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VARIATIONAL METHODS AND SPECTRAL ASYMPTOTICS OF TWO PARAMETER ELLIPTIC EIGENVALUE PROBLEMS IN A BALL

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ABSTRACT. We consider the following nonlinear two-parameter elliptic eigenvalue problem

$$\begin{cases} \Delta u + \mu u^p = \lambda u & \text{in } B = \{x \in R^N : |x| < 1\}, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $N \geq 2$, $p > 1$ and $\mu, \lambda > 0$ are eigenvalue parameters. We apply two different kind of variational methods to this problem and define the variational eigenvalues $\lambda = \lambda(\mu)$ and $\mu = \mu(\lambda)$. Then we shall establish the asymptotic formulas of $\lambda(\mu)$ and $\mu(\lambda)$ as $\mu \to \infty$ and $\lambda \to \infty$, respectively, and the close relationship between the two asymptotic formulas are confirmed.

1. Introduction. We consider the following nonlinear two-parameter elliptic eigenvalue problems in a ball:

$$(1.1) \qquad \begin{cases} \Delta u + \mu u^p = \lambda u & \text{in } B = \{x \in R^N : |x| < 1\}, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $N \geq 2$, $p > 1$ and $\mu, \lambda \in R$ are eigenvalue parameters.

Since all positive solutions of (1.1) are radially symmetric (cf. Gidas, Ni and Nirenberg [**3**]), we consider the ordinary differential equation

$$(1.2) \qquad \begin{cases} u''(r) + \frac{N-1}{r} u'(r) + \mu u^p = \lambda u, & 0 < r < 1, \\ u(r) > 0 & 0 \leq r < 1, \\ u'(0) = 0, \qquad u(1) = 0, \end{cases}$$

where $r = |x|$.

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In order to describe and motivate the results of this paper, let us briefly recall some of the known facts concerning linear and nonlinear two-parameter problems.

There are many works concerning linear two-parameter problems. Especially, Binding and Browne [**2**] considered the following equation

$$u''(x) + \mu g_1(x)u(x) = \lambda g_2(x)u(x), \quad x \in I = (0,1). \tag{1.3}$$

Under the suitable boundary conditions and conditions on g_1, g_2, they established the following asymptotic formula:

$$\lim_{\mu\to\infty} \frac{\lambda_n(\mu)}{\mu} = \operatorname{ess\,sup}_{x\in I} \frac{g_2(x)}{g_1(x)}. \tag{1.4}$$

Here $\lambda_n(\mu)$ is the n-th eigenvalue for given $\mu > 0$. Motivated by these results, Shibata [**7**] considered the following nonlinear two-parameter Sturm-Liouville problem

$$\begin{cases} u''(x) + \mu u(x) = \lambda(1 + |u(x)|^{p-1})u(x) & x \in I, \\ u(0) = u(1) = 0. \end{cases} \tag{1.5}$$

By using the variational method on general level sets due to Zeidler [**10**], the following asymptotic formula was established:

$$\lim_{\mu\to\infty} \frac{\lambda_n(\mu,\alpha)}{\mu} = 1, \tag{1.6}$$

where $\lambda_n(\mu,\alpha)$ is the n-th variational eigenvalue and $\alpha > 0$ is a normalizing parameter of general level sets. In [**7**], the homogeneity of the lefthand side of the equation (1.5) played an important role. We note here that this property does not hold for (1.2).

In this paper we apply two different kinds of variational methods to (1.2). More precisely, as an eigenvalue problem, there are two ways of dealing with the problem (1.2). Firstly, for a given $\mu > 0$, we define $\lambda = \lambda(\mu)$ as a Lagrange multiplier by using the variational method introduced by Zeidler [**10**] on a general level set. Secondly, for a given $\lambda > 0$, $\mu = \mu(\lambda)$ is defined as a Lagrange multiplier by using the standard variational method.

The main object of this paper is to establish asymptotic formulas of $\lambda = \lambda(\mu)$ and $\mu = \mu(\lambda)$ as $\mu \to \infty$ and $\lambda \to \infty$, respectively, and to show the close relationship between the two formulas, that is, the same type of asymptotic formulas for $\lambda(\mu)$ and $\mu(\lambda)$ will be confirmed.

We use the following notations. Let $X := W_0^{1,2}(B)$ be the usual Sobolev space. Let

$$\|u\|_X^2 := \int_B |\nabla u|^2\, dx, \qquad \|u\|_q^q := \int_B |u|^q\, dx, \quad (q = 2,\ p+1).$$

We denote by $N_{\mu,\alpha}$ the general level set:

$$N_{\mu,\alpha} := \left\{ u \in X : H(u) := \frac{1}{2}\|u\|_X^2 - \frac{\mu}{p+1}\|u\|_{p+1}^{p+1} = -\alpha \right\}, \tag{1.7}$$

where $\alpha > 0$ is a fixed constant.

