ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 1, Winter 1997

TRAVELLING FRONTS IN CYLINDERS AND THEIR STABILITY

JERROLD W. BEBERNES, COMGMING LI AND YI LI

Dedicated to Louis Nirenberg on the occasion of his 70th birthday

1. Introduction. We study the qualitative properties of travelling fronts of the semilinear parabolic equation

$$\partial u/\partial t - \Delta u = f(u), \quad (t, x) \in (0, \infty) \times \Sigma,$$

where $\Sigma = \mathbf{R} \times \Omega$ with $\Omega \subset \mathbf{R}^{n-1}$ being a bounded smooth domain and $n \geq 2$. We often denote $x \in \Sigma$ by $x = (x_1, y)$ with $x_1 \in \mathbf{R}$ and $y \in \Omega$, and the outer unit normal to $\partial\Omega$ or to $\partial\Sigma$ by ν.

In the above equation the term $f(u)$ represents a source term with $f(0) = f(1) = 0$. Equations as above have been derived to model problems arising from applied sciences, such as population dynamics, genetics, combustion and flame propagation. In these situations one of the most interesting and natural questions is the behavior of solutions $u(x, t)$ as $t \to +\infty$; in particular, the question about the existence of travelling fronts, and whether the general solutions approach travelling fronts (i.e., the stability of travelling fronts). Travelling fronts are solutions of the form $u = u(x_1 + ct, y)$ satisfying $u \to 0$ or 1 as $x_1 \to -\infty$ or $+\infty$, respectively (here c is a real constant and is usually referred to as the speed of the front). In the past several decades, the questions of existence, nonexistence and stability of travelling fronts have attracted the attention of many mathematicians, leading to the production of a large literature on the subject. The interested readers may refer to the book [**17**] by P. Fife, the paper [**14**] by H. Berestycki and L. Nirenberg and the papers [**37, 38**] by A. Volpert (see also, [**1, 2, 4**–**13**, **15**–**26**, **29**–**36**, **39**–**42**]) for the history of problems related to travelling fronts.

Throughout this paper the homogeneous Neumann boundary conditions on $\partial\Sigma$ are assumed. Therefore, the following equation must be

Received by the editors on June 16, 1995.
The first author was supported by NSF DMS#9417238.
The second author was supported by NSF DMS#9401441 and a University of Colorado CRCW Junior Faculty Award.
The third author was supported by NSF DMS#9225145.

satisfied by (c,u):

$$(1.1)\qquad \begin{cases} \Delta u - c(\partial u/\partial x_1) + f(u) = 0 & \text{on } \Sigma, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Sigma, \\ u(-\infty, y) = 0, \quad u(+\infty, y) = 1, & \\ 0 < u < 1 & \text{on } \Sigma. \end{cases}$$

H. Berestycki and L. Nirenberg carried out the first systematic study of equation (1.1) and its generalization in higher dimension, $n \geq 2$, in a sequence of papers. Concerning the existence and the asymptotic behavior of travelling fronts of (1.1), they proved

Theorem A [**14**, pp. 503–504]. *Assume that $f \in C^{1,\alpha}([0,1], \mathbf{R})$ for some $\alpha \in (0,1)$ and $f(0) = f(1) = 0$, $f'(1) < 0$. Then*

(a) *If $f > 0$ in $(0,1)$, then there exists $c^* > 0$ such that there exists a solution u of* (1.1) *if and only if $c \geq c^*$. For every $c \geq c^*$, there is a solution with $\partial u/\partial x_1 > 0$ in Σ. Furthermore, if $f'(0) > 0$, then for each fixed $c \geq c^*$ the solution u is unique* (*modulo translation in the x_1 direction*) *and it decays exponentially at $x_1 = \pm\infty$.* (*See $D_\pm$ below.*)

(b) *If for some $\theta \in (0,1)$ $f = 0$ in $(0,\theta)$, $f > 0$ in $(\theta, 1)$, then there exists a unique solution (c,u) of* (1.1) *with $\partial u/\partial x_1 > 0$, i.e., if (c', u') is also a solution; then $c' = c$ and $u'(x_1, y) = u(x_1 + \tau, y)$ for some real τ. Furthermore, it decays exponentially at $x_1 = \pm\infty$.* (*See $D_\pm$ below.*)

(c) *If for some $\theta \in (0,1)$ $f < 0$ in $(0,\theta)$ and $f > 0$ in $(\theta,1)$ and Ω is convex, then there exists a solution (c,u) of* (1.1) *which decays exponentially at $x_1 = \pm\infty$ and $\partial u/\partial x_1 > 0$. Furthermore, if $f'(0) < 0$ or $f'(0) = 0$ and $\int_0^1 f(s)\,ds > 0$, then the solution (c,u) is unique.*

Remark 1.1. Actually in many cases [**14**] allows more general source term f than being $C^{1,\alpha}$. The requirement of $C^{1,\alpha}$ is for the simplicity of the statements.

Case (a) with $f'(0) = 0$ occurs in many models and deserves further study. In particular, we want to distinguish fast decay or stable travelling fronts from others. Note also that Case (a) with $f'(0) = 0$ is between Case (a) with $f'(0) > 0$ and Case (b) in some sense where all solutions are classified and they are all exponential decay solutions.

An open question about the structure of solutions of (1.1) for Case (a) with $f'(0) = 0$ was raised by H. Berestycki and L. Nirenberg in [**14**, p. 505]. In this paper we are able to show that (c^*, u^*) is the unique exponential decay solution which is also stable for compact supported initial-values (please see Theorems 1 and 4 for details) and all other solutions (c, u) with $c > c^*$ are not exponential decay solutions. This result implies in some sense that c^* is the "preferred" or natural speed at which solutions travel. The same result indicates in a certain sense that nonexponential decay solutions are unstable.

As in [**14**], in some applications, the dependence on c in (1.1) may be different, and the source term f may be spatially inhomogeneous. Therefore, we also treat a more general problem:

$$\begin{cases} \Delta u - \beta(c,y)(\partial u/\partial x_1) + f(y,u) = 0 & \text{on } \Sigma, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Sigma, \\ u(-\infty, y) = 0, \quad u(+\infty, y) = 1, & \\ 0 < u < 1 & \text{on } \Sigma. \end{cases} \tag{1.2}$$

We always assume throughout this paper that

(C1) $\beta(c, y)$ is continuous in $(c, y) \in \mathbf{R} \times \overline{\Omega}$ such that $\beta(0, y) \equiv 0$ and $\beta(c, y)$ is strictly increasing in c with $\beta(-\infty, y) = -\infty$ and $\beta(+\infty, y) = +\infty$ uniformly in $y \in \overline{\Omega}$.

(C2) $f \in C^{1,\alpha}(\overline{\Omega} \times [0,1], \mathbf{R})$ for some $\alpha \in (0,1)$ and $f(y, 0) = f(y, 1) \equiv 0$ in $\overline{\Omega}$. Let $k := \|f\|_{C^{1,\alpha}}$.

Definition 1. A *solution* u of (1.2) is said to decay exponentially at $-\infty$ if there exists some $\alpha_0 > 0$ and $M > 0$ such that

$$e^{-\alpha_0 x_1} u(x_1, y) \le M \quad \text{on } (-\infty, 0) \times \Omega. \tag{D$_-$}$$

Similarly, it decays exponentially at $+\infty$ if

$$e^{\alpha_0 x_1}(1 - u(x_1, y)) \le M \quad \text{on } (0, \infty) \times \Omega. \tag{D$_+$}$$

And any solution satisfying both $D_\pm$ is said to *decay exponentially.*

Definition 2. For a continuous $\Psi(y)$ on $\overline{\Omega}$, $\mu_1(\Psi)$ is defined to be the *principal eigenvalue* of the "linearized equation" of (1.2) at Ψ in y

$$\begin{cases} -\Delta_y \varphi - f_u(y, \Psi(y))\varphi = \mu\varphi & \text{on } \Omega, \\ \partial\varphi/\partial\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Our first main result concerns the existence of exponential decay solutions of (1.2). In particular, we prove the existence, uniqueness and monotonicity of exponential decay solutions in Theorem 2 under some mild nondegenerated conditions on f. In Theorem 1 we give a rather general study on the qualitative properties of exponential decay solutions. Theorem 4 treats the stability of exponential decay travelling fronts.

Because of Theorem 3 below, to have an exponential decay solution, one of $\mu_1(0), \mu_1(1)$ must be positive. Without loss of generality we will assume the following condition when we deal with exponential decay solutions.

$$\text{(C3)} \qquad \mu_1(1) > 0.$$

For otherwise we may use the transformation $v(x_1, y) = 1 - u(-x_1, y)$. Theorem A (a) assures the existence of exponential decay solution under the condition $\mu_1(0) < 0$. We now deal with the case $\mu_1(0) = 0$. More precisely we sometimes assume

(C4) $f(y, s) \geq 0$ on $\overline{\Omega} \times [0, 1]$ and $\max_{y \in \overline{\Omega}} f(y, s) > 0$ for every $s \in (0, 1)$, and $\mu_1(0) = 0$.

