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FACTORIZATIONS IN UNIVERSAL OPERATOR SPACES AND ALGEBRAS

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ABSTRACT. We build on previous work with V. Paulsen, using a notion of quantum variables. This enables us to give an explicit description of the norm in many universal C^*-algebras, operator algebras and operator spaces. This yields curious factorization results, for example a "generalized Fourier series" representation of all continuous functions on a compact group.

1. Introduction. Let f be a function in the Wiener algebra. Thus, we think of f as a function on the circle, and we may write $f = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$, where $\sum_{k=-\infty}^{\infty} |a_k| < \infty$. Now rewrite f in a slightly different but exactly equivalent way:

$$f = [\, b_1 \quad b_2 \quad b_3 \quad \cdots \,] \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & e^{i\theta} & 0 & 0 & \cdots \\ 0 & 0 & e^{-i\theta} & 0 & \cdots \\ 0 & 0 & 0 & e^{2i\theta} & \cdots \\ \vdots & \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix}.$$

This may be done, for example, by relabelling the double series as an ordinary series $d_1 + d_2 e^{i\theta} + d_3 e^{-i\theta} + d_4 e^{2i\theta} + \cdots$, and then taking the $b_k = |d_k|^{1/2}$ and c_k equal to b_k multiplied by a complex number of modulus 1. Then the row and column matrices above have bounded norm. Let us write the factorization above as $f = \underline{\mathrm{b}}^t Z(\theta)\underline{\mathrm{c}}$. The coefficients ($b_k$ and c_k) are no longer uniquely determined; however, the Wiener algebra norm may be obtained by

$$\|f\|_W = \sum_{k=-\infty}^{\infty} |a_k| = \min\{\|\underline{\mathrm{b}}^t\|\,\|\underline{\mathrm{c}}\| : f = \underline{\mathrm{b}}^t Z(\theta)\underline{\mathrm{c}}\}.$$

We can again rewrite f equivalently as $f = \underline{\mathrm{b}}^t Z(\theta)\underline{\mathrm{c}}$, but now we allow $Z(\theta)$ to be a diagonal matrix with powers $e^{i\overline{k}\theta}$ on the diagonal in any

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order, and allowing repetition. Again it is clear that $\|f\|_W$ equals the same minimum as above. So this is simply a curious way to describe the Wiener algebra.

Of course, the Wiener algebra is a proper $*$-subalgebra of $C(T)$, and the $\|f\|_W$ norm is very different from the uniform ($\|f\|_\infty$) norm.

Now consider a more general form of function. Consider functions which can be written as a product as above, except that we add in more matrices in the middle. Namely, we consider functions of the form

$$f = [\, b_1 \quad b_2 \quad \cdots \,]\, Z_1(\theta) D_1 Z_2(\theta) D_2 \cdots \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix}.$$

Here the matrices $Z_k(\theta)$ are as described above (diagonals with integer powers of $e^{i\theta}$ on the diagonal), and the D_k matrices are scalar matrices of norm 1, of infinite size but which are compact. If one supposes an infinite number of matrices in the product above, one needs a certain convention (described later) to make sense of the infinite product, which converges uniformly to a continuous function on the circle.

Very surprisingly, it turns out that all continuous functions f on the circle have such a form. Moreover, now the supremum norm $\|f\|_\infty$ is essentially achieved by some such factorization: if one again considers the numbers $\|\underline{\mathrm{b}}^t\| \, \|\underline{\mathrm{c}}\|$, then there is such a factorization of f with this number as close as we wish to $\|f\|_\infty$. Thus, factorizations with one middle term give the Wiener algebra norm, while factorizations with infinitely many middle terms give the uniform norm.

We shall see that there is a similar result for continuous functions on any compact group, or indeed for any compact quantum group (see Example 3.23 and [**7**]).

This type of factorization result is essentially a corollary of a characterization of operator algebras we gave several years ago with Z-j. Ruan and A.M. Sinclair. In work with V. Paulsen [**9**], we found the first such factorization formulae, at least for dense subalgebras of three particular universal operator algebras. Although it seemed clear that this should work more generally, we were unable at that time to 1) give a framework in which to fit all these types of results and 2) to move from the dense subalgebra to the full algebra completion.

Here we add these two ingredients and give a universal approach to operator spaces and algebras, which may be described as "varieties of operator algebras," and we construct what seems to be an operator algebraic version of the coordinate ring of a variety.