We shall define the variational eigenvalue $\lambda = \lambda(\mu)$ for given $\mu > 0$. We call $\lambda = \lambda(\mu)$ the variational eigenvalue if the associated eigenfunction $u = u_\mu(x) \in N_{\mu,\alpha}$ satisfies the following conditions (1.8)–(1.9):

$$(\mu, \lambda(\mu), u_\mu) \in R_+ \times R \times N_{\mu,\alpha} \quad \text{satisfies (1.1)}, \tag{1.8}$$

$$\Psi(u_\mu) = \inf_{u \in N_{\mu,\alpha}} \Psi(u), \tag{1.9}$$

where $R_+ = (0, \infty)$ and $\Psi(u) = (1/2)\|u\|_2^2$.

Next we shall define the variational eigenvalue $\mu = \mu(\lambda)$ for fixed $\lambda > 0$. Let

$$\begin{aligned} M_{\lambda,\beta} :&= \left\{ u \in X : g(u) := \frac{1}{p+1}\|u\|_{p+1}^{p+1} \right. \\ &\left. = \frac{1}{p+1}\beta_\lambda := \frac{1}{p+1}\beta\lambda^{-1/\gamma} \right\}, \end{aligned} \tag{1.10}$$

where $\beta > 0$ is a constant and $\gamma = 4/(N+2-p(N-2))$. We call $\mu = \mu(\lambda)$ the variational eigenvalue if the associated eigenfunction $u = u_\lambda(x)$ satisfies the following conditions (1.11)–(1.12):

$$(\mu(\lambda), \lambda, u_\lambda(x)) \in R \times R_+ \times M_{\lambda,\beta} \quad \text{satisfies (1.1)}, \tag{1.11}$$

$$\Phi_\lambda(u_\lambda) = \inf_{u \in M_{\lambda,\beta}} \Phi_\lambda(u), \tag{1.12}$$

where

$$\Phi_\lambda(u) = \frac{1}{2}(\|u\|_X^2 + \lambda\|u\|_2^2).$$

Now we state our main results.

Theorem 1.1. *Let $1 < p < 1 + 4/N$. Then there exists the variational eigenvalue $\lambda(\mu)$ for $\mu > 0$. Furthermore, the following asymptotic formula holds as $\mu \to \infty$:*

$$\lambda(\mu) = \left(\frac{N+2-p(N-2)}{N+4-Np}\frac{2\alpha}{\|w_\infty\|_{L^2(R^N)}^2}\right)^{\gamma(p-1)/2}\mu^\gamma + o(\mu^\gamma), \tag{1.13}$$

where w_∞ is the ground state of scalar field equation

$$\begin{cases} \Delta w + w^p - w = 0 & \text{in } R^N, \\ w > 0 & \text{in } R^N, \\ \lim_{|x|\to\infty} w(x) = 0. \end{cases} \tag{1.14}$$

Theorem 1.2. *Let $1 < p < (N+2)/(N-2)$. Then there exists the variational eigenvalue $\mu(\lambda)$ for $\lambda > 0$. Furthermore, the following asymptotic formula holds as $\lambda \to \infty$:*

$$\begin{aligned}\mu(\lambda) = &\left(\frac{2(p+1)}{((N+2)-p(N-2))\beta}\|w_\infty\|_{L^2(R^2)}^2\right)^{(p-1)/(p+1)} \\ &\cdot \lambda^{1/\gamma} + o(\lambda^{1/\gamma}).\end{aligned} \tag{1.15}$$

We note here that the formulas (1.13) and (1.15) are closely related, namely, we can choose $\beta > 0$ so that the coefficients of the top term of (1.13) and (1.15) are the same.

The remainder of this paper is organized as follows. In Section 2 we study the existence of $\lambda(\mu)$ and the positive radially symmetric eigenfunction associated with $\lambda(\mu)$. In Section 3 we prepare some fundamental lemmas for the proof of Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.1. Finally we shall prove Theorem 1.2 in Section 5.

2. Existence of $\lambda(\mu)$ and u_μ. Hereafter, C denotes various positive constants independents of μ. We shall show the existence of $\lambda(\mu)$ and the associated positive eigenfunction of $u_\mu(x)$. To this end, we shall apply the existence result of Zeidler [**10**, Proposition 6a]. We have only to check the following condition

$$L_{\mu,A} := \{u \in N_{\mu,\alpha} : (1/2)\|u\|_2^2 < A\} \subset X \tag{2.1}$$

is bounded for all constants $A > 0$ and fixed $\mu > 0$. All the other conditions assumed in [**10**, Proposition 6a] are easily checked. For $u \in L_{\mu,A}$, by using interpolation inequality

$$\|u\|_{p+1}^{p+1} \le C_1\|u\|_2^d\|u\|_X^{p+1-d}, \qquad d = N - (N-2)(p+1)/2 \tag{2.2}$$

we obtain by (1.7) and (2.1) that

$$\begin{aligned} \frac{1}{2}\|u\|_X^2 &= \frac{\mu}{p+1}\|u\|_{p+1}^{p+1} - 2\alpha \\ &\le \frac{\mu}{p+1}\|u\|_{p+1}^{p+1} \\ &\le \frac{\mu}{p+1}C_1(2A)^{d/2}\|u\|_X^{p+1-d}. \end{aligned} \tag{2.3}$$