Theorem 1. *Assume* (C1)–(C4). *Then there exists a unique* $c^* > 0$ *such that* (1.2) *has an exponential decay solution* (c^*, u^*) *with* $\partial u/\partial x_1 > 0$*, and there exists a solution* (c, u) *of* (1.2) *if and only if* $c \geq c^*$.

Furthermore u^* *is the unique solution for* $c = c^*$ *(modulo translation) and solutions* (c, u) *of* (1.2) *for* $c > c^*$ do not *decay exponentially at* $-\infty$.

Theorem 2. *Assume, in addition to* (C1)–(C3) *that* $\mu_1(0) \geq 0$. *Then the exponential decay solution of* (1.2) *is unique (if* (c, u) *and* (c', u') *are two such solutions, then* $c = c'$ *and* $u(x_1, y) = u'(x_1 + \tau, y)$ *for some real* τ*).*

Next we will prove a set of nonexistence of exponential decay solutions.

Theorem 3. (i) *If* $\mu_1(0) \leq 0$ *and*

$$\int_\Omega \int_0^1 f(y,s)\, ds\, dy \leq 0, \tag{1.4}$$

then (1.2) *does not have any solution decaying exponentially at* $-\infty$ (D_-).

(ii) *If* $\mu_1(1) \leq 0$ *and* $\int_\Omega dy \int_0^1 f(y,s)\, ds \geq 0$, *then* (1.2) *does not have any solution* u *decaying exponentially at* $+\infty$ (D_+).

(iii) *If* $\mu_1(0) \leq 0$ *and* $\mu_1(1) \leq 0$, *then* (1.2) *does not have any exponential decay solution in* Σ.

In [**14**] the following results on the asymptotic behavior of solutions of (1.2) are given.

Theorem B [**14**, Theorem 3.1]. (i) *Any solution* u *of* (1.2) *decays exponentially at* $-\infty$ *if either* $\mu_1(0) \neq 0$ *or* $\mu_1(0) = 0$ *with* $f(y,s) - f_u(y,0)s \leq 0$ *near* $s = 0$ *and* $\int_\Omega dy \int_0^1 f(y,s)\, ds > 0$.

(ii) *Any solution* u *of* (1.2) *decays exponentially at* $+\infty$ *if either* $\mu_1(1) \neq 0$ *or* $\mu_1(1) = 0$ *with* $f(y,1-s) + f_u(y,1)(1-s) \geq 0$ *near* $s = 0$ *and* $\int_\Omega dy \int_0^1 f(y,s)\, ds < 0$.

Combining the above results with Theorem B we are able to prove the nonexistence of solution (1.2) in several situations.

Corollary 1. (i) *If* $\mu_1(0) < 0$, *then* $\int_\Omega dy \int_0^1 f(y,s)\, ds \leq 0$ *implies that* (1.2) *possesses no solution.*

(ii) *If* $\mu_1(1) < 0$, *then* $\int_\Omega dy \int_0^1 f(y,s)\, ds \geq 0$ *implies that* (1.2) *possesses no solution.*

(iii) *Assume that* $\mu_1(0) < 0$. *Also assume that* $\mu_1(1) < 0$. *Then* (1.2) *possesses no solution.*

Remark 1.2. The conditions in the above corollary are optimal in the sense that if $f(y,s) = f(s) > 0$ in $(0,1)$ so that $f'(0) \geq 0$ and $\int_\Omega dy \int_0^1 f(s)\, ds > 0$, then Theorem A (a) in [**14**] proves the existence of solutions of (1.1) for all c sufficiently large.

Finally we come to the stability of exponential decay travelling front solutions. Consider the following parabolic equation

$$
\begin{cases} u_t - \Delta u = f(y,u) & \text{in } (0,\infty)\times\Sigma, \\ \partial u/\partial\nu = 0 & \text{on } (0,\infty)\times\partial\Sigma, \\ u(0,x) = \phi(x). \end{cases} \tag{1.5}
$$

Theorem 4. *Assume* (C1)–(C2) *and that* $f_u(y,0)\le 0$, $f_u(y,1)<0$, $\int_\Omega dy\int_0^1 f(y,s)\,ds > 0$ *and* (1.5) *has a travelling front* (c^*,u^*) *that decays exponentially. Assume also that the initial value* $\phi(x)$ *is compact supported and satisfies* $\phi(x_1-\xi_0,y)\ge\underline{\alpha}_0$, *for some* $\underline{\alpha}_0\in(0,1)$, $R>0$, $\xi_0\in\mathbf{R}$, $y\in\Sigma$ *and* $|x_1|\le R$. *Then there exist positive constants* β, q_0 *and* $\xi_1,\xi_2\in\mathbf{R}$ *such that for* $(t,x)\in(0,\infty)\times\Sigma$,

$$
\begin{aligned} u^*(x_1+c^*t+\xi_1,y) + u^*(-x_1+c^*t+\xi_1,y) - 1 - q_0e^{-\beta t} &\\ \le u(t,x_1,y)&\\ \le u^*(x_1+c^*t+\xi_2,y)&\\ + u^*(-x_1+c^*t+\xi_2,y) - 1 + q_0e^{-\beta t}.& \end{aligned} \tag{1.6}
$$

Please refer to Lemma 4.3 *for the precise definition of* $\underline{\alpha}_0$ *and* R.

Remark 1.3. Inequality (1.6) shows that the x_1-profile of $u(t,x)$ approaches that of the travelling fronts. In particular, it shows that the interval on which u is close to 1 is expanding at the speed of c^*.

Remark 1.4. If $\int_\Omega\int_0^1 f(y,s)\,ds\,dy < 0$, then an analogous stability result can be obtained (where the solution approaches 0 at the speed c^*).

Remark 1.5. The stability in the case of one spatial dimension, $n=1$, is treated in [**3**].

This paper is organized as follows. In Section 2 we give a proof of Theorem 3 and Corollary 1. In Section 3, Theorem 2 and Theorem 1 are proved; in addition, we state and prove several propositions using the method developed in [**14, 27, 28**]. Section 4 is devoted to the stability of exponential decay travelling fronts.

2. Nonexistence results. In this section we prove Theorem 3 and its corollaries.

Proof of Theorem 3.

Part (i) *of Theorem* 3. Assume that u is a solution of (1.2) which satisfies (D_-) for some $c \in \mathbf{R}$, $M > 0$ and $\alpha_0 > 0$. We can apply the L^p estimates to get

$$\lim_{|x_1|\to\infty} \sup_{\Omega} \left| \frac{\partial u_t}{\partial x_1}(x_1, y) \right| = 0. \tag{2.1}$$

Now multiplying (1.2) by $\partial u/\partial x_1$ and integrating it over Σ, we get

$$-\int_\Sigma \beta(c,y)\left(\frac{\partial u}{\partial x_1}\right)^2 + \int_\Omega dy \int_0^1 f(y,s)\,ds = 0, \tag{2.2}$$

which gives by (1.4)

$$\int_\Sigma \beta(c,y)\left(\frac{\partial u}{\partial x_1}\right)^2 = \int_\Omega dy \int_0^1 f(y,s)\,ds \le 0.$$

Hence (C1) implies that $\beta(c,y) \le 0$.

In $\Sigma^- = (-\infty, 0) \times \Omega$, we have

$$f(y, u(x_1, y)) = f_u(y,0)u(x_1,y) + d(x)u^{1+\alpha}(x_1,y), \tag{2.3}$$

where $d(x)$ may depend on u but $\|d\|_{L^\infty} \le \|f\|_{C^{1,\alpha}} \equiv k$.

Let φ be a positive eigenfunction of (1.3) for $\psi \equiv 0$ with $\mu_1(0) \le 0$ and $u(x) = \varphi(y)w(x)$. It is easy to see that:

$$0 < w(x) = 0(e^{\alpha_0, x_1}) \quad \text{at } x_1 = -\infty, \tag{2.4}$$

and

$$\Delta w - \beta(c,y)\frac{\partial w}{\partial x_1} + 2\frac{\nabla_y\varphi\nabla_y w}{\varphi} + d(x)\varphi^\alpha w^{1+\alpha} - \mu_1(0)w = 0. \tag{2.5}$$

Now for some R large, we have

$$\alpha_0^2/4 + d(x)\varphi^\alpha(y)w^\alpha(x) - \mu_1(0) > 0 \quad \text{in } (-\infty, -R] \times \overline{\Omega}$$

since $w \to 0$ as $x_1 \to -\infty$ and $\mu_1(0) \leq 0$.

