbf{m} \otimes \partial$ under this map.

Lemma 3.2. *Let* $V_K \in \mathrm{Grass}_f(M)$ *correspond to* K*, and set* $V_K(R) := V_K \cap R^f$*.*

(1) $\phi(V_K) * D(\phi) \subset \boldsymbol{\delta}^K$.

(2) $(\mu^{-1}D_{in}(\phi)^f) \cap (\phi(V_K(R)) \otimes D(\phi)) = \{\varepsilon \in \phi(V_K(R)) \otimes D(\phi)$: *there exists a* $\rho \in R$*,* $\rho \neq 0$*, with* $\phi(\rho)\varepsilon \in \phi(V_K(R)) \otimes D_{in}(\phi)\}$.

Proof. (1) For $\sum \mathbf{m}_h \otimes \partial_h \in \phi(V_K(R)) \otimes D(\phi)$, any $a \in A$, and any $\mathbf{k} \in K$,

$$\Big(\sum_h \mathbf{m}_h * \partial_h(a)\Big)(\mathbf{k}) = \sum_h \partial_h(a)[\mathbf{m}_h(\mathbf{k})] = 0,$$

since each $\mathbf{m}_h \in \phi(V_K(R))$, and thus $\mathbf{m}_h(\mathbf{k}) = 0$. This establishes (1).

(2) (Containment $\supset$). If $\phi(\rho)\varepsilon \in \phi(V_K(R)) \otimes D_{in}(\phi)$, then by the proof of Proposition 1.3, $\phi(\rho) * \mu(\varepsilon) = \mu(\phi(\rho)\varepsilon) \in D_{in}(\phi)^f$. By Proposition 1.3 applied componentwise, we see that $\phi(\rho) * \mu(\varepsilon) \in D_{in}(\phi)^f \Rightarrow \mu(\varepsilon) \in D_{in}(\phi)^f$, as desired.

(Containment $\subset$). Now fix an M-basis $\mathbf{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_\kappa\} \subset R^f$ of V_K which has a diagonal initial block, say. Note that for any $\mathbf{m} = (m_1, \dots, m_f) \in \phi(V_K(R))$, there are elements $\rho, \rho_1, \dots, \rho_\kappa \in R$, $\rho \neq 0$, such that

$$\phi(\rho)\mathbf{m} = \sum_{j=1}^{\kappa} \phi(\rho_j)\phi(\mathbf{b}_j).$$

Now say that the image of $\varepsilon := \sum \mathbf{m}_h \otimes \partial_h$ is inner, i.e., that each component of $\sum \mathbf{m}_h * \partial_h$ is inner, say

$$\sum \partial_h(a)(\mathbf{m}_h) = (\delta_\phi^{(S_1)}(a), \dots, \delta_\phi^{(S_f)}(a)), \qquad \forall\, a \in A.$$

Then, since $\phi(\rho)$ commutes with each $\phi(a)$, $a \in A$, we have

$$\Big(\sum \mathbf{m}_h^* \partial_h\Big)(a)\phi(\rho) = (\delta_\phi^{(S_1\phi(\rho))}(a), \dots, \delta_\phi^{(S_f\phi(\rho))}(a)). \tag{17}$$

But, when we express each $\mathbf{m}_h$ via $\phi(\rho)\mathbf{m}_h = \sum \phi(\rho_{hj})\phi(\mathbf{b}_j)$, the lefthand side of (17) can be rewritten as

$$\sum_j \Big(\sum_h \partial_h(a)\phi(\rho_{hj})\Big)\phi(\mathbf{b}_j).$$

Now, by assumption, for $j = 1, \ldots, \kappa$, $\mathbf{b}_j$ has a nonzero entry λ_j among the first κ entries only in the jth position. In particular, then, for $j = 1, \ldots, \kappa$, our hypothesis implies that the map

$$a \longmapsto \sum_h \partial_h(a)\phi(\rho_{hj})\phi(\lambda_j)$$

is an inner biderivation. By the automorphism of H^*_{DR} induced by $\phi(\lambda_j)^*$, (Proposition 1.3) this implies that for $j = 1, \ldots, \kappa$,