We give a long list of examples which fit into our framework. Some of these examples are old, and some have not been described elsewhere in this generality, as far as we know (such as the crossed product of a discrete semigroup by an operator algebra). This section should also be of interest to nonspecialists in that it assembles together most of the basic important universal constructions with operator spaces and algebras. Our theorems show that they all have explicit factorization type norm formulae. To save space we leave it to the reader to extract, and in some cases simplify, the form of the factorization (for example, the norm in the crossed product mentioned above can be given over factorizations alternating diagonal matrices with entries in the semigroup and matrices of norm ≤ 1 from the operator algebra). By considering the finite factorizations we get norm formulae for a dense subalgebra, respectively subspace. In certain particular examples (such as the projective operator space tensor product and the maximal operator space of a normed space) we recover useful norm formulae which have been discovered by other authors [**15, 20**]. Indeed, in these two examples, knowing the norm formulae is essential in some situations. To see this more clearly, recall that it is essential for many purposes to know the explicit formulae $\inf\{\sum_i \|x_i\|\|y_i\| : u = \sum_i x_i \otimes y_i\}$ for the Banach space projective tensor product (as opposed to simply knowing the norm as a supremum over a family of bounded bilinear functionals).

In fact, we not only describe the norm on these universal spaces and algebras X, we describe the norms $\|\cdot\|_n$ on the space of $n \times n$ matrices with entries in X.

Our notation is standard: see [**1, 19, 14, 12**]. We assume a certain familiarity with some basic concepts of this theory. Throughout $\mathcal{H}$ is a Hilbert space and the algebra of bounded linear operators on $\mathcal{H}$ is denoted $B(\mathcal{H})$. An operator space X is a linear subspace of $B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. These have an elegant characterization by Ruan [**23, 17**]. An operator algebra is a (not necessarily self-adjoint) subalgebra of $B(\mathcal{H})$, and we will assume the presence of an identity of norm 1. These have been characterized in [**10**]. The spaces $M_n(X)$ of

$n \times n$ matrices with entries in X inherits a norm as a subspace of

$$M_n(B(\mathcal{H})) = B\,(\underbrace{\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n}).$$

This enables us to refer to an operator space without referring to $\mathcal{H}$ by keeping note of the sequence of matrix norms (the norms on $M_n(X)$). Of course, the completion of an operator space is an operator space, and we usually assume all spaces are complete. The appropriate morphisms between operator spaces are the completely bounded maps (cb) [**1, 19**]. If X and Y are two operator spaces, and if $T : X \to Y$ is a linear transformation, then we write T_n for the linear transformation mapping $M_n(X)$ to $M_n(Y)$ defined by $T_n([x_{ij}]) = [T(x_{ij})]$. If each T_n is bounded, and if $\sup_n \|T_n\|_n$ is finite, then we say that T is *completely bounded* (or cb) and we define $\|T\|_{\mathrm{cb}}$ the completely bonded (cb) norm of T to be the supremum. We say that T is *completely contractive* or cc in case $\|T\|_{\mathrm{cb}} \leq 1$, and we say that T is *completely isometric*, or is a *complete isometry*, if each T_n is an isometry.

We identify operator spaces and algebras which are completely isometrically and algebraically isomorphic,

2. A framework and some theorems. Let Γ be a set, and let $n : \Gamma \to \{\text{Sets}\}$ be a function with $n(\gamma) = n_\gamma$. In most cases and examples below n_γ is a finite or countable set, in which case we write n_γ also for the cardinality. Let Λ be a set of variables (or formal symbols) x^γ_{ij}, one variable for each $\gamma \in \Gamma$, $i, j \in n_\gamma$. If $n_\gamma = 1$ we call the variable an *ordinary variable*. We differentiate two cases, universal operator algebras and universal operator spaces and proceed as one might expect (see [**2**]). Form the free associative algebra (respectively free vector space) Φ on Λ. Assume in addition some relations between the variables, as follows. Let $\mathcal{R}$ be a set of polynomial identities (respectively linear identities) $P = 0$ in the variables in Λ. Regard $\mathcal{R}$ as a subset of Φ. We give many examples below, but for now note that the most common relation in the algebra case is that there is an element $e \in \Gamma$ with $n_e = 1$, such that $x^e x^\gamma_{ij} = x^\gamma_{ij} x^e = x^\gamma_{ij}$ for all γ, i, j. We write x^e as 1 and say that Λ has identity. It is unnecessary, but sometimes helpful intuitively, if at this point one takes a quotient of Φ by the ideal (respectively subspace) generated by $\mathcal{R}$. We define a semi-norm on $M_n(\Phi)$ by

$$\|[u_{ij}]\|_\Lambda = \sup\{\|[\pi(u_{ij})]\|\}$$

where the supremum is taken over all algebra, respectively linear, representations π of Φ on a Hilbert space satisfying the condition $\|[\pi(x^\gamma_{ij})]\| \leq 1$ for all γ. This latter matrix is indexed on rows by i and on columns by j, for all $1 \leq i,\, j \leq n_\gamma$.