Since $p+1-d<2$ is equivalent to $p < 1+4/N$, our assertion immediately follows from (2.3). Therefore, we can apply [**10**, Proposition 6a1)] to obtain the existence of $\lambda(\mu)$ and associated eigenfunction $u_\mu \in N_{\mu,\alpha}$. Then, by a standard argument of regularity (cf. Berestycki and Lions [**1**, Lemma 1]), we have $u_\mu \in C^2(B)$.

Next we shall show the existence of positive radial eigenfunction $u_\mu(x)$ associated with $\lambda(\mu)$. Put $v_\mu = |u_\mu|$. Since $\|v_\mu\|_X^2 \le \|u_\mu\|_X^2$ (cf. Gilberg and Trudinger [**4**, Lemma 7.6]), we obtain

$$-\alpha_1 := H(v) \le H(u) = -\alpha;$$

namely, $\alpha_1 \ge \alpha$. Now we shall show that $\alpha_1 = \alpha$. To this end, we assume that $\alpha_1 > \alpha$ and derive a contradiction. We define $v_{\mu,t} := tv_\mu$, $0 \le t \le 1$, and $h(t) := H(v_{\mu,t})$. Then

$$h(t) = \frac{1}{2}t^2\|v_\mu\|_X^2 - \frac{\mu}{p+1}t^{p+1}\|v_\mu\|_{p+1}^{p+1}.$$

It is clear that $h(0) = 0$, $h(1) = -\alpha_1$ and $h(t)$ is strictly decreasing if $h(t) < 0$. Therefore, we see that there exists $0 < t_0 < 1$ uniquely satisfying $h(t_0) = -\alpha$, namely, $v_{\mu,t_0} \in N_{\mu,\alpha}$. Then, by (1.9),

$$\Psi(v_{\mu,t_0}) = (t_0)^2\Psi(v_\mu) = (t_0)^2\Psi(u_\mu) < \inf_{z\in N_{\mu,\alpha}} \Psi(z).$$

This is a contradiction. Hence, we obtain $\alpha_1 = \alpha$, that is, $v_\mu \in N_{\mu,\alpha}$. Moreover, it is obvious that $\Psi(v_\mu) = \Psi(u_\mu)$. Therefore, we see that v_μ also satisfies the equation in (1.1) with the same $\lambda(\mu)$ as that of u_μ, since $\lambda(\mu)$ is represented explicitly by

$$\lambda(\mu) = \frac{2\alpha + \mu(p-1)\|u_\mu\|_{p+1}^{p+1}/(p+1)}{\|u_\mu\|_2^2}. \tag{2.4}$$

Thus, we obtain the existence of nonnegative solutions of (1.1) which satisfy (1.8)–(1.9).

Next we shall show the existence of nonnegative nonincreasing radially symmetric solutions of (1.1) which satisfy (1.8)–(1.9). By the argument above, we may assume that $u_\mu \ge 0$. Let u_μ^* be the Schwarz spherical rearrangement of u_μ. Then $u_\mu^* \in X$ and is a nonnegative nonincreasing function of $r = |x|$, $x \in B$, which satisfies

$$\|u_\mu^*\|_{p+1} = \|u_\mu\|_{p+1}, \qquad \|u_\mu^*\|_2 = \|u_\mu\|_2, \qquad \|u_\mu^*\|_X \le \|u_\mu\|_X. \tag{2.5}$$

For these properties, we refer to Berestycki and Lions [**1**, Appendix]. Therefore,

$$-\alpha_2 := H(u_\mu^*) \le H(u_\mu) \le \alpha.$$

Now, applying the same argument as that used to derive the nonnegativity of u_μ just above, we can also obtain $\alpha_2 = \alpha$, that is, $u_\mu^* \in N_{\mu,\alpha}$. Then, by (2.5), we find that u_μ^* is a desired nonnegative solution of (1.1) which satisfies (1.8)–(1.9).

Finally, we show that $u_\mu^* > 0$ in B. Since u_μ^* is radially symmetric and nonnegative, u_μ^* satisfies the equation (1.2). Let

$$r_0 := \sup\{r \in [0,1] : u_\mu^*(s) > 0 \text{ for all } s < r\} > 0.$$

Then it is clear that $u_\mu^* > 0$ for $0 < r < r_0$ and $u_\mu^*(r) = 0$ for $r_0 \le r \le 1$. If $r_0 < 1$, then since the equation (1.2) is a regular ODE near $r = r_0$, by the uniqueness theorem of ODE, we obtain that $u_\mu^* \equiv 0$ near $r = r_0$. This is a contradiction. Thus, we obtain that $u_\mu^* > 0$ in B.