Therefore w satisfies

$$(2.6)\quad \Delta w - \beta(c, y_1)\frac{\partial w}{\partial x_1} + 2\frac{\nabla_y \varphi \nabla_y w}{\varphi} - \frac{\alpha_0^2}{4} w < 0 \quad \text{in } (-\infty, -R] \times \Omega.$$

But $e^{(\alpha_0/2)x_1}$ is a subsolution of (2.6) since $\beta(c, y) \leq 0$. The maximum principle and the Hopf boundary lemma imply that

$$w(x) - \Big\{\min_{y \in \bar{w}} w(-R, y)\Big\} e^{(\alpha_0/2)(x_1+R)} > 0 \quad \text{in } (-\infty, -R] \times \overline{\Omega},$$

which is a contradiction to (2.4). Part (i) of Theorem 3 is proved.

Part (ii) *and* (iii) *of Theorem* 3. Let $v(x_1, y) = 1 - u(-x_1, y)$. Then v satisfies (i) if u satisfies (ii). This proves part (ii). Part (iii) is a direct consequence of (i) and (ii). Theorem 3 is proved. □

Corollary 1 is a direct consequence of Theorem 3 and Theorem B.

3. Existence, uniqueness and monotonicity properties of exponential decay solutions. We prove Theorems 1 and 2, always assuming (C1)–(C3) in this section.

Lemma 3.1. *If* (c, u) *and* (c, u') *are two exponential decay solutions of* (1.2) *with the same* c, *then* $u'(x_1, y) = u(x_1 + \tau_1, y)$ *for some* $\tau \in \mathbf{R}$. *Furthermore,* u *is increasing in* x_1, *namely* $\partial u/\partial x_1 > 0$ *in* Σ.

Proof. Let (c, u) and (c, u') satisfy $(\mathrm{D}_\pm)$ for some $\alpha_0 > 0$. Theorem 3.2 in [**14**] shows that there exist some $c_1 > 1$, $\alpha > 0$ such that

$$(3.1)\quad \begin{aligned} c_1^{-1} e^{-\alpha x_1} \leq u(-x_1, y),\ (1 - u(x_1, y)),& \\ u'(x_1, y), \quad \text{and} \quad u'(-x_1, y) \leq c_1 e^{-\alpha x_1},& \\ \text{in } \Sigma^+ = (0, \infty) \times \overline{\Omega}.& \end{aligned}$$

Therefore, by Theorem 4.1 in [**14**], we have the following asymptotic

expansions:
(3.2)

$$\begin{cases} u(x_1,y)=\alpha_1 e^{\lambda_1 x_1}\phi_1(y)+o(e^{\lambda_1 x_1}) & \text{at } x_1=-\infty,\\ u'(x_1,y)=\alpha_2 e^{\lambda_1 x_1}\phi_1(y)+o(e^{\lambda_1 x_1}) & \text{at } x_1=-\infty,\\ \text{or} & \\ u(x_1,y)=\alpha_1 e^{\lambda_1 x_1}(\phi_2(y)(-x_1)+\phi_3(y))+o(e^{\lambda_1 x_1}) & \text{at } x_1=-\infty,\\ u'(x_1,y)=\alpha_2 e^{\lambda_1 x_1}(\phi_2(y)(-x_1)+\phi_4(y))+o(e^{\lambda_1 x_1}) & \text{at } x_1=-\infty, \end{cases}$$

$$\begin{cases} 1-u(x_1,y)=\alpha_3 e^{-\lambda_2 x_1}\phi_5(y)+o(e^{-\lambda_2 x_1}) & \text{at } x_1=+\infty,\\ u'(x_1,y)=\alpha_4 e^{-\lambda_2 x_1}\phi_5(y)+o(e^{-\lambda_2 x_1}) & \text{at } x_1=+\infty. \end{cases} \tag{3.3}$$

where ϕ_1, ϕ_2 and ϕ_5 are smooth, positive functions on $\bar{\Sigma}$.

These asymptotic expansions imply that the proof of Theorem 7.1 in [**14**] can be applied here to prove Lemma 3.1. □

Theorem 3.1. *In addition to* (C1)–(C3), *suppose that* $\mu_1(0) \geq 0$. *If* (c_0, u_0) *is an exponential decay solution of* (1.2), *then there exists no solution of* (1.2) *for* $c < c_0$.

Proof. By Lemma 3.1, we have $\partial u_0/\partial x_1 > 0$ in Σ and Theorem 3 implies that if $\mu_1(0) = 0$, then

$$\int_\Omega dy \int_0^1 f(y,s)\,ds > 0. \tag{3.4}$$

This, (C1) and (2.2) imply that $c_0 > 0$. Thus we proved that either $c_0 > 0$ or $\mu_1(0) > 0$. Now suppose to the contrary that (1.2) has a solution (c, u) for some $c < c_0$.

Let $\varphi_0(y)$ and $\varphi_1(y) > 0$ be the eigenfunctions of (1.3) with respect to $\mu_1(0)$ and $\mu_1(1)$. Choose $\delta > 0$, $\varepsilon \in [0, \alpha_0/2)$, $L_0 > 0$ so that

$$\begin{aligned} &|f(y,1-s)+f_u(y,1)(1-s)|+|f(y,s)-f_u(y,0)s| \\ &\qquad \leq k\delta^\alpha \quad \text{for } s\in[0,\delta], \end{aligned} \tag{3.5}$$

$$u_0(x_1,y)+1-u_0(-x_1,y) \leq \delta/2 \quad \text{for } x_1 \leq -L_0 \tag{3.6}$$

$$\begin{cases} \varepsilon^2 - \varepsilon\beta_- + k\delta^\alpha - \mu_1(0) \le -(\varepsilon^2 + \mu_1(0)/2) \\ k\delta^\alpha - \mu_1(1) \le -\mu_1(1)/2 \end{cases} \tag{3.7}$$

where $\beta_- = \min_{y\in\overline{\Omega}} \beta(c,y)$.

Note that if $\mu_1(0) > 0$ then one can simply take $\varepsilon = 0$, and if $\mu_1(0) = 0$ then $\beta_- > 0$ which also ensures the existence of $\varepsilon > 0$ and $\delta > 0$ satisfying (3.7).

Define $w^R(x_1, y) = u(x_1 + R, y) - u_0(x_1, y)$.

Notice that

$$\lim_{R\to+\infty} w^R(x_1,y) = 1 - u_0(x_1,y) > 0$$

and

$$\lim_{R\to-\infty} w^R(x_1,y) = -u_0(x_1,y) < 0$$

uniformly on $[-L_0, L_0] \times \overline{\Omega}$.

Therefore there is an $R_0 > 0$ such that, for $R > R_0$,

$$u(x_1 + R, y) \ge 1 - \delta/2 \quad \text{for } x_1 \ge L_0,$$

and

$$w^R(x_1,y) > 0 \quad \text{and} \quad w^{-R}(x_1,y) < 0 \quad \text{on } [-L_0, L_0] \times \overline{\Omega}.$$

Let

$$w^R(x_1,y) = \Phi(x_1,y)v^R(x_1,y) \quad \text{on } (-\infty,+\infty) \times \overline{\Omega}, \tag{3.8}$$

where

$$\Phi(x_1,y) = \begin{cases} e^{\varepsilon x_1}\varphi_0(y) & x_1 \le -L_0, \\ \varphi_1(y) & x_1 \ge L_0 \\ \text{positive and smooth} & \text{otherwise.} \end{cases} \tag{3.9}$$

Let $\Sigma^R = \{(x_1,y) \in \Sigma \mid w^R(x_1,y) < 0\}$. Using the fact that $\beta_- \le \beta(c,y) < \beta(c_0,y)$ and $\partial u_0/\partial x_1 > 0$, we obtain, for some $\theta \in (0,1)$,

$$\Delta w^R - \beta(c,y)\frac{\partial w^R}{\partial x_1} + f_u(y, \theta u^R + (1-\theta)u_0)w^R < 0,$$

and

$$\Delta v^R - \beta(c,y)\frac{\partial v^R}{\partial x_1} + 2\frac{\nabla\Phi\nabla v^R}{\Phi} + d(x_1,y)v^R < 0 \tag{3.10}$$

in Σ^R, where

$$\begin{aligned} d(x_1,y) &= \frac{\Delta\Phi}{\Phi} - \beta(c,y)\frac{\partial\Phi}{\partial x_1} + f'_u(y,\theta u^R + (1-\theta)u_0) \\ &\leq -\frac{\min(\mu_1(0),\mu_1(1))}{2} \end{aligned} \tag{3.11}$$

for $|x_1| \geq L_0$ on Σ^R. For $R \geq R_0$, $|x_1| \leq L_0$, we have

$$w^R(x_1,y) > 0.$$

It is easy to check that in Σ^R

$$\lim_{|x_1|\to+\infty} v^R(x_1,y) \geq 0. \tag{3.12}$$

Thus, if Σ^R is not empty, we can find

$$(\bar{x}_1,\bar{y}) \in \Sigma^R,$$

such that

$$\begin{aligned} v^R(\bar{x}_1,\bar{y}) = \min_{(x_1,y)\in(-\infty,+\infty)\times\overline{\Omega}} v^R(x_1,y) < 0 \\ \text{for } |\bar{x}_1| > L_0. \end{aligned} \tag{3.13}$$

Combined with the fact that $\partial v^R/\partial\nu = 0$ on $(-\infty,\infty)\times\partial\Omega$, we get

$$\nabla v^R(\bar{x}_1,\bar{y}) = 0, \qquad \Delta v^R(\bar{x}_1,\bar{y}) \geq 0.$$

From (3.1), we derive

$$d(\bar{x}_1,\bar{y})v^R(\bar{x}_1,\bar{y}) < 0,$$

which contradicts (3.11) and (3.13). Thus, we have proved that $v^R \geq 0$ or $w^R \geq 0$ for $R \geq R_0$. Strong maximal principle implies that $w^R > 0$.