That is, a matrix of polynomials (respectively linear combinations) is normed (in general only 'seminormed') by taking a supremum of the norms one obtains by substituting (Hilbert space) operators T^γ_{ij} in for the variables. However, the T^γ_{ij} must form a complete solution set in $B(\mathcal{H})$ of the relations in $\mathcal{R}$, and also $\|[T^\gamma_{ij}]\| \leq 1$ for all $\gamma \in \Gamma$. The identity $\|[T^\gamma_{ij}]\| \leq 1$ in the case that n_γ is infinite is interpreted as saying that the finite square submatrices are uniformly bounded. Now quotient by the nullspace of this semi-norm to obtain an operator algebra (respectively operator space) $FA(\Lambda, \mathcal{R})$ (respectively $FS(\Lambda, \mathcal{R})$). One may use our characterization of operator algebras [**10, 6**], respectively Ruan's theorem [**23, 17**], here if one wishes to avoid set theoretic difficulties. The completion of this space is denoted by $OA(\Lambda, \mathcal{R})$ (respectively $OS(\Lambda, \mathcal{R})$). We call this the *free operator algebra* (respectively *space*) on Λ with relations $\mathcal{R}$. We continue however to write x^γ_{ij} for the equivalence class of the generator, even if by now it has vanished. If the completion is not the zero space, then $(\Lambda, \mathcal{R})$ is said to be *admissible*. This condition implies, in the algebra case, that if Λ has identity then $OA(\Lambda, \mathcal{R})$ will be an operator algebra with identity of norm 1. It will be a C^*-algebra if we know further that $(x^\gamma_{ij})^* \in \Lambda$ for all γ, i, j. This may be ensured by specifying conditions such as: for each $\gamma_1 \in \Gamma$, there exists $\gamma_2 \in \Gamma$ such that $x^{\gamma_1}_{ij}$ and $x^{\gamma_2}_{ij}$ satisfy the algebraic conditions which force the matrix $[x^{\gamma_1}_{ij}]$ to have inverse $[x^{\gamma_2}_{ij}]$. In this case, since these matrices have norm 1, they are unitary so that $(x^{\gamma_1}_{ij})^* = x^{\gamma_2}_{ji} \in \Lambda$.

Remark 1. This framework can be extended to also include (or replace the contraction inequalities by) a set of inequalities which must be satisfied by certain matrices of polynomials (linear combinations) in the variables of Λ as in [**2**], and to include "approximate relations," that is, nets of polynomials which converge to zero, but we will not use this further generality.

Remark 2. A special case of the free operator algebra construction above was used in [**6**] to give a useful characterization of *all* operator algebras up to completely bounded isomorphism.

Now we state some theorems. The first significance of these theorems are that they allow a more tractable (or at least more appealing) expression for the norms in these universal spaces. Some comments on the notations below: for concreteness you may wish to assume that all of the n_γ are finite, although that is not significant. Secondly, in many concrete examples, the method of proof gives the form of the factorization in a more appealing looking form; in particular, we can often rearrange the forms of the block matrices (see [**9**] for example).

Theorem 2.1. *Suppose that* $(\Lambda, \mathcal{R})$ *is admissible. If* $U \in M_n(FS(\Lambda, \mathcal{R}))$, *then* $\|U\| < 1$ *if and only if we can write* U *in* FS *as a product* AXB, *where* A *and* B *are scalar matrices* (*with a finite number of nonzero entries*), *with* $\|A\| < 1$, $\|B\| < 1$, *and where* X *is a block diagonal matrix with a finite number of blocks, and each block equal to a matrix* $[x_{ij}^{\gamma}]$ *for some* $\gamma \in \Gamma$.

Proof. Let $U \in M_n(FS(\Lambda, \mathcal{R}))$. That some factorization $U = AXB$ exists as above (with no attempt to control $\|A\|, \|B\|$) is an elementary exercise (see the proof of Theorem 2.3). By appealing to Ruan's theorem as in [**9**], one can show that defining $|U|$ equal to the infimum of $\|A\|\|B\|$ over all representations as above, and taking a quotient by the nullspace, gives a new operator space structure on $FS(\Lambda, \mathcal{R})$. It is clear that $|[x_{ij}^{\gamma}]| \leq 1$ for all γ, that is, the finite square submatrices are uniformly bounded. Hence, by definition $|\cdot| \leq \|\cdot\|$. However, the other direction that $\|\cdot\| \leq |\cdot|$ is obvious. Hence, these matrix norms $\|\cdot\|$ and $|\cdot|$ are identical, from which the assertion follows. □

Note that if $A = [A_1 A_2 \cdots]$ and $B = [B_0 B_1 \cdots]^t$ are compact matrices, and if each A_i, B_i has only a finite number of nonzero entries, then of course $\|[A_n A_{n+1} \cdots]\|\|[B_n B_{n+1} \cdots]^t\| \to 0$ as $n \to \infty$. Thus, if $\|z_i\| \leq 1$ for all i, then $\sum_{n=1}^{\infty} A_n z_n B_n$ is Cauchy and hence converges. We write the limit as AZB, where $Z = \text{diag}\,\{z_1, z_2 \ldots\}$. We say that A_i and B_i are blocks corresponding to z_i. This notation is used in the next theorem.