Therefore, in what follows, we consider the positive radially symmetric nonincreasing function $u_\mu(r)$ as the associated eigenfunction with $\lambda(\mu)$ and consider the equation (1.2).

3. Fundamental lemmas. In this section we prepare some useful lemmas for the proof of Theorem 1.1.

Lemma 3.1. *As* $\mu \to \infty$, $\lambda(\mu) \to \infty$.

Proof. Let u_0 be the unique positive radial solution of the equation

$$\begin{cases} -\Delta u = u^p & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{3.1}$$

Then it is clear that $u_\mu \not\equiv u_0$. We defined a function $J(t)$, $t \geq 0$, by

$$J(t) := H(tu_0) = \frac{1}{2}t^2\|u_0\|_X^2 - \frac{1}{p+1}\mu t^{p+1}\|u_0\|_{p+1}^{p+1}. \tag{3.2}$$

Then by simple calculation we find that there exists $t = t_\mu$ uniquely such that $J(t_\mu) = -\alpha$, that is, $t_\mu u_0 \in N_{\mu,\alpha}$. Then it follows from (1.9) that

$$\Psi(u_\mu) = \frac{1}{2}\|u_\mu\|_2^2 \leq \Psi(t_\mu u_0) = \frac{1}{2}t_\mu^2\|u_0\|_2^2. \tag{3.3}$$

We shall show that there exists a constant $C > 0$ such that for $\mu \gg 1$,

$$t_\mu \leq C\mu^{-1/(p+1)}. \tag{3.4}$$

It is easy to see that $J(t)$ is decreasing when $J(t) < 0$. Therefore, $J(C\mu^{-1/(p+1)}) \leq J(t_\mu) = -\alpha$ implies that $C\mu^{-1/(p+1)} \geq t_\mu$. We can choose $C > 0$ so large that for $\mu \gg 1$

(3.5)

$$\begin{aligned} J(C\mu^{-1/(p+1)}) &= C^2\left(\frac{1}{2}\mu^{-2/(p+1)}\|u_0\|_X^2 - \frac{1}{p+1}C^{p-1}\|u_0\|_{p+1}^{p+1}\right) \\ &\leq C^2\left(\|u_0\|_X^2 - \frac{1}{p+1}C^{p-1}\|u_0\|_{p+1}^{p+1}\right) \\ &\leq -\alpha. \end{aligned}$$

Thus, we obtain (3.4) by (3.5). Now it follows from (2.4), (3.3) and (3.4) that

$$\lambda(\mu) \geq \frac{2\alpha}{\|u_\mu\|_2^2} \geq \frac{2\alpha}{\|u_0\|_2^2 t_\mu^2} \geq \frac{2\alpha}{C^2\|u_0\|_2^2}\mu^{2/(p+1)}.$$

Thus the proof is complete. □

We put $v_\mu(r) = (\lambda(\mu)/\mu)^{-1/(p-1)}u_\mu(r)$. Furthermore, let $t = \lambda(\mu)^{1/2}r$ and $w_\mu(t) = v_\mu(r)$. Then it follows from (1.2) that $w_\mu(t)$ satisfies the following equation:

$$(3.6) \qquad \begin{cases} w_\mu''(t) + \frac{N-1}{t}w_\mu'(t) + w_\mu(t)^p & \\ \qquad -w_\mu(t) = 0 & t \in I_\mu = (0, \lambda(\mu)^{1/2}), \\ w_\mu(t) > 0 & t \in I_\mu, \\ w_\mu'(0) = 0, \qquad w_\mu(\lambda(\mu)^{1/2}) = 0. & \end{cases}$$

Put $\gamma_\mu := w_\mu(0) = \max_{0\leq r\leq \lambda(\mu)^{1/2}} w_\mu(t)$.

Lemma 3.2. *The following identity holds for $t \in \bar{I}_\mu$:*

$$(3.7) \qquad \begin{aligned} \frac{1}{2}(w_\mu'(t))^2 &+ \int_0^t \frac{N-1}{s}w_\mu'(s)^2\,ds \\ &+ \frac{1}{p+1}w_\mu(t)^{p+1} - \frac{1}{2}w_\mu(t)^2 \\ &= \frac{1}{p+1}\gamma_\mu^{p+1} - \frac{1}{2}\gamma_\mu^2 \\ &= \frac{1}{2}(w_\mu'(\lambda(\mu)^{1/2}))^2 + \int_0^{\lambda(\mu)^{1/2}} \frac{N-1}{s}w_\mu'(s)^2\,ds > 0. \end{aligned}$$