$$\sum_h \phi(\rho_{hj}) * \partial_h \in D_{in}(\phi).$$

Then, however,

$$\begin{aligned} \phi(\rho)\varepsilon &= \phi(\rho)\Big(\sum_h \mathbf{m}_h\Big) \otimes \partial_h \\ &= \sum_h \sum_j \phi(\rho_{hj})\phi(\mathbf{b}_j) \otimes \partial_h \\ 0 &= \sum_j \phi(\mathbf{b}_j) \otimes \Big(\sum_h \phi(\rho_{hj}) * \partial_h\Big) \in \phi(V_K(R)) \otimes D_{in}(\phi), \end{aligned}$$

as desired to establish equality (2). $\square$

Proposition 3.3. *Let K be a connected A-closed reduced algebraic subgroup K of $\mathcal{D}^f$ of codimension κ with corresponding subspace $V_K \subset M^f$. Let $\mathbf{B} \subset R^f$ be an M-basis for V_K with a diagonal $\kappa \times \kappa$ block. For each $\mathbf{b}_i \in \mathbf{B}$, let $\boldsymbol{\beta}_i = \phi_i(\mathbf{b}_i)$. Then*

$$W_K = \sum_i \boldsymbol{\beta}_i * D_{sr}(\phi) + \boldsymbol{\delta}_K$$

is a maximal subspace of $D(\phi)^f$ satisfying properties (1) *and* (2) *of the quasi-periodic classification Theorem* 2.1. *It corresponds to a minimal*

A-closed reduced algebraic subgroup H of G with $\pi(H) = K$ and $\operatorname{codim} H = \kappa r$.

Proof. We first show that W_K satisfies the following two properties of Theorem 2.1.

Property 1. Now $\boldsymbol{\delta}_K \subset \boldsymbol{\delta}^K$ by definition. We show that each $\boldsymbol{\beta} * D_{sr}(\phi) \subset \boldsymbol{\delta}^K$: Note that for $\boldsymbol{\beta} = (\beta_1, \dots, \beta_f) \in \phi(V_K(R))$ the additive polynomial $\boldsymbol{\beta}\mathbf{X} := \sum \beta_i X_i$ vanishes on K. Hence, for $\partial \in D(\phi)$, $a \in A$,

$$\partial(a) \sum \beta_i X_i = \boldsymbol{\beta} * \partial(a)\mathbf{X} = \partial(a) \cdot \boldsymbol{\beta}\mathbf{X} = 0$$

on K. Thus, $W_K \subset \sum \boldsymbol{\beta}_i D(\phi) + \boldsymbol{\delta}_K \subset \boldsymbol{\delta}^K$, as desired.

Property 2. By definition, $(W_K)_{in} \supset \boldsymbol{\delta}_K$. Now let $\boldsymbol{\delta} \in (W_K)_{in}$, say $\boldsymbol{\delta} = \boldsymbol{\delta}_0 + \boldsymbol{\delta}_1$, $\boldsymbol{\delta}_0 \in \boldsymbol{\delta}_K$, $\boldsymbol{\delta}_1 \in (\sum_i \boldsymbol{\beta} *_i D_{sr}(\phi))_{in}$.

Let us assume for simplicity of notation that the diagonal block of the hypothesis occurs in the first columns: $\boldsymbol{\beta}_i = (\beta_{ij})$ with β_{ij}, $1 \le i, j \le \kappa$, nonzero exactly when $i = j$. Then if $\sum_i c_i \boldsymbol{\beta}_i^* \partial_i = (\delta_\phi^{(S_j)}) \in D(\phi)_{in}^f$, we find that $c_i \beta_{ii}^* \partial_i = (\delta_\phi^{(S_i)})$, $i = 1, \dots, \kappa$. However, we know by Proposition 1.3 that each β_{ii}^* induces an automorphism of $H_{DR}^*(\phi)$. So in each case, $c_i \delta_i \in \mathcal{D}_{in}(\phi)$. But, by hypothesis, each $\partial_i \in D_{sr}(\phi)$,and thus $c_i \partial_i = 0$.

Consequently, $(W_K)_{in} \subset \boldsymbol{\delta}_K$. Together with the opposite inclusion noted at the beginning of this case, we find $(W_K)_{in} = \boldsymbol{\delta}_K$, as desired.

W_K maximal. To show that W_K is maximal, we show that it corresponds to a minimal H. The key to this is to note that the argument of case (2) above shows that the map sending the κ-tuple $(\partial_1, \dots, \partial_\kappa)$ of strictly reduced biderivations to the first κ strictly reduced components of $\mathbf{b}_1^* \partial_1 + \cdots + \mathbf{b}_\kappa^* \partial_\kappa$ is injective. Thus the dimension of the image $\operatorname{mod} \boldsymbol{\delta}_K$ is $\kappa(r-1)$, and the codimension of the corresponding subgroup in G is at least $\kappa(r-1) + \kappa = \kappa r$.