Now let $\overline{R}$ be the smallest number such that $v^R \geq 0$ for $R \geq \overline{R}$. Obviously, $\overline{R} > -R_0$. Continuity and the maximum principle imply $v^{\overline{R}}(x_1, y) > 0$ since it cannot be identically 0. In particular,

$$\begin{aligned} 1 - u(x_1 + \overline{R}, y) &\leq 1 - u_0(x_1, y) \\ &\leq 1 - \frac{\delta}{2} \quad \text{for } x_1 \geq L_0. \end{aligned}$$

Thus, there exists some $\varepsilon_1 > 0$ such that

$$1 - u(x_1 + R, y) \leq 1 - \delta \quad \text{for } x_1 \geq L_0,$$

and $v^R(x_1, y) > 0$ for $|x_1| \leq L_0$ for $R \geq \overline{R} - \varepsilon$. A similar argument as before leads to

$$v^R(x_1, y) > 0 \quad \text{for } R \geq \overline{R} - \varepsilon$$

on $(-\infty, +\infty) \times \Omega$ which contradicts the fact that $\overline{R}$ is the smallest one.

This completes the proof of Theorem 3.1. □

Remark 3.1. Theorem 3.1 holds even if $\mu_1(1) = 0$, since then $\mu_1(0) > 0$ by Theorem 3 for the existence of u_0. It is clear that Theorem 3.1 implies Theorem 2.

Next we will give a proof of Theorem 1 which will be divided into several steps.

Let $\varphi(s)$ be a smooth monotone function such that $\varphi((-\infty, 1]) \equiv 0$, $\varphi([2, +\infty)) \equiv 1$ and strictly positive otherwise and $f_\varepsilon(y, s) = \varphi(s/\varepsilon) f(y, s)$. First we have

Proposition 3.1. *Assume conditions* (C1)–(C4). *For each* $\varepsilon \in (0, 1/4)$*, there exists a unique exponential decay solution* $(c_\varepsilon, u_\varepsilon)$ *of* (1.2) *with* f *replaced by* f_ε *and* $\partial u_\varepsilon / \partial x_1 > 0$ *in* $(-\infty, \infty) \times \overline{\Omega}$.

The proof of Proposition 3.1 follows from the one given in [**14**], with some slight modifications. The solution for a given ε is constructed by solving the corresponding problems in finite cylinders $\Sigma_a = (-a, a) \times \Omega$ and then letting $a \to \infty$.

Proposition 3.2. *Assume conditions* (C1)–(C4). *Then there exists a sequence* $\varepsilon_n \to 0$ *such that* $(c_{\varepsilon_n}, u_{\varepsilon_n})$ *converges to a solution* (c_0, u_0) *of* (1.2) *with* $c_0 \in \mathbf{R}$. *Furthermore,* u_0 *decays exponentially.*

Proof of Proposition 3.2. First we show that c_ε are bounded. By translation along the x_1 direction we may assume that $\max_\Omega u_\varepsilon(0, y) = \delta$ for some fixed $\delta \in (0, 1)$. Similar to the proof of Theorem 3, we can conclude that $c_\varepsilon > 0$. Therefore, we only need to obtain an upper bound for c_ε. The arguments on page 559 of [**14**] apply here to give us the upper bound of c_ε with only the modification of f_θ in [**14**] being replaced by $H_\varepsilon \equiv \max_{y\in\overline{\Omega}} f_\varepsilon(y, s)$.

With the bounds on c_ε, we can apply the standard elliptic estimates to conclude that there exists a sequence $\varepsilon_n \to 0$ such that $(c_{\varepsilon_n}, u_{\varepsilon_n})$ converges to a solution (c_0, u_0) of

$$\begin{cases} \Delta u - \beta(c, y)(\partial u/\partial x_1) + f(y, u) = 0 & \text{on } \Sigma, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Sigma, \\ u(-\infty, y) = \psi_-(y) \le \delta \le u(+\infty, y) = \psi_+(y) \le 1 & \\ \partial u/\partial x_1 \ge 0, & \end{cases}$$

with $c_0 \in \mathbf{R}$. It is easy to see that $f(y, \psi_-) \equiv 0 \equiv f(y, \psi_+)$ and (C4) implies that $\psi_- = 0$ and $\psi_+ = 1$. Therefore u_0 is a solution of (1.2).

Let $\infty > \beta_+ \equiv \sup_{1<n<\infty, y\in\overline{\Omega}} \beta(c_{\varepsilon_n}, y) \ge \beta_- \equiv \inf_{1<n<\infty, y\in\overline{\Omega}} \beta((c_{\varepsilon_n}, y) > 0$. By (1.2) and (C4) we obtain

$$\Delta u_{\varepsilon_n} - \beta(c_{\varepsilon_n}, y)\frac{\partial u_{\varepsilon_n}}{\partial x_1} \le 0$$

and hence

$$\Delta u_{\varepsilon_n} - \beta_+ \frac{\partial u_{\varepsilon_n}}{\partial x_1} \le 0$$

since

$$\frac{\partial u_{\varepsilon_n}}{\partial x_1} > 0$$

which gives for $\bar{u}(x_1) \equiv \int_\Omega u(x_1, y)\, dy$ that

$$\bar{u}''_{\varepsilon_n} - \beta_+ \bar{u}'_{\varepsilon_n} \le 0 \tag{3.14}$$

and integrating it from $-\infty$ to x_1 we have

$$\bar{u}'_{\varepsilon_n} - \beta_+ \bar{u}_{\varepsilon_n} \le 0. \tag{3.15}$$

Note that since $f_u(y,0) \ge 0$ and $\mu_1(0) = 0$ we have $f_u(y,0) \equiv 0$ and 1 can be taken as the corresponding eigenfunction φ. Hence similar to (2.3) and (2.5) we have

$$\Delta u_{\varepsilon_n} - \beta_- \frac{\partial u_{\varepsilon_n}}{\partial x_1} + k u_{\varepsilon_n}^{1+\alpha} \ge 0 \quad \text{in } (-\infty, 0) \times \Omega.$$

Therefore we have

$$\Delta u_{\varepsilon_n} - \beta_- \frac{\partial u_{\varepsilon_n}}{\partial x_1} + k\delta^{\alpha} u_{\varepsilon_n} \ge 0 \quad \text{in } (-\infty, 0) \times \Omega$$

and hence

$$\bar{u}''_{\varepsilon_n} - \beta_- \bar{u}'_{\varepsilon_n} + k\delta^{\alpha} \bar{u}_{\varepsilon_n} \ge 0 \quad \text{in } (-\infty, 0) \times \Omega. \tag{3.16}$$

Now choose δ so that $k\delta^{\alpha} < \beta_-^2/4$.

Claim 1.

$$\frac{\beta_-}{2} \bar{u}_{\varepsilon_n} \le \bar{u}'_{\varepsilon_n} \le \beta_+ \bar{u}_{\varepsilon_n} \quad \text{in } (-\infty, 0) \times \Omega. \tag{3.17}$$

From (3.15) we need only to show the left half of (3.17).

Since $f_{\varepsilon_n} \equiv 0$ for $0 < u < \varepsilon_n$, there exists $x_{\varepsilon_n} < 0$ such that

$$\Delta u_{\varepsilon_n} - \beta(c_{\varepsilon_n}, y) \frac{\partial u_{\varepsilon_n}}{\partial x_1} = 0 \quad \text{in } (-\infty, x_{\varepsilon_n}) \times \Omega$$

and hence

$$\bar{u}''_{\varepsilon_n} - \beta_- \bar{u}'_{\varepsilon_n} \ge 0 \quad \text{in } (-\infty, x_{\varepsilon_n}) \times \Omega,$$

which implies

$$\bar{u}'_{\varepsilon_n} - \beta_- \bar{u}_{\varepsilon_n} \ge 0 \quad \text{in } (-\infty, x_{\varepsilon_n}) \times \Omega.$$

Define $R_{\varepsilon_n} = \sup\{R \in (-\infty, 0] \mid \bar{u}'_{\varepsilon_n}(x_1) \ge (\beta_-/2)\bar{u}_{\varepsilon_n}(x_1)$ for all $x_1 \le R\}$.