Proof. Multiplying (3.6) by $w_\mu'(t)$ we obtain

$$w_\mu''(t)w_\mu'(t) + \frac{N-1}{t}w_\mu'(t)^2 + w_\mu(t)^p w_\mu'(t) - w_\mu(t)w_\mu'(t) = 0,$$

that is,

$$\frac{d}{dt}\left\{\frac{1}{2}(w_\mu'(t))^2 + \int_0^t \frac{N-1}{s}w_\mu'(s)^2\,ds + \frac{1}{p+1}w_\mu(t)^{p+1} - \frac{1}{2}w_\mu(t)^2\right\} = 0;$$

this implies that

$$\frac{1}{2}(w_\mu'(t))^2 + \int_0^t \frac{N-1}{s} w_\mu'(s)^2\,ds + \frac{1}{p+1} w_\mu(t)^{p+1} - \frac{1}{2} w_\mu(t)^2 \equiv \text{constant}. \tag{3.8}$$

Now put $t = 0$, $\lambda(\mu)^{1/2}$ in (3.8) to obtain (3.7). □

From now on, for a subsequence of $\{w_\mu\}$ we will use the same notation for simplicity. Furthermore, we regard $w_\mu$as a function in R by 0-extension.

Lemma 3.3. *The sequence $\{w_\mu\}$ converges to w_∞ as $\mu \to \infty$ on any compact subset in $[0, \infty)$.*

Proof. We know from Yanagida [**9**] and Lemma 3.1 that $\{\gamma_\mu\}$ is bounded for $\mu \gg 1$. More precisely, if $\lambda(\mu_1) < \lambda(\mu_2)$, then $\gamma_{\mu_1} > \gamma_{\mu_2}$. Then it follows from (3.7) that, for $\mu \gg 1$,

$$\begin{aligned} \frac{1}{2} w_\mu'(t)^2 &\le \frac{1}{p+1}(\gamma_\mu^{p+1} - w_\mu(t)^{p+1}) - \frac{1}{2}(\gamma_\mu^2 - w_\mu(t)^2) \\ &\le \frac{1}{p+1}(\gamma_\mu^{p+1} - w_\mu(t)^{p+1}) \\ &\le \frac{1}{p+1}\gamma_\mu^{p+1} < C. \end{aligned} \tag{3.9}$$

Next we shall show that $|w_\mu''(t)|$ is bounded for $\mu \gg 1$. Obviously, the equation (3.6) is equivalent to:

$$(t^{N-1} w_\mu(t)')' + t^{N-1}(w_\mu(t)^p - w_\mu(t)) = 0, \tag{3.10}$$

which implies that, for $t \in R_+$,

$$\begin{aligned} \left|\frac{w_\mu'(t)}{t}\right| &= \left| t^{-N} \int_0^t s^{N-1}(w_\mu(s) - w_\mu(s)^p)\,ds \right| \\ &\le t^{-N} \left| \int_0^t C s^{N-1}\,ds \right| \le C. \end{aligned} \tag{3.11}$$

Then it follows from (3.6) and (3.11) that $\{w''_\mu\}$ is bounded. Since we obtain by (3.9) and (3.11) that $\{w_\mu\}$, $\{w'_\mu\}$, $\{w''_\mu\}$ are bounded, we can apply Ascoli-Arzela's theorem to extract a subsequence of $\{w_\mu\}$ such that as $\mu \to \infty$,

$$w_\mu(t) \longrightarrow w_1(t), \qquad w'_\mu(t) \longrightarrow w_2(t) \tag{3.12}$$

uniformly on any compact subsets in $[0, \infty)$. Clearly, for fixed $t > 0$

$$w_\mu(t) - w_\mu(0) = \int_0^t w'_\mu(s)\,ds;$$

by letting $\mu \to \infty$, we obtain

$$w_1(t) - w_1(0) = \int_0^t w_2(s)\,ds;$$

this implies that $w'_1(t) = w_2(t)$ and $w_1(t) \in C^1(R)$. For any $k \in C_0^\infty([0,\infty))$, we obtain by (3.10) and integration by parts that

$$-\int_0^\infty t^{N-1} w'_\mu(t) k'(t)\,dt + \int t^{N-1}(w_\mu(t)^p - w_\mu(t))k(t)\,dt = 0; \tag{3.13}$$

by letting $\mu \to \infty$ in (3.13), we obtain

$$-\int_0^\infty t^{N-1} w'_1(t) k'(t)\,dt + \int t^{N-1}(w_1(t)^p - w_1(t))k(t)\,dt = 0. \tag{3.14}$$

This implies that w_1 is a weak solution of the equation in (1.14). Since $w_1 \in C^1(R)$, by a standard regularity argument, we see that $w_1 \in C^2(R)$. We shall show that $w_1 = w_\infty$, the unique ground state of (1.14). We denote by $w = w(r, \delta)$ the solution of the initial value problem

$$\begin{cases} w''(t) + \frac{N-1}{r} w'(t) + w(t)^p - w(t) = 0 & t > 0, \\ w(0) = \delta > 0, \qquad w'(0) = 0. \end{cases} \tag{3.15}$$