Theorem 2.2. *Suppose* $(\Lambda, \mathcal{R})$ *is admissible. If* $U \in M_n(OS(\Lambda, \mathcal{R}))$, *then* $\|U\| < 1$ *if and only if we can write* U *in* OS *as a product* AXB, *where* X *is a block diagonal matrix, with each block equal to a matrix* $[x_{ij}^{\gamma}]$ *for some* $\gamma \in \Gamma$, *and where* A *and* B *are compact scalar matrices with* $\|A\| < 1$, $\|B\| < 1$, *and the corresponding blocks in* A *and* B (*corresponding to the diagonal blocks in* X) *have only finitely many nonzero entries.*

Proof. The discussion above the theorem demonstrates the one direction. The other direction proceeds as follows. Suppose that $U \in M_n(OS(\Lambda, \mathcal{R}))$, $\|U\| < 1$. Given $\varepsilon > 0$ there exists a U_1 in $M_n(FS(\Lambda, \mathcal{R}))$, $U_1 = A_1X_1B_1$ as in Theorem 2.1, with $\|U - U_1\|_n < \varepsilon$, and $\|A_1\| = \|B_1\| < 1$. Now choose $U_2 = A_2X_2B_2$ with $\|U - U_1 - U_2\|_n < \varepsilon/2$, and $\|A_2\| = \|B_2\| < \sqrt{\varepsilon}$. Choose $U_3 = A_3X_3B_3$ with $\|U - U_1 - U_2 - U_3\|_n < \varepsilon/4$, and $\|A_3\| = \|B_3\| < \sqrt{\varepsilon/2}$. Continue in this manner. From these inequalities it follows that $\|[A_1A_2 \cdots]\| < 1 + 2\varepsilon$, $\|[B_1B_2 \cdots]^t\| < 1 + 2\varepsilon$, and that $\sum_{n=1}^{\infty} A_nX_nB_n$ converges uniformly to U. □

We remark that this also gives an explicit norm formula for elements in $OS(\Lambda, \mathcal{R})$. Namely, $\|U\| = \inf\{\|A\|\|B\| : U = AXB \text{ as above}\}$.

We now turn to the algebra case.

Theorem 2.3. *Suppose that* $(\Lambda, \mathcal{R})$ *is admissible and has identity. If* $U \in M_n(FA(\Lambda, \mathcal{R}))$, *then* $\|U\| < 1$ *if and only if there exists* $k \in \mathbf{N}$ *such that we can factor* U *in* FA *as a product* $A_0X_1A_1X_1 \cdots X_kA_k$, *where the* A_i *are scalar matrices* (*with a finite number of nonzero entries*), *each* $\|A_i\| < 1$, *and where each* X_i *is a block diagonal matrix with a finite number of blocks, and each block equal to a matrix* $[x_{ij}^{\gamma}]$ *for some* $\gamma \in \Gamma$.

Proof. Again, that any $U \in M_n(FA(\Lambda, \mathcal{R}))$ has some such factorization $A_0X_1A_1X_1 \cdots X_kA_k$ (with no attempt to control the norms of the A_i) is a simple exercise. For the reader's convenience, we sketch how this may be seen. Firstly, note that if U has only one nonzero entry in the $i - j$ position, and if this entry is a monomial (a product of a finite number of elements of Λ), or a monomial times a scalar, then it is clear that U may be written in form $A_0X_1A_1X_1\cdots X_kA_k$ (in fact we can take each X_r to be of the form $[x^\gamma_{ij}]$ for some γ, and the A_r to consist only of zeros and ones). Now a general $U \in M_n(FA(\Lambda,\mathcal{R}))$ may be written as a finite sum $\sum_p U_p$, where U_p is of the form just described, namely $U_p = A^p_0X^p_1A^p_1X^p_1\cdots X^p_kA^p_k$. We have assumed here that a single k will suffice for all p, which is justified since we can add copies of the identity matrix in the products to ensure that they all have the same length. We now set $A_0 = [A^1_0A^2_0\cdots]$, $A_k = [A^1_kA^2_k\cdots]^t$, $X_r = X^1_r\oplus X^2_r\oplus\cdots X^k_r$, $A_r = A^1_r\oplus A^2_r\oplus\cdots A^k_r$. Here $\oplus$ means the block diagonal direct sum of matrices. Clearly $U = A_0X_1A_1X_1\cdots X_kA_k$.

By appealing to the characterization of operator algebras [**10**] as in [**9**], one can show that defining $|U|$ equal to the infimum of $\|A_0\|\cdots\|A_k\|$ over all k and all representations as above, gives (after taking a quotient by the nullspace) a new operator algebra structure on $FA(\Lambda,\mathcal{R})$. Now proceed as in Theorem 2.1 to deduce that $|U| = \|U\|$, which proves the result stated. □