Note $R_{\varepsilon_n} \geq x_{\varepsilon_n}$. And we want to show $R_{\varepsilon_n} \equiv 0$. If not, say $R_{\varepsilon_n} < 0$. Then (3.16) and the definition of R_{ε_n} imply that

$$\begin{cases} \bar{u}'_{\varepsilon_n}(x) \geq (\beta_-/2)\bar{u}_{\varepsilon_n}(x_1) \\ \\ \bar{u}''_{\varepsilon_n}(x) - \beta_-\bar{u}'_{\varepsilon_n} + \kappa\delta^\alpha \bar{u}_{\varepsilon_n} \geq 0 \end{cases} \quad \text{in } (-\infty, R_{\varepsilon_n}] \times \Omega,$$

and hence

$$\bar{u}''_{\varepsilon_n} - \beta_-\bar{u}'_{\varepsilon_n} + \frac{2\kappa\delta^\alpha}{\beta_-}\bar{u}'_{\varepsilon_n} \geq 0.$$

And after integrating it from $-\infty$ to x_1, we have

$$\bar{u}'_{\varepsilon_n} \geq \left(\beta_- - \frac{2\kappa\delta^\alpha}{\beta_-}\right)\bar{u}_{\varepsilon_n} > \frac{\beta_-}{2}\bar{u}_{\varepsilon_n} \quad \text{in } (-\infty, R_{\varepsilon_n}]$$

which contradicts the definition of R_{ε_n} and the claim is proved. From (3.17), we have for $\bar{\delta} = \int_\Omega u(0,y)\,dy \simeq \delta$ that

$$\bar{\delta}e^{\beta_+ x_1} \leq \bar{u}_{\varepsilon_n}(x_1) \leq \bar{\delta}e^{(\beta_-/2)x_1} \quad \text{in } (-\infty, 0).$$

Now the L^p estimates imply that, for some $D > 1$ independent of u_{ε_n} such that

$$u_{\varepsilon_n}(x_1, y) \leq D\delta e^{(\beta_-/2)x_1} \quad \text{in } \Sigma^-, \tag{3.18}$$

and taking the limit as $\varepsilon_n \to 0^+$, we have

$$u_0(x_1, y) \leq D\delta e^{(\beta_-/2)x_1} \quad \text{in } \Sigma^-,$$

which shows that $u_0(x_1, y)$ decays exponentially at $-\infty$. Exponential decay of $u_0(x_1, y)$ at $+\infty$ is a direct consequence of (C3) and Theorem B. Thus, the proof of Proposition 3.2 is completed. □

Proposition 3.3. *Case* (1.2) *has a solution* (c, u) *if* $c \geq c_0$.

In Section 9.1 of [**14**], H. Berestycki and L. Nirenberg proved the existence of solutions in the case f independent of y. However, their proof can be used to prove Proposition 3.3 without any change.

Proof of Theorem 1. Proposition 3.2 shows that (c_0, u_0) is an exponential decay travelling front. Theorem 3.1 proves that if $c < c_0$ then (1.2) has no solution. Proposition 3.3 implies that $c_0 = c^*$. And Theorem 2 means that any solution of (1.2) with $c > c^*$ cannot decay exponentially at $-\infty$. Theorem 1 is proved. □

4. Stability. In this section we will give a proof of Theorem 4 in the introduction, namely, the stability of travelling fronts with exponential decays at $x_1 = \pm\infty$. Hence the hypotheses of Theorem 4 are assumed throughout this section.

Under the assumptions of Theorem 4, we have $\mu_1(1) > 0$ by Theorem 3, and then Theorem 3.1 implies that the exponential decay solution is unique. Let us assume that (c^*, u^*) is the unique travelling front with exponential decay of

$$\begin{cases} \Delta u^* - c^*(\partial u^*/\partial x_1) + f(y, u^*) = 0 & \text{in } \Sigma, \\ u^*(-\infty) = 0, \quad u^*(+\infty) = 1 & \\ 0 < u^* < 1 & \text{in } \Sigma, \end{cases} \tag{4.1}$$

where $f_u(y, 0) \le 0$ and without loss of generality we may assume that $f_u(y, 1) \le -1$. Then we have that $\partial u^*/\partial x_1 > 0$ for all $x \in \Sigma$, and

$$u^*(x_1, y) \le k_0 e^{\alpha_0 x_1} \quad \text{in } (-\infty, 0) \times \Omega \tag{4.2}$$

$$1 - u^*(-x_1, y) \le k_0 e^{\alpha_0 x_1} \quad \text{in } (-\infty, 0) \times \Omega \tag{4.3}$$

for some $\alpha_0, k_0 > 0$.

From $\int_\Omega dy \int_0^1 f(y, s)\, ds > 0$, we have $c^* > 0$. And, for convenience, we choose α_0, k_0 such that

$$\alpha_0 \le c^*, \qquad k = \|f\|_{C^{1,\alpha}} \le k_0, \tag{4.4}$$

and hence

$$\begin{cases} f(y, a) + f(y, b) - f(y, a + b) \le k_0 a^\alpha b \\ f(y, 1 - a) + f(y, 1 - b) - f(y, 1 - a - b) \le k_0 a^\alpha b \end{cases} \tag{4.5}$$

for $a, b, a + b \in [0, 1]$.

By suitable rescaling of the variables, we may assume that $f_u(y,0) \le -1$ for $y \in \Sigma$. Let $\delta_0 > 0$ be such that

$$f_u(y,s) \le -\frac{1}{2} \quad \text{if } s \in [1-\delta_0, 1] \tag{4.6}$$

and

$$f_u(y,s) \le \frac{\theta c^*}{8} \quad \text{if } s \in [0, \delta_0] \tag{4.7}$$

where

$$\theta = \min\left\{\frac{c^*}{2}, \frac{\alpha_0}{4}, \frac{1}{5c^*/2 + \sqrt{4 + 25c^{*2}/4}}\right\}. \tag{4.8}$$

After a translation in the x_1-direction, we may assume that

$$\min_{y \in \overline{\Omega}} u^*(0,y) \ge 1 - \left(\frac{\delta_0}{12k_0^2}\right)^{1/\alpha} \ge 1 - \frac{\delta_0}{3}. \tag{4.9}$$

Let $M > 1$ such that

$$u^*(x) \le \frac{\delta_0}{2} \quad \text{in } (-\infty, -M) \times \overline{\Omega}, \tag{4.10}$$

and we define here some positive constants

$$\begin{aligned} l &= \min_{-M \le x_1 \le 1} \frac{\partial u^*}{\partial x_1}(x) > 0, \\ \beta &= \min\left\{\frac{1}{4}, \frac{\theta c^*}{4}, \alpha_0 c^*, \frac{\alpha_0}{4}\right\}, \qquad q_0 = \frac{\delta_0}{3}. \end{aligned} \tag{4.11}$$

For the construction of a lower solution of

$$\frac{\partial u}{\partial t} - \Delta u = f(y,u) \quad \text{in } (0,\infty) \times \Sigma, \tag{4.12}$$

we first let

$$q(x,t) = q_0 e^{-\beta t} \min[1, e^{-\theta x_1 - \theta c^* t}, e^{\theta x_1 - \theta c^* t}], \tag{4.13}$$

and

$$\zeta(t) = \zeta_0 e^{-\beta t} \quad \text{with } \zeta' \le 0, \tag{4.14}$$

where

$$\zeta_0 = \max\left\{ \frac{2k_0^2 + k_0\delta_0}{l\beta}, \left(\frac{48k_0^2}{\delta_0 \theta c^*}\right)^{2/\alpha_0^2} \right\}. \tag{4.15}$$

Then let

$$\begin{aligned} \underline{\theta}(x,t) &= u^*(-x_1 + c^*t + \zeta(t), y) \\ &\quad + u^*(x_1 + c^*t + \zeta(t), y) - 1 - q(x,t) \\ &= u^*_-(x,t) + u^*_+(x,t) - 1 - q(x,t), \end{aligned} \tag{4.16}$$

and

$$\underline{\alpha}(x,t) = \max\{0, \underline{\theta}(x,t)\}. \tag{4.17}$$

Lemma 4.1. *$\underline{\alpha}$ is a subsolution to* (4.12).

Proof. Since 0 is a solution of (4.12) and $\underline{\alpha}$ is symmetric in x_1, we only need to check when $x_1 \ge 0$ and $\underline{\theta}(x,t) > 0$. Now let $N(w) = \partial w/\partial t - \Delta w - f(w)$ and $z = -x_1 + c^*t$.