On the other hand, let W' denote the vector space of biderivations corresponding via Theorem 2.1 to a minimal A-closed subgroup H

contained in the one corresponding to W_K. Then $\pi_{sr}(W_K) \subset \pi_{sr}(W')$ and so $\dim\pi_{sr}(W_K) \leq \kappa(r-1)$ according to Lemma 3.1. Combining this with the opposite inequality shows that $\dim\pi_{sr}(W) = \kappa(r-1)$ as claimed. Thus, the A-closed algebraic subgroup defined by the various L_δ, for $\boldsymbol{\delta} \in W_K$ has

$$\text{codim} = \kappa r.$$

By the argument used to show unicity in the quasi-periodic classification theorem, $W_K = W'$ and W_K corresponds to H. □

Since W_K is a maximal subspace of $D(\phi)^f$ corresponding to a reduced A-closed algebraic subgroup projecting onto K, all the remaining A-closed algebraic subgroups containing H and projecting onto K correspond to subspaces of W_K.

Proposition 3.4. *Fix a basis $\mathbf{b}_1, \dots, \mathbf{b}_\kappa \subset V_K(R)$ determining the reduced A-closed algebraic subgroup K of $\mathcal{D}^f$ of codimension κ and such that the initial $\kappa \times \kappa$ submatrix of entries of the $\mathbf{b}_i$ is diagonal. Then there are bijections between the three sets*

(1) *the $\bar{k}_\infty$-subspaces Δ of $D_{sr}(\phi)^\kappa$,*

(2) *the $\bar{k}_\infty$-subspaces W of W_K containing $\boldsymbol{\delta}_K$, and*

(3) *the connected reduced A-closed algebraic subgroups H' of G containing $H =$ zeros (L_{W_K}) with $\pi(H') = K$. The correspondence is the following*:

$$(\delta_1, \dots, \delta_\kappa) \in \Delta \longleftrightarrow \boldsymbol{\delta} = \sum_{i=1}^{\kappa} \phi(\mathbf{b}_i)^* \delta_i \in W \longleftrightarrow L_\delta \text{ vanishes on } H.$$

Proof. The equivalence 2 $\Leftrightarrow$ 3 follows from the quasi-periodic classification Theorem 2.1 and the fact that W_K corresponds to the minimal reduced A-closed algebraic subgroup H projecting onto K.

The equivalence 1 $\Leftrightarrow$ 2 follows from the fact established in Property 2 and the proof of maximality of W_K in Proposition 3.3 that

$$(\partial_1, \dots, \partial_\kappa) \longmapsto \sum \phi(\mathbf{b}_i)^* \partial_i$$

induces an isomorphism $D_{sr}(\phi)^\kappa \to W_K/\boldsymbol{\delta}_\kappa$. $\square$

General case. The general case of power products corresponding to nonisogenous A-Drinfeld modules necessitates also a minor adjustment of notation, to group together copies of the same $\mathcal{Q}_i$.

Common hypothesis. *From now on,*

$$G = \mathcal{Q}_1^{f_1} \times \cdots \times \mathcal{Q}_s^{f_s},$$

where the underlying A-Drinfeld modules $\mathcal{D}_i$ are nonisogenous for distinct indices i. The notation H, R_i, M_i, r_i otherwise retains its meaning, while

$$\begin{aligned} \mathbf{X} &= (X_{11}, \dots, X_{1f_1}; \dots; X_{s1}, \dots, X_{sf_s}), \\ \mathbf{Y} &= (\mathbf{Y}_{11}, \dots, \mathbf{Y}_{1f_1}; \dots; \mathbf{Y}_{s1}, \dots, \mathbf{Y}_{sf_s}), \\ \mathbf{Y}_{ij} &= (Y_{ij2}, \dots, Y_{ijr_i}), \qquad i = 1, \dots, s, \ j = 1, \dots, f_i, \end{aligned}$$

and $\pi_i : G \to \mathcal{D}^{f_i}$ denotes the projection. Let $\kappa_i = \mathrm{codim}_{\mathcal{D}_i^{f_i}} \pi_i(H)$.