We now give a method giving a formula for the norm in $M_n(OA(\Lambda,\mathcal{R}))$. To extend the previous theorem to the closure we need to note that the method of the proof actually gives the norm of an element $U \in M_n(FA(\Lambda,\mathcal{R}))$ equal to an infimum over representations of U of form $\sum_{k=1}^N A^k_0X^k_1A^k_1X^k_2\cdots A^k_{m_k}$, where $\|A^k_i\| \le 1$ for all k and $1 \le i \le m_k - 1$. Here, of course, each A^k_j has finitely many nonzero entries, and X^k_i is a block diagonal matrix with a finite number of blocks, each block is $[x^\gamma_{ij}]$ for some γ. We shall term X^k_i a major block. We will adopt the following convention in all that follows to make all the m_ks equal; namely, add identity matrices between the last X and last A. Once all the ms are equal, the usual direct sum trick (see proof of Theorem 2.3) is used to write $U = [A^1_0A^2_0\cdots]X_1D_1X_2\cdots X_m[A^1_mA^2_m\cdots]^t$. Here the X_i are block diagonals, each block equal to an x^γ, and D_i is a block diagonal matrix of scalars, with blocks corresponding to major blocks in the X_i. Simply for notational convenience we also call the blocks in D_i major blocks.

It makes sense to talk about such a product even if there are infinitely many matrices. Write $A_0X_1D_1X_2\cdots D_{\infty-1}X_\infty A_\infty$ for an infinite such product. Here all matrices have an infinite number of major blocks, but major block positions correspond, and also given m there exists an N such that, for $i > N$, the mth major blocks of D_i and X_i are copies of the identity matrix. All subblocks of the D_i are the identity matrix or have a finite number of entries and norm ≤ 1. In addition, A_0 and A_∞ must be compact, with each sub-block having a finite number of entries. Labelling the major block entries in the matrices as A_i^k and X_i^k, we see that the product $A_0 X_1 D_1 \cdots D_\infty A_\infty$ is simply the uniform limit of the Cauchy sequence $\{\sum_{k=1}^N A_0^k X_1^k A_1^k X_2^k \cdots A_{m_k}^k\}_N$. This limit is an element of $M_n(OA(\Lambda, \mathcal{R}))$ and also equals the uniform limit $\lim_{k\to\infty} A_0 X_1 D_1 \cdots D_{k-1} X_k A_\infty$.

This infinite factorization is somewhat reminiscent of the Blaschke product. Indeed, applying the next theorem in the disk algebra, which is the unital universal operator on one generator, we see that we have an infinite matrix factorization for functions in the disk algebra.

Theorem 2.4. *Suppose that $(\Lambda, \mathcal{R})$ is admissible and has identity. If $U \in M_n(OA(\Lambda, \mathcal{R}))$, then $\|U\| < 1$ if and only if there exists a factorization $U = A_0 X_1 D_1 \cdots X_\infty A_\infty$ as above, with $\|A_0\| < 1$, $\|A_\infty\| < 1$. (By definition, the $\|D_i\| \leq 1$.)*

Proof. The argument above the theorem establishes the one direction. The other direction follows the proof of Theorem 2.2. We use Theorem 2.3 to construct successive approximations U_k as in Theorem 2.2. We use copies of the identity matrix, with the convention above, to equalize the lengths of representations of the successive approximations U_k. Finally, $U = \sum_{n=1}^\infty U_n$ uniformly. This may, by the direct sum procedure described in the beginning of the proof of Theorem 2.3, or in the paragraph after the proof of Theorem 2.3, be written in the form $U = A_0 X_1 D_1 \cdots X_\infty A_\infty$. The reader should check that we are obeying the conventions described above for such an infinite product. □

Again, the infimum over factorizations gives the norm formula. Namely, $\|U\| = \inf\{\|A_0\| \|A_\infty\| : U = A_0 X_1 D_1 \cdots X_\infty A_\infty \text{ as above}\}$.

3. Examples. We now list a host of examples of universal construction which fit into the framework described above. In the interests of brevity we do not prove many of our assertions, and the method of proof is in most cases clear. We also assume all operator spaces and algebras are complete.

Remark 3.1. The theorems above give interesting information in most cases, but as we observed earlier before Theorem 2.1, we may need to simplify the block matrices in specific examples: the appearance of naked generators is frightening until one quotients out by the relations and simplifies.

Remark 3.2. We observe that (assuming that one has already taken a quotient by $\mathcal{R}$) it is unnecessary to take the discussed quotient of Φ in nearly all of the examples below (or, in other words, the semi-norm is already a norm).

Example 3.3. Let X be an operator space (respectively unital operator algebra) define $\Gamma_X = \cup_n \text{BALL}\,(M_n(X))$ and take Λ_X to be the set of all matrix entries, considered as variables x_{ij}^{γ} indexed by $\gamma = [x_{ij}^{\gamma}] \in \Gamma_X$ and i, j. Let $\mathcal{R}$ be the set of all linear (respectively polynomial) identities satisfied in X by elements in Λ_X. Then $OS(\Lambda_X, \mathcal{R}) = X$ (respectively for OA). If X is simply a matrix normed space (respectively algebra with identity of norm 1) which does not satisfy the L^∞ condition of Ruan [**23**] then the construction produces the maximal operator space (respectively algebra) envelope. We observe that this is possibly inadmissible. An interesting case is the operator algebra envelope $OA(\text{MAX}\,(L^1(G)))$ for a compact group G (with convolution multiplication), which is $C^*(G)$. These envelopes have the expected universal property: given any cc morphism $X \to W$, into an operator space (respectively algebra) W there exists a unique cc extension from the envelope into W.