Then the positive initial data can be classified as follows (cf. Kwong [**5**, Theorem]):

$$N = \{\delta : \text{ there exists } R > 0 \text{ such that } w(R,\delta) = 0\} = \{\delta > \delta_0\},$$
$$G = \{\delta : w(r,\delta) > 0 \text{ for } r \ge 0 \text{ and } \lim_{r\to\infty} w(r,\delta) = 0\} = \{\delta_0\},$$
$$P = \{\delta : w(r,\delta) > 0 \text{ for } r \ge 0 \text{ and there exists } R > 0 \text{ s.t. } w'(R,\delta) = 0\}$$
$$= \{0 < \delta < \delta_0\}.$$

Clearly, $\{\gamma_\mu\} \subset N$ and since $\lambda(\mu) \to \infty$ as $\mu \to \infty$ by Lemma 3.1, we obtain that $\delta_0 = \gamma_\infty := \lim_{\mu\to\infty} \gamma_\mu$, that is, $\gamma_\infty \in G$. Since $\gamma_\infty = w_1(0)$, we obtain that $w_1 = w_\infty$. Thus the proof is complete. □

Lemma 3.4. *There exists $z(t) \in L^2(R^N) \cap L^{p+1}(R^N)$ such that $w_\mu(t) \le z(t)$ for $\mu \gg 1$ and $t = |x| \ge 0$.*

Proof. We know from Berestycki and Lions [**1**, Lemma 2] that there exist constants $C, \delta > 0$ such that for $r \ge 0$

$$0 < w_\infty(r) \le Ce^{-\delta r}. \tag{3.16}$$

We have from (3.7) that, for $0 \le t \le \lambda(\mu)^{1/2}$,

$$-w'_\mu(t) = \sqrt{H(t)}, \tag{3.17}$$

where

$$\begin{aligned} H(t) &:= w_\mu(t)^2 - \frac{2}{p+1} w_\mu(t)^{p+1} + w'_\mu(\lambda(\mu)^{1/2})^2 \\ &\quad + 2\int_t^{\lambda(\mu)^{1/2}} \frac{N-1}{s} w'_\mu(s)^2\, ds. \end{aligned} \tag{3.18}$$

Let $z_1(t) = C(t+1)^{1/(1-q)}$, where $1 < q < 1 + 1/N$ and $C > 0$ is a sufficiently large constant. Then $z_1(t)$ satisfies the following initial value problem

$$\begin{cases} -w'(t) = \sqrt{w(t)^{2q}/\{(q-1)^2 C^{2(q-1)}\}}, \\ w(0) = C. \end{cases} \tag{3.19}$$

Let $0 < \varepsilon \ll 1$ be fixed. We choose $\mu > 0$ so large that $\lambda(\mu)^{1/2} > t_0 := \sqrt{C/\varepsilon} - 1$. If $|w| \le \varepsilon$, then

$$w^2 - \frac{2}{p+1} w^{p+1} - w^{2q}/\{(q-1)^2 C^{2(q-1)}\} > 0. \tag{3.20}$$

Hence we obtain by (3.17), (3.19), (3.20) and the comparison theorem of ODE that there exists no interval $J_\mu = [t_\mu, s_\mu] \subset [t_0, \infty)$ such that

$w_\mu(t_\mu) = z_1(t_\mu)$, $w_\mu(s_\mu) = z_1(s_\mu)$ and $w_\mu(t) > z_1(t)$ for $t \in J_\mu$. Furthermore, if there exists $t_\mu \in [t_0, \infty)$ such that $w_\mu(t_\mu) = z_1(t_\mu)$ and $w_\mu(t) < z_1(t)$ for $t \in [t_\mu, \lambda(\mu)^{1/2})$, then for $t \in [t_0, t_\mu]$, we have $w_\mu(t) \geq z_1(t)$. Consequently, for $\mu \gg 1$, we obtain $w_\mu(t_0) \geq z_1(t_0)$. However, this is impossible because of Lemma 3.3, (3.16) and the definition of $z_1(t)$. Thus, we obtain that, for $\mu \gg 1$, $w_\mu(t) \leq z_1(t)$ for $t \in [t_0, \infty)$.

Now we put, for a sufficiently large $C > 0$,

$$z(t) = \begin{cases} C & t \in [0, t_0), \\ z_1(t) & t \in [t_0, \infty). \end{cases}$$

This is the desired function. □

Now by Lemma 3.4 and a standard argument of compactness, we obtain that $w_\infty(t) = \lim_{\mu\to\infty} w_\mu(t)$ in $L^2(R^N)$ and $L^{p+1}(R^N)$.