Case 1. $z + \zeta(t) \ge 0$, $x_1 \ge 0$. In this case, we have

$$\begin{aligned} u^*_+ \ge u^*_- &\ge u^*_+ + u^*_- - 1 - q \ge 2u^*(0,y) - 1 - \delta_0/3, \\ &\ge 2\left(1 - \frac{\delta_0}{3}\right) - 1 - \frac{\delta_0}{3} = 1 - \delta_0 \quad \text{by (4.9)}. \end{aligned}$$

Then we obtain

$$\begin{aligned} f(y,u^*_+) + f(y,u^*_-) - f(y,\underline{\theta}) \qquad & \\ &= f(y,u^*_+) + f(y,u^*_-) - f(y, u^*_+ + u^*_- - 1) \\ &\quad + f(y, u^*_+ + u^*_- - 1) - f(y,\underline{\theta}) \\ &\le k_0(1-u^*_+)(1-u^*_-)^\alpha - q/2 \qquad \text{by (4.5) and (4.6)} \\ &\le k_0 k_0 e^{\alpha_0(-x_1 - c^*t - \zeta(t))}(1 - u^*(0))^\alpha - q/2 \\ &= (3k_0^2/\delta_0)(1-u^*(0))^\alpha q_0 e^{-\alpha_0(x_1 + c^*t + \zeta(t))} - q/2 \\ &\le q/4 - q/2 = -q/4, \end{aligned} \tag{4.18}$$

since $\alpha_0(x_1 + c^* t + \zeta(t)) \geq \max\{\beta t, \beta t + \theta x_1 + \theta c^* t, \beta t - \theta x_1 + \theta c^* t\}$ by (4.8), (4.11) and $(3k_0^2/\delta_0)(1 - u^*(0))^\alpha \leq 1/4$ from (4.9).

On the other hand, we derive easily by (4.13) that

$$\frac{\partial q}{\partial t} - \Delta q \geq -(\beta + \theta c^* + \theta^2) q \geq -q/4$$

by our choice of θ. Therefore we have

$$\begin{aligned} N(\underline{\theta}) = \zeta' \Big(& \frac{\partial u^*}{\partial x_1}(-x_1 + c^* t + \zeta(t), y) + \frac{\partial u^*}{\partial x_1}(x_1 + c^* t + \zeta(t), y) \Big) \\ & + f(y, u_-^*) + f(y, u_+^*) - f(y, \underline{\theta}) - \left(\frac{\partial q}{\partial t} - \Delta q \right) \\ \leq & -\frac{1}{4} q + \frac{1}{4} q = 0. \end{aligned}$$

Case 2. $-M \leq z + \zeta(t) < 0$, $x_1 \geq 0$. In this region we have $(\partial u^*/\partial x_1)(-x_1 + c^* t + \zeta(t), y) \geq l > 0$ and $z < 0$ since $\zeta > 0$. Hence, by (4.13) we have

$$q(x, t) = q_0 e^{-\beta t} e^{\theta z} = q_0 e^{(\theta c^* - \beta) t - \theta x_1}$$

such that

$$\begin{aligned} \partial q/\partial t - \Delta q = (\theta c^* - \beta - \theta^2) q & \geq \left(\frac{\theta c^*}{2} - \beta \right) q \\ & \geq \frac{\theta c^*}{4} q \geq 0 \end{aligned} \tag{4.19}$$

by (4.8), (4.11) and as before we obtain

$$\begin{aligned} f(y, u_+^*) + f(y, u_-^*) - f(y, \underline{\theta}) & \leq k_0 (1 - u_+^*)(1 - u_-^*)^\alpha \\ & \quad + \|f_u\|_{L^\infty} q \\ & \leq k_0 (1 - u_+^*) + k_0 q \\ & \leq k_0^2 e^{-\alpha_0 (x_1 + c^* t + \zeta(t))} \\ & \quad + k_0 q_0 e^{-\beta t + \theta z} \\ & \leq (k_0^2 + k_0 q_0) e^{-\beta t}. \end{aligned} \tag{4.20}$$

Therefore we derive that

$$\begin{aligned} N(\underline{\theta}) &= \zeta' \left(\frac{\partial u^*}{\partial x_1}(-x_1 + c^* t + \zeta(t), y) + \frac{\partial u^*}{\partial x_1}(x_1 + c^* t + \zeta(t), y) \right) \\ &\quad + f(y, u_-^*) + f(y, u_+^*) - f(y, \underline{\theta}) - \left(\frac{\partial q}{\partial t} - \Delta q \right) \\ &\le \zeta' \frac{\partial u^*}{\partial x_1}(-x_1 + c^* t + \zeta(t), y) \\ &\quad + f(y, u_-^*) + f(y, u_+^*) - f(y, \underline{\theta}) - \left(\frac{\partial q}{\partial t} - \Delta q \right) \\ &\le \left[-l\beta\zeta_0 + k_0^2 + k_0 q_0 \right] e^{-\beta t} \\ &\le 0 \quad \text{by (4.15)}. \end{aligned}$$

Case 3. $z + \zeta(t) < -M$, $x_1 \ge 0$. Then by (4.10) and (4.7) we obtain

$$\underline{\theta} = u_+^* + u_-^* - 1 - q \le u_+^* + u_-^* - 1 \le u_-^* \le \delta_0/2,$$

and

$$f(y, u_+^* + u_-^* - 1) - f(y, \underline{\theta}) \le (\theta c^*/8) q,$$

and as in (4.18) we have

$$\begin{aligned} f(y, u_+^*) + f(y, u_-^*) - f(y, \underline{\theta}) &\le k_0(1 - u_+^*) + (\theta c^*/8) q \\ &\le k_0^2 e^{-\alpha_0 (x_1 + c^* t + \zeta(t))} + (\theta c^*/8) q \\ &\equiv \big[(k_0^2/q_0) e^{-\alpha_0 (x_1 + c^* t + \zeta(t)) + \theta x_1 - (\theta c^* - \beta) t} \\ &\quad + (\theta c^*/8) \big] q \\ &\le (\theta c^*/8 + \theta c^*/8) q \\ &= (\theta c^*/4) q, \end{aligned} \tag{4.21}$$

by choosing M sufficiently large that depends only on c^*, α_0, k_0 and δ_0. Thus, by (4.19) we obtain that

$$\begin{aligned} N(\underline{\theta}) &= \zeta' \left(\frac{\partial u^*}{\partial x_1}(-x_1 + c^* t + \zeta(t), y) + \frac{\partial u^*}{\partial x_1}(x_1 + c^* t + \zeta(t), y) \right) \\ &\quad + f(y, u_-^*) + f(y, u_+^*) - f(y, \underline{\theta}) - \left(\frac{\partial q}{\partial t} - \Delta q \right) \\ &\le \frac{\theta c^*}{4} q - \frac{\theta c^*}{4} q \\ &= 0 \end{aligned}$$

which shows that $\underline{\alpha}$ is a subsolution. □

Next let $\eta(t) = -\zeta(t) = -\zeta_0 e^{-\beta t}$ with $\eta' \geq 0$ and $q(x,t)$ as before. Let

$$\begin{aligned}\bar{\theta}(x,t) &= u^*(-x_1 + c^*t + \eta(t), y) \\ &\quad + u^*(x_1 + c^*t + \eta(t), y) - 1 + q \\ &= u^*_- + u^*_+ - 1 + q\end{aligned}$$

and

$$\bar{\alpha}(x,t) = \min\{1, \bar{\theta}(x, t + t_0)\}$$

where $t_0 = \max\{2\zeta_0/c^*, (1/\beta)\ln\zeta_0, 2/(\alpha_0 c^*)\}$. Then we have

Lemma 4.2. *$\bar{\alpha}$ is an upper solution to* (4.12)*.*

Proof. As in the proof of Lemma 4.1, we only need to check when $x_1 \geq 0$, $t \geq 0$ and $\bar{\theta}(x,t) < 1$ since 1 is a solution of (4.12).

Case 1. $z + \eta(t) \geq 0$, $x - 1 \geq 0$. In this case we have

$$\begin{cases} u^*_+ \geq u^*_- \geq u^*(0) \geq 1 - \delta_0/3 \\ \bar{\theta} \geq u^*_+ + u^*_- - 1 \geq 2u^*(0) - 1 \geq 1 - 2\delta_0/3. \end{cases}$$

Similar to (4.18) we have by (4.5) and (4.6) that

$$\begin{aligned} f(y,u^*_+) + f(y,u^*_-) - f(y,\bar{\theta}) &\\ &= f(y,u^*_+) + f(y,u^*_-) - f(y, u^*_+ + u^*_- - 1) \\ &\quad + f(y, u^*_+ + u^*_- - 1) - f(y,\bar{\theta}) \\ &\geq -k_0(1-u^*_+)(1-u^*_-)^\alpha + q/2 \\ &\geq -k_0^2 e^{\alpha_0(-x_1 - c^*t - \eta(t))}(1-u^*(0))^\alpha + q/2 \\ &= \big[(1/2)\cdot(k_0^2/q_0)(1-u^*(0))^\alpha \\ &\qquad\qquad e^{-\alpha_0 x_1 - \alpha_0 c^* t - \alpha_0\eta(t) + \beta t}\big] q \\ &\geq \big[1/2\cdot(k_0^2/q_0)(1-u^*(0))^\alpha \\ &\qquad\qquad e^{-\alpha_0 c^* t_0 + \alpha_0\zeta_0 e^{-\beta t_0} + \beta t_0}\big] q \\ &\geq q/4 \end{aligned}$$

since $-\alpha_0 c^* t_0 + \alpha_0 \zeta_0 e^{-\beta t_0} + \beta t_0 \leq 0$ by the definition of t_0.