Example 3.4 (Quotient spaces [**23, 10**]). Let X be an operator space (or algebra), E a closed subspace (respectively two-sided ideal) of X, and form Λ_X as above. Let $\mathcal{R}$ be the algebraic relations satisfied in X by elements of Λ_X, together with the relations $y = 0$ for all $y \in Y$. Then $OS(\Lambda_X, \mathcal{R})$ is the quotient operator space (respectively OA is the quotient operator algebra) [**23, 10**]. This has universal property: given any completely contractive morphism ϕ from X into an operator space (respectively algebra) W, with ϕ annihilating E, then there exists a unique morphism from the quotient into W completing the commutative diagram.

Example 3.5 (The universal operator space for a row (column) contraction). Let Γ have one element γ, and let the associated matrix entries be zero except for the first row (this statement constitutes the relations). Then $OS(\Lambda, \mathcal{R})$ is column Hilbert space [**8, 16, 4**]. Similarly, if the associated matrix entries are zero except for the first column, then $OS(\Lambda, \mathcal{R})$ is a row Hilbert space.

Example 3.6 (The universal operator space for a matrix contraction). Let Γ have one element γ, and let $n_\gamma = n$ (possibly infinite). Then $OS(\Lambda)$ is T_n, the standard pre-dual [**3**] of M_n.

Example 3.7 (The universal operator space for m contractions). Let Γ have m elements, with each $n_\gamma = 1$. Then $OS(\Lambda)$ is l^1_m with MAX matrix norm structure (see Example 3.11).

Example 3.8 (Coproduct). Suppose that X_α is a family of complete operator spaces. Let $\Gamma_{X_\alpha}, \Lambda_{X_\alpha}$ be as in Example 3.3, let $\Gamma = \cup_\alpha \Gamma_{X_\alpha}$, let Λ be the collection of entries in the matrices in Γ and let $\mathcal{R}$ be the union of the algebraic relations satisfied in X_α by elements of Λ_{X_α}. Then $OS(\Lambda, \mathcal{R})$ is the L^1 direct sum operator space defined in [**3**]. This has the coproduct universal property for the operator spaces and cc maps. See [**5**] for a proof of this, and for a proof of the fact that this construction commutes with the quotient construction.

Example 3.9. Let X and Y be operator spaces, and let Γ be the set of formal symbols $[x_{ij} \otimes y_{kl}]$ for all $[x_{ij}]$ and $[y_{kl}]$ in BALL$\,(M_n(X))$. Note that $[x_{ij} \otimes y_{kl}]$ is a matrix with $(i,k),(j,l)$ entry $x_{ij} \otimes y_{kl}$, rows numbered by i,k, columns by j,l. Let Λ be the set of these entries. Let $\mathcal{R}$ include

1) for fixed $y \in \Lambda_Y$, relations between tensors of form $\cdot \otimes y$ corresponding to relations satisfied in X by elements of Λ_X;

2) corresponding relations for fixed $x \in \Lambda_X$;

Thus we include all relations of the form

$$(\lambda_1 x_1 + \lambda_2 x_2) \otimes y = \lambda_1 (x_1 \otimes y) + \lambda_2 (x_2 \otimes y),$$
$$x \otimes (\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 (x \otimes y_1) + \lambda_2 (x \otimes y_2)$$

In these identities we have used symbols x_k, x, y_k, y to represent elements of Λ_X and Λ_Y. Then $OS(\Lambda, \mathcal{R})$ is the operator space projective tensor product [**8, 15**]. In [**5**] we show this commutes with the previous two constructions. We remark that the theorem above gives the form of the norm given in [**15**].

Example 3.10. Proceed as in the previous example, but now Γ are the matrices $[\sum_{k=1}^n x_{ik} \otimes y_{kj}]$. Then $OS(\Lambda, \mathcal{R})$ is the Haagerup tensor product [**8, 16**].

Example 3.11 (The maximal universal operator space of a normed space [**3, 8, 20**]). Let X be a Banach space and $\Gamma = \Lambda = \mathrm{BALL}\,(X)$, each $n_\gamma = 1$. Let $\mathcal{R}$ be the linear relations satisfied in $\mathrm{BALL}\,(X)$. Then $OS(\Lambda, \mathcal{R}) = \mathrm{MAX}\,(X)$. The theorem above gives the form of the norm given in [**20**].

Example 3.12. Take an operator space X, and take Γ to be the set of completely contractive maps of X into $B(\mathcal{H})$, where $\mathcal{H}$ is separable. Fix an orthonormal basis of $\mathcal{H}$. Let $n_\gamma = \dim (H)$. For each $T \in \Gamma$ consider the coefficients of T, namely the entries in its matrix, and let Λ be the collection of such scalar valued functions on X. For relations identify functions that are equal on $\mathrm{BALL}\,(X)$. Then $OS(\Lambda, \mathcal{R})$ is the standard dual [**3**] of X.