Then generalizing Proposition 3.3, we find the following result:

Proposition 3.5. *For $i = 1, \dots, s$, let K_i be an A-closed reduced algebraic subgroup of $\mathcal{D}^{f_i}$ of codimension κ_i with corresponding $V_{K_i} \in M_i^{f_i}$. Let $\mathbf{B}_i \subset R_i^{f_i}$ be an M_i-basis for V_{K_i} with a diagonal $\kappa_i \times \kappa_i$ block. For each $\mathbf{b}_{ij} \in \mathbf{B}_i$, let $\boldsymbol{\beta}_{ij} := \phi_i(\mathbf{b}_{ij})$. Then, considering each $D(\phi_i)^{f_i}$ as a subspace of $\Pi D(\phi_j)^{f_j}$,*

$$W_K = \prod_i \Big(\sum_j \boldsymbol{\beta}_{ij}^* D_{sr}(\phi_i) + \boldsymbol{\delta}_{K_i} \Big) \tag{18}$$

is a maximal subspace of $\Pi D(\phi_i)^{f_i}$ satisfying properties (1) *and* (2) *of the quasi-periodic classification theorem. Then W_K corresponds to a minimal A-closed reduced algebraic subgroup H of G with $\pi(H) = K = \Pi K_i$. In particular,* $\mathrm{codim} H = \sum \kappa_i r_i$.

Proof. By Theorem 1.1, the definition of $\boldsymbol{\delta}_K$ and the assumption that the ϕ_i are nonisogenous, we se that $\boldsymbol{\delta}_K = \prod \boldsymbol{\delta}_{K_i}$ and $W_K = \prod W_{K_i}$.

Therefore, the verification of properties (1) and (2) reduce to the verification for the block of variables corresponding to each $\mathcal{Q}_i^{f_i}$, which was done in Proposition 3.3. As the product in (18) is direct, it is maximal by Proposition 3.3 and Theorem 1.1. $\square$

As before, this allows us to classify all the reduced A-closed subgroups of G projecting onto a fixed K.

Theorem 3.6. *Fix a basis $\mathbf{b}_{i1}, \dots, \mathbf{b}_{i\kappa_i} \subset V_{K_i}(R_i)$ determining the reduced A-closed algebraic subgroup K_i of $\mathcal{D}_i^f$ and such that the matrix of entries of the $\mathbf{b}_{ij}$ contains a $\kappa_i \times \kappa_i$ diagonal submatrix. Then there is bijection between*

(1) *the products $\Pi\Delta_i$ of $\bar{k}_\infty$-subspaces Δ_i of $D_{sr}(\phi_i)^{\kappa_i}$, $i = 1, \dots, s$,*

(2) *the $\bar{k}_\infty$-subspaces W of W_K containing $\Pi\boldsymbol{\delta}_{K_i}$, and*

(3) *the connected reduced A-closed algebraic subgroups $H' = H_1' \times \cdots \times H_s'$ of G with $\pi(H') = K$ containing the minimal A-closed subgroup corresponding to W_K.*

The correspondence is the following:

$$(\delta_{i1}, \dots, \delta_{i\kappa_i}) \in \Delta_i, \forall\, i \longleftrightarrow \boldsymbol{\delta} = \sum_{ij} \phi_i(\mathbf{b}_{ij})^* \delta_{ij} \in W \longleftrightarrow L_\delta \text{ vanishes on } H.$$

Proof. As remarked in the proof of Proposition 3.5, the members of the three sets are all parallel direct products. Therefore it suffices to establish the correspondences in each factor, which was done in Proposition 3.4. $\square$

Coda 3.7. *Using the notation of Proposition 3.5, let $K_i = \{\mathbf{X}_i : \mathbf{b}_{ij}\mathbf{X}_i = 0\}$, $i = 1, \dots, s$. There is a bijection between products ΠS_i of $\bar{k}_\infty$-subspaces S_i of $\mathbf{B}_i \otimes D_{sr}(\phi_i)$ and the connected, reduced A-closed algebraic subgroups H of $G = \mathcal{Q}_1^{f_1} \times \cdots \times \mathcal{Q}_s^{f_s}$ containing ΠH_i, where H_i denotes the zeros of $L_\delta, \delta \in \boldsymbol{\delta}_{K_i}$ with*

$$\pi(H) = \bigoplus K_i.$$

Proof. According to Proposition 3.3 and the proof of Lemma 3.2, the image of
$$(\phi(R)\mathbf{b}_1 \oplus \cdots \oplus \phi(R)\mathbf{b}_\kappa) \otimes D_{sr}(\phi)$$
in $\pi_{sr}(W_K)$ has dimension equal to $(r-1)\kappa$, the dimension of $\pi_{sr}(W_K)$ itself.