4. Proof of Theorem 1.1. With the aid of the lemmas proved in the previous section, we shall prove Theorem 1.1.

Proof of Theorem 1.1. Let $|S_1|$ denote the measure of the unit sphere in R^N. By definition of w_μ, we have

$$\begin{aligned} \|w_\mu\|_2^2 &= |S_1| \int_0^{\lambda(\mu)^{1/2}} t^{N-1} w_\mu(t)^2 \, dt \\ &= |S_1| \lambda(\mu)^{N/2} \int_0^1 r^{N-1} v_\mu(r)^2 \, dr \\ &= |S_1| \lambda(\mu)^{N/2-2/(p-1)} \mu^{2/(p-1)} \int_0^1 r^{n-1} u_\mu(r)^2 \, dr \\ &= \lambda(\mu)^{N/2-2/(p-1)} \mu^{2/(p-1)} \|u_\mu\|_2^2, \end{aligned} \tag{4.1}$$

$$
\begin{aligned}
\|w_\mu\|_{p+1}^{p+1} &= |S_1| \int_0^{\lambda(\mu)^{1/2}} t^{N-1} w_\mu(t)^{p+1}\, dt \\
&= |S_1| \lambda(\mu)^{N/2} \int_0^1 r^{N-1} v_\mu(r)^{p+1}\, dr \\
&= |S_1| \lambda(\mu)^{N/2-(p+1)/(p-1)} \mu^{(p+1)/(p-1)} \\
&\quad \cdot \int_0^1 r^{N-1} u_\mu(r)^{p+1}\, dr \\
&= \lambda(\mu)^{N/2-(p+1)/(p-1)} \mu^{(p+1)/(p-1)} \|u_\mu\|_{p+1}^{p+1}.
\end{aligned} \tag{4.2}
$$

Furthermore, we have, by Pohozaev's identity (cf. Strauss [**8**]), that

$$
\|w_\infty\|_{p+1}^{p+1} = \frac{2(p+1)}{2(p+1) - N(p-1)} \|w_\infty\|_2^2. \tag{4.3}
$$

It follows from Lemmas 3.3, 3.4 and Lebesgue's convergence theorem that, as $\mu \to \infty$,

$$
\|w_\mu\|_2 \longrightarrow \|w_\infty\|_2, \qquad \|w_\mu\|_{p+1} \longrightarrow \|w_\infty\|_{p+1}. \tag{4.4}
$$

Now (2.4) and (4.1)–(4.4) imply that, as $\mu \to \infty$,

$$
\begin{aligned}
\lambda(\mu)^{(p+1)/(p-1)-N/2} \mu^{-2/(p-1)} &= \frac{2\alpha}{\|w_\mu\|_2^2 - (p-1)\|w_\mu\|_{p+1}^{p+1}/(p+1)} \\
&\longrightarrow \frac{N+2-p(N-2)}{N+4-Np} \frac{2\alpha}{\|w_\infty\|_2^2}.
\end{aligned} \tag{4.5}
$$

Thus the proof is complete. □

5. Proof of Theorem 1.2. We begin with the existence of $\mu(\lambda)$ for $\lambda > 0$. Let us recall that $g(u)$ and $\Phi_\lambda(u)$ are defined in (1.10) and (1.12), respectively. Let g' and Φ'_λ denote the Frechet derivative of g and Φ_λ, respectively.

Lemma 5.1. *Φ_λ satisfies Palais-Smale condition on $M_{\lambda,\beta}$: namely, any sequence $\{u_n\}_{n\in N} \subset M_{\lambda,\beta}$ satisfying*

$$
\Phi_\lambda(u_n) \le C, \tag{5.1}
$$

$$\Phi'_{\lambda,\beta}(u_n) := \Phi'_\lambda(u_n) - \frac{\Phi_\lambda(u_n)u_n}{g'(u_n)u_n} g'(u_n) \longrightarrow 0 \quad \textit{in } X' \tag{5.2}$$

contains a convergent subsequence in X. Here X' denotes the dual space of X.

Proof. We define operators B_0, B_1, A in X by the rules

$$(B_0 u, v) = \int_B \nabla u \nabla v \, dx, \qquad (Au, v) = \int_B uv \, dx,$$
$$(B_1 u, v) = \int_B |u|^{p-1} uv \, dx.$$

Then (5.2) implies that, as $\mu \to \infty$,

$$\Phi'_{\lambda,\beta}(u_n) = B_0 u_n + \lambda A u_n - \frac{\Phi_\lambda(u_n)u_n}{B_\lambda} B_1 u_n \longrightarrow 0 \quad \text{in } X. \tag{5.3}$$

By Sobolev's embedding theorem, it is obvious that B_1 and A are compact operators. By (5.1) we see that $\|u_n\|_X^2 \le C$. Hence, we can choose a weakly convergent subsequence: $u_{n_k} \to u_0$ weakly in X as $k \to \infty$. Then we obtain as $k \to \infty$,

$$A u_{n_k} \longrightarrow A u_0, \qquad B_1 u_{n_k} \longrightarrow B_1 u_0 \quad \text{in } X. \tag{5.4}$$

Since $\Phi_\lambda(u_n)$ is bounded by (5.1), by using (5.2) and (5.4) we can choose a convergent subsequence from $\{B_0 u_{n_k}\}$ in X. By the Lax-Milgram theorem, there exists a bounded inverse B_0^{-1} of B_0. Hence, we obtain that a subsequence of $\{u_{n_k}\}$ converges in X as $k \to \infty$. Thus the proof is complete. □

Since $\Phi_\lambda(u)$ is bounded below on $M_{\lambda,\beta}$, we can apply the existence theorem of Rabinowitz [**6**, Theorem 2.10, Remark 2.8(ii)] to obtain that there exists $(\mu(\lambda), \lambda, u_\lambda(x)) \in R \times R_+ \times M_{\lambda,\beta}$ which satisfies (1.11)–(1.12), although we must show the positivity of u_λ.