Example 3.13. Direct limits of operator spaces (and algebras) are defined in the obvious way, see [**6**]. To keep in the context of this paper we insist that morphisms are cc (respectively and unital). We take Γ as in Example 3.8 but add to $\mathcal{R}$ the condition that $x = y$ if $f_{kn}(x) = f_{km}(y)$ (here $x \in X_n, y \in X_m$).

Example 3.14 (The universal operator algebra on matrix units). Let Γ have two elements, one corresponding to an identity, the other having $n_\gamma = n$ and let Λ be the entries in a matrix $[e_{ji}]$. Let $\mathcal{R}$ be the relations $e_{ij} e_{kl} = \delta_{jk} e_{il}$, $\sum_{k=1}^n e_{kk} = 1$. Then $OA(\Lambda, \mathcal{R}) = M_n$.

Example 3.15 (The universal operator algebra on n noncommuting (commuting) unitaries). Let Λ have identity and also contain $2n$ ordinary variables u_k and v_k, $1 \leq k \leq n$, and let $\mathcal{R}$ be the relations $u_k v_k = v_k u_k = 1$, $1 \leq k \leq n$. Then $OA(\Lambda, \mathcal{R}) = C^*(F_n)$, the C^*-algebra of the free group on n generators.

If you now add to $\mathcal{R}$ the relations $u_k v_l = v_k u_l$, $1 \leq k, l \leq n$, then $OA(\Lambda, \mathcal{R}) = C(\mathbf{T}^n)$, the continuous functions on the n-torus.

Example 3.16. Let Λ have identity and also contain the entries in a row matrix $[u_1, \dots, u_n]$ and a column matrix $[v_1, \dots, v_n]^t$ (so there are some relations forcing the other entries to be zero), let $\mathcal{R}$, in addition, have relations $\sum_{k=1}^n u_k v_k = 1$, and $v_k u_k = 1$, $1 \leq k \leq n$. Then $OA(\Lambda, \mathcal{R}) = \mathcal{O}_n$, the Cuntz C^*-algebra. A similar approach gives the irrational rotation C^*-algebras.

Example 3.17 (Universal matrix unitaries). Let Λ have identity and also contain the entries of the $n \times n$ matrices $[u_{ij}]$ and $[v_{ij}]$, and let $\mathcal{R}$ be the relations $\sum_{k=1}^n u_{ik} v_{kj} = \delta_{ij} 1 = \sum_{k=1}^n v_{ik} u_{kj}$ for all i and j. Then $OA(\Lambda, \mathcal{R}) = U_n^{nc}$, Brown's universal noncommuting unitary C^*-algebra [**11**]. If we further add to $\mathcal{R}$ the polynomial relations making the transposed matrices $[u_{ji}]$ a unitary with inverse $[v_{ji}]$, then $OA(\Lambda, \mathcal{R}) = A(n)$, Wang's universal compact quantum group [**26**]. If we further add to $\mathcal{R}$ the relations making all variables commute, then $OA(\Lambda, \mathcal{R}) = C(U_n)$, the continuous functions on the unitary group.

Example 3.18 (Universal operator algebra on a discrete contraction semi-group). Let G be a discrete group. Put $\Gamma = \Lambda = G$, $n_\gamma = 1$ for all γ. Let $\mathcal{R}$ be the group relations. Then $OA(\Lambda, \mathcal{R}) = C^*(G)$, the group C^*-algebra. If G is a semigroup with identity, then we obtain $OA(\Lambda, \mathcal{R}) = OA(G)$ as defined in [**9**].

Example 3.19 (Free product with amalgamation). Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be unital operator algebras, and let ϕ and ψ be unital completely isometric homomorphisms from $\mathcal{C}$ into $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\Gamma = \Gamma_{\mathcal{A}} \cup \Gamma_{\mathcal{B}}$, let Λ be the collection of entries in matrices in Γ, and let $\mathcal{R}$ be the collection of relations satisfied as in 3.8, but now together with relations $\phi(x) = \psi(x)$ for elements $x \in \mathrm{Ball}(\mathcal{C})$. Then $OA(\Lambda, \mathcal{R}) = \mathcal{A} *_{\mathcal{C}} \mathcal{B}$, the amalgamated free product operator algebra discussed in [**9**]. If $\mathcal{C} = \mathbf{C}$ then this is the coproduct in the associated category.

Example 3.20. The maximal tensor product operator algebra $\mathcal{A} \otimes_{\max} \mathcal{B}$ [**9, 21**]. This is described on the set Γ as in Example 3.19, $\mathcal{C} = \mathbf{C}$, but with additional relations $ab = ba$.