Now

$$N(\bar\theta) \geq \frac{1}{4}q + \frac{\partial q}{\partial t} - \Delta q = \frac{1}{4}q - \beta q \geq 0.$$

Case 2. $-M \leq z + \eta(t) < 0$, $x_1 \geq 0$. In this region again we have $(\partial u^*/\partial x_1)(-x_1 + c^* t + \eta(t), y) \geq l > 0$. Similar to (4.20) we have

$$\begin{aligned} f(y, u_+^*) + f(y, u_-^*) - f(y, \bar\theta) &\geq -k_0(1-u_+^*)(1-u_-^*)^\alpha - \|f_u\|_{L^\infty} q \\ &\geq -k_0(1-u_+^*) - k_0 q \\ &\geq -k_0^2 e^{\alpha_0(-x_1(t-\eta(t))} - k_0 q \\ &\geq -k_0^2 e^{-\beta t} - k_0 q_0 e^{-\beta t} \\ &= -(k_0^2 + k_0 q_0) e^{-\beta t} \end{aligned}$$

and since $q = q_0 e^{-\beta t} \min\{1, e^{+\theta z}\}$, we obtain

$$\frac{\partial q}{\partial t} - \Delta q \geq -\beta q_0 e^{-\beta t} \geq -k_0 q_0 e^{-\beta t}.$$

Thus,

$$\begin{aligned} N(\bar\theta) &\geq \eta' \left(\frac{\partial u^*}{\partial x_1}(-x_1 + c^* t + \eta(t), y) + \frac{\partial u^*}{\partial x_1}(x_1 + c^* t + \eta(t), y) \right) \\ &\quad + f(y, u_-^*) + f(y, u_+^*) - f(y, \bar\theta) + \left(\frac{\partial q}{\partial t} - \Delta q \right) \\ &\geq \eta' \frac{\partial u^*}{\partial x_1}(-x_1 + c^* t + \eta(t), y) + f(y, u_-^*) \\ &\quad + f(y, u_+^*) - f(y, \bar\theta) + \left(\frac{\partial q}{\partial t} - \Delta q \right) \\ &\geq (l\zeta_0, \beta - k_0^2 - 2k_0 q_0) e^{-\beta t} \\ &\geq 0 \quad \text{by (4.15).} \end{aligned}$$

Case 3. $z + \eta(t) < -M$, $x_1 \geq 0$. Then $z < -M - \eta(t) < -1 + \zeta_0 e^{-\beta t_0} \leq 0$ which implies that $q = q_0 e^{-\beta t} e^{\theta z}$ and hence by (4.19),

$$\frac{\partial q}{\partial t} - \Delta q \geq \frac{\theta c^*}{4} q.$$

Similar to (4.21) we obtain

$$\begin{aligned} f(y,u_+^*)+f(y,u_-^*)-f(y,\bar\theta) &\geq -k_0(1-u_+^*)-(\theta c^*/8)q \\ &\geq -k_0^2 e^{\alpha_0(-x_1-c^*t-\eta(t))}-(\theta c^*/8)q \\ &= -\big[(k_0^2/q_0)e^{-\alpha_0(x_1+c^*t+\eta(t))+\beta t-\theta z} \\ &\qquad\qquad + \theta c^*/8\big]q \\ &\geq -(\theta c^*/4)q, \end{aligned}$$

as before. Therefore,

$$N(\bar\theta) \geq -\frac{\theta c^*}{4}q + \frac{\theta c^*}{4}q \geq 0. \qquad \square$$

After the constructions above, we need the following comparison results to prove Theorem 4.

Lemma 4.3. *Let* $\underline{\alpha}_0 = \max_{x\in\Sigma}\underline{\alpha}(x,0) \in (0,1)$ *and* $R = \max(\zeta, \zeta + (1/(\alpha_0-\theta))\ln(k_0/\delta_0))$. *If there exists* $\xi_0 \in \mathbf{R}$ *such that* $\phi(x_1-\xi_0,y) \geq \underline{\alpha_0}$ *for* $(x_1,y)\in(-R,R)\times\Omega$, *then*

$$u(t,x) \geq \underline{\alpha}(x_1+\xi_0,y,t) \quad \textit{in } (0,\infty)\times\Sigma.$$

Proof. Since $\phi \geq 0$ and $\underline{\alpha}$ is symmetric in x_1, we only need to check that $\underline{\alpha} \leq 0$ on the interval $(-\infty,-R]$.

Since $R \geq \zeta$, (4.2) and (4.3) imply

$$\begin{aligned} \underline{\alpha}(x,0) &= \max\{0,\underline{\theta}(x,0)\} \\ &= \max\{0, u^*(-x_1+\zeta)+u^*(x_1+\zeta)-1-q(x,t)\} \\ &= \max\{0, u^*(-x_1+\zeta)+u^*(x_1+\zeta)-1-q_0e^{\theta x_1}\} \\ &\leq \max\{0, k_0e^{\alpha_0(x_1+\zeta)}-q_0e^{\theta x_1}\} \\ &= 0. \end{aligned}$$

Therefore, the maximum principle for parabolic equations shows $u(t,x_1-\xi_0,y) \geq \underline{\alpha}(x_1,y,t)$ which finishes the proof. $\square$

Next we will use $\bar{\alpha}(x,t)$ to obtain some upper bound on $u(t,x)$.

Lemma 4.4. *For each $0 \le \phi \le 1$ with compact support, there exists some $T \ge 0$ such that $\bar{\alpha}(x,T) \ge \phi(x)$ for $x \in \Sigma$, and*

$$u(t,x) \le \bar{\alpha}(x,t+T) \text{ in } (0,\infty) \times \Sigma.$$

Proof. Assume that the support of ϕ is in $[-M,M]$ for some $M > 0$. From the choice of t_0, we have that $\bar{\alpha} > 0$. Hence we only need to check that $\bar{\theta}(x,t) > 1$ on $[-M,M] \times \Omega$.

Since $\bar{\theta}$ is symmetric in x_1, we will assume that $x_1 \in [-M,0]$. But for $t > (M+\zeta)/c^*$ we have from (4.2) and (4.3) that

$$\begin{aligned}\bar{\theta}(x,t) &= u^*(-x_1 + c^*t + \eta(t)) + u^*(x_1 + c^*t + \eta(t)) - 1 + q \\ &\ge 1 - k_0 e^{-\alpha_0(-x_1 + c^*t + \eta(t))} + 1 - k_0 e^{-\alpha_0(x_1 + c^*t + \eta(t))} - 1 \\ &\quad + q_0 e^{-\beta t + \theta x_1 - \theta c^* t} \\ &\ge 1 - 2k_0 e^{-\alpha_0(-M + c^*t - \zeta_0)} + q_0 e^{-\beta t - M - \theta c^* t} \\ &\ge 1 - 2k_0 e^{-\alpha_0(-M + c^*t - \zeta_0)} + q_0 e^{-\beta t - M - \theta c^* t}.\end{aligned}$$

From (4.8) and (4.11), we have

$$\beta + \theta c^* \le \frac{\theta c^*}{4} + \theta c^* = \frac{5\theta c^*}{4} \le \frac{5\alpha_0 c^*}{16} < \alpha_0 c^*,$$

which implies that $\bar{\theta}(x,t) > 1$ on $[-M,0] \times \Omega$ for all t large, and hence

$$\bar{\theta}(x,t) > 1 \quad \text{on } [-M,M] \times \Omega \times [T,\infty),$$

for some $T \ge 0$. The maximum principle for parabolic equations derives the assertion of this lemma. □

Proof of Theorem 4. From Lemmas 4.3 and 4.4, for such $0 \le \phi \le 1$ with compact support in the x_1 direction we can find some $T \ge 0$ and $\xi_0 \in \mathbf{R}$ such that

$$\underline{\alpha}(x_1 + \xi_0, y, t) \le u(t,x) \le \bar{\alpha}(x, t+T) \quad \text{in } (0,\infty) \times \Sigma. \tag{4.22}$$