According to the "initially diagonal" choice of our $\mathbf{b}_j$, the first κ projections of $\phi(R)\mathbf{b}_j \otimes D_{sr}(\phi)$ are

$$\pi_{j'}(\mathbf{b}_j^* D_{sr}(\phi)) = \begin{cases} D_{sr}(\phi) & j = j' \\ 0 & j \neq j'. \end{cases}$$

Thus $\dim_{\bar{k}_\infty}(\mathbf{b}_j^* D_{sr}(\phi)) \geq (r-1)\kappa$. Since we have already established the opposite inequality and that $\pi_{sr} : \mathbf{b}_j^* D_{sr}(\phi) \to (W_K)_{sr}$ is injective, we see that
$$D_{sr}(\phi) \simeq (W_K)_{sr} \otimes \sum \mathbf{b}_j.$$
Now the result follows from Theorem 3.6. $\square$

Using Theorem 3.6, this result extends immediately to arbitrary products of quasi-periodic A-modules: $\mathcal{Q} = \mathbf{G}_a^e \times \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_s$, where A acts by multiplication on the first e factors and by the extended Drinfeld actions Φ_i on the quasi-periodic modules $\mathcal{Q}_i$ associated with the Drinfeld modules ϕ_i, $i = 1, \ldots, s$.

REFERENCES

1. G.W. Anderson, *t-Motives*, Duke Math. J. **53** (1986), 457–502.

2. W.D. Brownawell and K.K. Kubota, *The algebraic independence of Weierstrass functions and some related numbers*, Acta Arith. **33** (1977), 111–149.

3. W.D. Brownawell, *Drinfeld exponential and quasi-periodic functions*, in *Advances in number theory* (F.Q. Gouvêa and N. Yui, eds.), Oxford University Press, Oxford, 1993.

4. R.F. Coleman, *On a stronger version of the Schanuel-Ax theorem*, Amer. J. Math. **102** (1980), 595–624.

5. P. Deligne and D. Husemöller, *Survey of Drinfeld modules*, Contemp. Math. **67** (1987), 25–91.

6. V.G. Drinfeld, *Elliptic modules*, Mat. Sbornik **94** (1974), 594–627, English translation Math. USSR Sbornik **23** (1974), 561–592.

7. E.-U. Gekeler, *On the de Rham isomorphism for Drinfeld modules*, J. Reine Angew Math. **401** (1989), 188–208.

8. D. Goss, *Drinfeld modules: Cohomology and special functions*, in *Motives*, Proc. Symp. Pure Math. **55** (1994), 309–362.

9. D.R. Hayes, *A brief introduction to Drinfeld modules*, in *The arithmetic of function fields* (D. Goss, D.R. Hayes, M. Rosen, eds.), Walter De Gruyter, New York, 1992.

10. E. Kolchin, *Algebraic groups and algebraic independence*, Amer. J. Math. **90** (1968), 1151–1164.

11. S. Lang, *Algebra*, Third edition, Addison-Wesley, Reading, MA, 1993.

12. A. Thiery, *Indépendance algébrique des périodes et quasi-périodes d'un module de Drinfeld*, in *The arithmetic of function fields* (D. Goss, D.R. Hayes and M. Rosen, eds.), Walter De Gruyter, New York, 1992.

13. M. Waldschmidt, *Transcendence problems connected with Drinfeld modules*, Istanbul Univ. Fen Fak. **49** (1993), 57–75.

14. J. Yu, *Transcendence and Drinfeld modules: Several variables*, Duke Math. J. **58** (1989), 559–575.

15. ———, *On periods and quasi-periods of Drinfeld modules*, Compositio Math. **74** (1990), 235–245.

16. ———, *Transcendence and special zeta values in characteristic p*, Annals of Math. **134** (1991), 1–23.

17. ———, *Analytic homomorphisms into Drinfeld modules*, preprint, 1991.

18. ———, *Transcendence in finite characteristic*, in *The arithmetic of function fields* (D. Goss, D.R. Hayes and M. Rosen, eds.), Walter De Gruyter, New York, 1992.

Penn State University, University Park, PA 16802
E-mail address: wdb@math.psu.edu