Example 3.21. Let Λ have two elements, one of which is the identity. Then $OA(\Lambda, \mathcal{R}) = A(\mathbf{D})$, the disk algebra. Now if Λ has three elements $1, u, v$, and if $uv = vu$ then $OA(\Lambda, \mathcal{R}) = A(\mathbf{D} \times \mathbf{D})$ the bi-disk algebra [**9**].

Example 3.22. Let G be a discrete semigroup with identity, $\mathcal{A}$ a unital operator algebra and α a unital homomorphism of G into $\operatorname{Aut}(\mathcal{A})$, the set of completely contractive automorphisms of $\mathcal{A}$. Let $\Gamma = G \cup \Gamma_{\mathcal{A}}$, and let Λ be the entries. Thus, Λ is the union of the ordinary variables represented by entries in G, together with the matrix entry variables coming from matrices with entries in $\mathcal{A}$ and norm ≤ 1. We take $\mathcal{R}$ to include the group identities, the relations satisfied in $\mathcal{A}$, the relation $1_G = 1_{\mathcal{A}}$, together with the identities $g \cdot a = \alpha(g)(a) \cdot g$. We call $OA(\Lambda, \mathcal{R})$ the maximal crossed product operator algebra. For a discrete group and a C^*-algebra, this is the usual crossed product.

Example 3.23. Let G be a compact group, or more generally let $G = (\mathcal{A}, \Phi)$ be a compact quantum group [**27**]. Let $T = \hat{G}$ be a complete set of representatives of equivalence classes of unitary matrices $[\mu_{ij}^{\gamma}]$ coming from the Peter-Weyl theorem. Define Λ to be the set of entries (which generate a dense *-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$) and their adjoints. Let $\mathcal{R}$ be all polynomial relations satisfied in $\mathcal{A}_0$. Then $OA(\Lambda, \mathcal{R})$ is a C^*-algebra and, indeed, a compact quantum group (or more precisely, a Woronowicz Hopf C^* algebra). We note that OA is the maximal Woronowicz C^*-algebra of the quantum group. An example to keep in mind is the various C^*-algebras of the free group on two generators. We do not know when $OA = \mathcal{A}$, there certainly are plenty of examples where they differ, but it has the same dense *-subalgebra and maps onto. We have been told that these should be viewed as the same quantum group, even if the C^*-algebras are different. In many cases the C^*-algebras are equal, and perhaps a more restrictive condition would ensure it, such as some type of amenability. In the non-quantized case there is no problem, of course; this is just $C(G)$. As a consequence of Theorems 3.3 and 3.4, we get factorizations and norm formulae for elements in OA and its important dense *-subalgebra. This is all discussed thoroughly in [**7**].

Example 3.24. Let $\mathcal{A}$ be a separable matrix normed algebra and an h-coalgebra (operator convolution algebra [**6, 18**]). Suppose that $\mathcal{A}$ has counit of norm 1. Then $\mathcal{A}^*$ is an operator algebra. The Fourier-Stieltjes algebra of coefficient operators form a subalgebra of $\mathcal{A}^*$ (see [**18**]). Fix a Hilbert space $\mathcal{H}$ of dimension large enough, and pick a fixed orthonormal basis. If we take Γ to be the set of all completely contractive representations γ of $\mathcal{A}$ on $\mathcal{H}$, $n_\gamma = \dim(\mathcal{H})$, Λ to be the set of coefficients of the representations (the entries in the matrix of the representation) and $\mathcal{R}$ to be the relations identifying functions if they are equal on $\mathcal{A}$ together with the convolution multiplication relations, then $OA(\Lambda, \mathcal{R})$ is the enveloping operator algebra of the Fourier-Stieltjes algebra. In the commutative compact group case this is all of $\mathcal{A}^*$ by the Stone-Weierstrass theorem.

We finish by showing that T_n, the $n \times n$ trace class matrices, with operator space structure as the standard pre-dual of M_n, is (completely isometrically) embedded in U_n^{nc} by the map $e_{ij} \to u_{ij}$, see Examples 3.6 and 3.17. Certainly, by the universal properties, this map is a complete contraction. However, if $T = [T_{ij}]$ is any matrix of operators with $\|[T_{ij}]\| \leq 1$, then we may dilate T to an $n \times n$ unitary matrix by the 2×2 dilation trick, for instance. Thus, there is a map from U_n^{nc} onto the C^*-algebra generated by the elements of the unitary matrix, and hence there is by compression a complete contraction taking u_{ij} to T_{ij}. This establishes the result.

This result should be compared with Paulsen's result [**20**] embedding $\mathrm{MAX}(l_n^1)$, the standard pre-dual of l_n^∞, completely isometrically into the C^*-algebra of the free group on $n-1$ generators.

We note that $\mathrm{MIN}(T_n)$ is completely isometrically embedded in $C(U_n)$, and from Example 3.17 it is clear that we may assign new operator space structures to T_n (and keeping the original Banach space norm) by considering the embedding into the quantum groups $A(n)$ and $A_0(n)$ of Wang [**26**]. Based on a suggestion of his we have checked that these matrix norm structures differ from the ordinary matrix norm structures.

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