Since u^* is increasing in x_1 we obtain from (4.22) and (4.13) that

$$\begin{aligned}\underline{\alpha}(x_1+\xi_0,y,t) &= \max\{0,\underline{\theta}(x_1+\xi_0,y,t)\}\\ &\geq \underline{\theta}(x_1+\xi_0,y,t)\\ &= u^*(-x_1-\xi_0+c^*t+\zeta(t),y)\\ &\quad + u^*(x_1+\xi_0+c^*t+\zeta(t),y)-1\\ &\quad - q(x_1+\xi_0,y,t)\\ &\geq u^*(-x_1-\xi_0+c^*t,y)+u^*(x_1+\xi_0+c^*t,y)\\ &\quad -1-q_0e^{-\beta t},\end{aligned}$$

and

$$\begin{aligned}\bar{\alpha}(x_1+\xi_0,y,t+T) &= \min\{1,\bar{\theta}(x_1+\xi_0,y,t+T+t_0)\}\\ &\leq \underline{\theta}(x_1+\xi_0,y,t+T+t_0)\\ &= u^*(-x_1+c^*(t+T+t_0)+\zeta(t+T+t_0),y)\\ &\quad + u^*(x_1+c^*(t+T+t_0)+\zeta(t+T+t_0),y)-1\\ &\quad + q(x_1,y,t+T+t_0)\\ &\leq u^*(-x_1+c^*(t+T+t_0),y)\\ &\quad + u^*(x_1+c^*(t+T+t_0),y)-1\\ &\quad + q_0e^{-\beta(t+T+t_0)}\\ &\leq u^*(-x_1+c^*(t+T+t_0),y)\\ &\quad + u^*(x_1+c^*(t+T+t_0),y)-1\\ &\quad + q_0e^{-\beta t}.\end{aligned}$$

Therefore, Theorem 4 is proved with $\xi_1 = c^*(T+t_0)$. □

REFERENCES

1. D.G. Aronson and H.F. Wenberger, *Nonlinear diffusion in population genetics, combustion and nerve propagation*, in *Partial differential equations and related topics*, Lecture Notes in Math. **446** (1975), 5–49.

2. ———, *Multidimensional nonlinear diffusions arising in population genetics*, Adv. in Math. (China) **30** (1978), 35–58.

3. J.W. Bebernes and A. Lacey, *The approach of combustion model solutions to travelling fronts*, preprint.

4. H. Berestycki, L.A. Caffarelli and L. Nierenberg, *Uniform estimates for regularization of free boundary problems*, in *Analysis and partial differential equations* (C. Sadosky, ed.), Marcel Dekker, 1990.

5. H. Berestycki and B. Larrouturou, *A semilinear elliptic equation in a strip arising in a two-dimensional flame propagation model*, J. Reine Angew. Math. **396** (1989), 14–40.

6. ———, *Mathematical aspects of planar flame propagation*, Longman Press, in preparation.

7. H. Berestycki, B. Larrouturou and P.-L. Lions, *Multidimensional traveling-front solutions of a flame propagation model*, Arch. Rational Mech. Anal. **111** (1990), 33–49.

8. H. Berestycki, B. Larrouturou and J.M. Roquejoffre, *Stability of travelling fronts in a curved flame model*, Part I, *Linear analysis*, Arch. Rational Mech. Anal. **117** (1991), 97–117.

9. H. Berestycki, B. Nicolaenko and B. Scheurer, *Travelling wave solutions to combustion models and their singular limits*, SIAM J. Math. Anal. **16** (1985), 1207–1242.

10. H. Berestycki and L. Nirenberg, *Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains*, in *Analysis, etc.*, (volume dedicated to J. Moser) (P.H. Rabinowitz and E. Zehnder, eds.), Academic Press, New York, 1990.

11. ———, *Travelling front solutions of semilinear equations in higher dimensions*, Frontiers Pure Appl. Math., North Holland (1991), 31–41.

12. ———, *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat. **22** (1991), 1–37.

13. ———, *Asymptotic behavior via the Harnack inequality*, in *Nonlinear Analysis*, (A tribute in honor of Giovanni Prodi) (A. Ambrosetti et al., eds.), Sc. Normale Sup., Pisa, 1991.

14. ———, *Travelling fronts in cylinders*, Ann. Inst. H. Poincaré Probab. Statist. **9** (1992), 497–572.

15. M. Bramson, *Convergence of solutions of the Kolmogorov equation to travelling waves*, Memoirs Amer. Math. Soc. **44** (1983), 1–190.

16. R.G. Casten and C.J. Holland, *Instability results for reaction-diffusion equations with Neumann boundary conditions*, J. Differential Equations **27** (1978), 266–273.

17. P.C. Fife, *Mathematical aspects of reacting and diffusing systems*, Lecture Notes in Biomathematics **28**, Springer-Verlag, 1979.

18. P.C. Fife and J.B. McLeod, *The approach of solutions of nonlinear diffusions to travelling front solutions*, Arch. Rational Mech. Anal. **65** (1977), 355–361.

19. M. Freidlin, *Limit theorems for large deviations and reaction-diffusion equations*, Ann. Probab. **13** (1985), 639–675.

20. ———, *KPP in cylinders*, to appear.

21. R. Gardner, *Existence of multidimensional traveling wave solutions of an initial-boundary value problem*, J. Differential Equations **61** (1986), 355–375.

22. S. Heinze, *Traveling waves for semilinear parabolic partial differential equations in cylindrical domains*, preprint.

23. W.E. Johnson and W. Nachbar, *Laminar flame theory and the steady linear burning of a monopropellant*, Arch. Rational Mech. Anal. **12** (1963), 58–91.

24. Ja I. Kanel, *Stabilization of solution of the Cauchy problem for equations encountered in combustion theory*, Mat. Sb. **59** (1962), 245–288.

25. ———, *On steady state solutions to systems of equations arising in combustion theory*, Dakl. Akad. Nauk **149** (1963), 367–367.

26. A.N. Kolmogorov, I.G. Petrovsky and N.S. Piskunov, *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, (Bjul. Moskowskogo Gos. Univ.) Série internationale, section A, Bulletin Université dÉtat à Moscow **1** (1937), 1–26; English translation: *Study of the diffusion equation with growth of the quantity of matter and its application to a biology problem*, in *Dynamics of curved fronts, Perspectives in physics series* (R. Pelcé, ed.), Academic Press, 1988.

27. C. Li, *Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains*, Comm. Partial Differential Equations **16** (1991), 585–615.

28. Y. Li, *On the positive solutions of the Matukuma equation*, Duke Math. J. **70** (1993), 575–589.

29. H. McKean, *An application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piscounov*, Comm. Pure Appl. Math. **28** (1975), 323–331.

30. ———, *An application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piscounov*, II, Comm. Pure Appl. Math. **29** (1976), 553–554.

31. J.M. Roquejoffre, *Stability of traveling fronts in a curved flame model*, Part II, *Nonlinear orbital stability*, Arch. Rational Mech. Anal. **117** (1992), 119.

32. ———, *Convergence to travelling waves for solutions of a class of semilinear parabolic equations*, J. Differential Equations **108** (1994), 262–295.

33. J.M. Roquejoffre and D. Terman, *The asymptotic stability of travelling wave solution arising from a combustion model*, Nonlinear Anal. **22** (1994), 137–154.

34. J.M. Roquejoffre, D. Terman and V.A. Volpert, *Global stability of travelling fronts and convergence towards wave trains in monotone parabolic systems*, preprint.

35. J.M. Vega, *The asymptotic behavior of the solutions of some semilinear elliptic equations in cylindrical domains*, J. Differential Equations **102** (1993), 119–152.

36. ———, *Travelling wavefronts of reaction-diffusion equations in cylindrical domains*, Comm. Partial Differential Equations **18** (1993), 505.

37. A.I. Vol'pert, *On the propagation of waves described by nonlinear parabolic equations*, (Comments on paper No. 6 by Kolmogorov, Petrowsky and Piskunov), in *I.G. Petrovsky's Collected works in differential equations and probability theory* (V.I. Arnold et al., ed.), Akad. Nauk, Moscow, 1988.

38. ———, *Wave solutions of parabolic equations*, Inst. Chemical Physics, Acad. Sci., Chernogolovka, 1983.

39. A.I. Volpert and V.A. Volpert, *Applications of the rotation theory of vector fields to the study of wave solutions of parabolic equations*, Trud. Moscow Math. Soc. **52** (1989); English translation: Trans. Moscow Math. Soc. **52** (1990), 59–108.

40. ———, *Existence and stability of traveling waves in chemical kinetics. Dynamics of chemical and biological systems*, Novosibirsk Nauka (1989) , 56–131.

41. ———, *Traveling waves in monotone parabolic systems*, Branch of the Inst. of Chemical Physics, Chernogolovka (1990), 49.

42. J.B. Zeldovich and D.A. Frank-Kamenetskii, *A theory of thermal propagation of flame*, Acta Physiochima URSS **9** (1938); English translation: *Dynamics of curved fronts*, in *Perspectives in physics series* (R. Pelcé, ed.), Academic Press, New York, 1988.

Applied Mathematics, University of Colorado at Boulder

Applied Mathematics, University of Colorado at Boulder

Department of Mathematics, University of Rochester,