Lemma 5.2. *There exists a positive solution $u_\lambda \in M_{\lambda,\beta}$ which satisfies* (1.11)–(1.12).

Proof. At first we shall show the existence of the nonnegative solution. Put $y_\lambda := |u_\lambda|$. Let y_λ^* be the Schwarz spherical symmetrization of y_λ. Then $y_\lambda^* \in X$ and by Gilberg-Trudinger [**4**, Lemma 7.6] and Berestycki and Lions [**1**, appendix] that

$$\begin{aligned} &\|y_\lambda^*\|_X^2 \le \|y_\lambda\|_X^2 \le \|u_\lambda\|_X^2, \\ &\|y_\lambda^*\|_q^q = \|y_\lambda\|_q^q = \|u_\lambda\|_q^q, \quad (q = 2,\ p+1). \end{aligned} \tag{5.5}$$

Therefore, $y_\lambda^* \in M_{\lambda,\beta}$ and, by (1.9), (1.12) and (5.5),

$$\begin{aligned} I_\lambda &:= 2\Phi_\lambda(u_\lambda) = \|u_\lambda\|_X^2 + \lambda\|u_\lambda\|_2^2 \\ &\le 2\Phi_\lambda(y_\lambda^*) = \|y_\lambda^*\|_X^2 + \lambda\|y_\lambda^*\|_2^2 \\ &\le 2\Phi_\lambda(u_\lambda); \end{aligned} \tag{5.6}$$

this implies that $2\Phi_\lambda(y_\lambda^*) = I_\lambda$ and, consequently, y_λ^* is nonnegative, radially symmetric nondecreasing and satisfies the equation (1.1) and (1.9) for the same $\mu = \mu(\lambda)$ as that of u_λ, since multiplying u_λ by (1.1) and integrating by parts we obtain

$$\mu(\lambda) = \frac{I_\lambda}{\beta_\lambda}. \tag{5.7}$$

By a standard regularity argument it follows that $y_\lambda^* \in C^2([0,1])$. Then by the same argument as that used in Section 3, we obtain that $y_\lambda^* > 0$ for $0 \le r < 1$. Thus, the proof is complete. □

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. We consider the equation (1.2). We put

$$\begin{aligned} &v_\lambda(r) := \left(\frac{\lambda}{\mu(\lambda)}\right)^{-1/(p-1)} u_\lambda(r), \qquad t := \lambda^{1/2} r, \\ &w_\lambda(t) := v_\lambda(r). \end{aligned}$$

Then by replacing $\lambda(\mu)$ with λ, we see that $w_\lambda(t)$ satisfies (3.6). Then, by (5.7),

$$\begin{aligned} \mu(\lambda) &= \beta_\lambda^{-1}(\|u_\lambda\|_X^2 + \lambda\|u_\lambda\|_2^2) \\ &= \beta^{-1}\lambda^{1/\gamma}\lambda^{1-N/2}\left(\frac{\lambda}{\mu(\lambda)}\right)^{2/(p-1)} (\|w_\lambda\|_X^2 + \|w_\lambda\|_2^2). \end{aligned} \tag{5.8}$$

By Lemma 3.3 and Lemma 3.4, we have that, as $\lambda \to \infty$,

$$\|w_\lambda\|_2^2 \longrightarrow \|w_\infty\|_2^2, \qquad \|w_\lambda\|_{p+1}^{p+1} \longrightarrow \|w_\infty\|_{p+1}^{p+1}. \tag{5.9}$$

Multiplying w_λ by (3.6) and integrating by parts, we obtain by (4.3) and (5.9) that, as $\lambda \to \infty$,

$$\begin{aligned} \|w_\lambda\|_X^2 &= \|w_\lambda\|_{p+1}^{p+1} - \|w_\lambda\|_2^2 \longrightarrow \|w_\infty\|_{p+1}^{p+1} - \|w_\infty\|_2^2 \\ &= \frac{N(p-1)}{2(p+1) - N(p-1)}\|w_\infty\|_2^2. \end{aligned} \tag{5.10}$$

Combining (5.8)–(5.10), we get, as $\lambda \to \infty$,

$$\mu(\lambda)^{(p+1)/(p-1)}\lambda^{N/2-(p+1)/(p-1)-1/\gamma} \longrightarrow \beta^{-1}\frac{2(p+1)}{2(p+1) - N(p-1)}\|w_\infty\|_2^2. \tag{5.11}$$

This implies Theorem 1.2. Thus, the proof is complete. □

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