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EXTENDING FINITE SUBSETS OF AN IMMUNE SET

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ABSTRACT. We shall deal with nonnegative integers (*numbers*), collections of numbers (*sets*) and collections of sets (*classes*). If σ is a finite set, $|\sigma|$ denotes its cardinality. We write $P_{\text{fin}}(\nu)$ for the class of all finite subsets of the set ν. If, given any set $\sigma \in P_{\text{fin}}(\nu)$, we can effectively extend σ to a larger set $\sigma^* \in P_{\text{fin}}(\nu)$, then ν is *recursively infinite*, i.e., ν has an infinite recursively enumerable (r.e.) subset. This paper deals with a more general property of $P_{\text{fin}}(\nu)$. Let f be a strictly increasing, recursive function with range α. Then the finite subsets of ν are *f-extendible*, if given any set $\sigma \in P_{\text{fin}}(\nu)$ with $|\sigma| \notin \alpha$, we can effectively extend σ to a larger set $\bar{\sigma} \in P_{\text{fin}}(\nu)$, so that

$$|\bar{\sigma}| = \text{the first number} > |\sigma| \text{ in the enumeration } f_0, f_1, \dots \text{ of } \alpha.$$

For many choices of the strictly increasing, recursive function f, this relation between the finite subsets of ν and the function f is not trivial. Let N be the RET (recursive equivalence type) of the immune set ν. Then the arithmetical properties of the infinite isol N are intimately related to the strictly increasing recursive functions with respect to which the finite subsets of ν are extendible. For example, N is even if and only if the finite subsets of ν are $2n$-extendible, while N is odd if and only if the finite subsets of ν are $2n+1$-extendible. In this paper we study the notion "the finite subsets of ν are f-extendible," in particular its relationship with Myhill's combinatorial operators [**7**] and Nerode's frames [**8**].

1. Notations and terminology. In addition to the notations and terminology mentioned in the abstract, we shall use the following. We write ε for the set $(0, 1, \dots)$ of all numbers, $\subset$ for inclusion and $\subset_+$ for proper inclusion. A *function* is a mapping from a subset of ε into ε. If p is a function, δp denotes its domain and ρp its range. We write Q for the class of all finite sets. If a_n is a function from ε into ε,

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"$a(n)$" and "a_n" mean the same. We need a Gödel-numbering of the denumerable class Q and we choose the *canonical enumeration* $\langle \rho_n \rangle$ of Q. It is defined by:

$$\rho_0 = \text{the empty set},$$
$$\rho_{n+1} = \begin{cases} (a_1, \dots, a_k), & \text{where } a_1, \dots, a_k \text{ are the distinct} \\ & \text{numbers s. t. } n+1 = 2^{a(1)} + \dots + 2^{a(k)}. \end{cases}$$

This enumeration of Q has no repetitions. If $\sigma \in Q$, the unique number i such that $\sigma = \rho_i$ is called the *canonical index* of σ and is denoted by $\operatorname{can} \sigma$ or $\operatorname{can}(\sigma)$. For $S \subset Q$ we define $\operatorname{can} S = \{\operatorname{can} \sigma | \sigma \in S\}$. A subclass S of Q is r.e. if the set $\operatorname{can} S$ is r.e. A mapping from a subclass of Q into Q is called a *procedure*. If Π is a procedure, we denote its domain by $\operatorname{Dom} \Pi$ and its range by $\operatorname{Ran} \Pi$. A procedure Π is *effective*, if the function $\operatorname{can} \sigma \to \operatorname{can} \Pi(\sigma)$, for $\sigma \in \operatorname{Dom} \Pi$, is partial recursive (p.r.). Since the domain and the range of a p.r. function are r.e. sets, the domain and the range of an effective procedure are r.e. *classes*. We let r be the recursive function with $r_i = |\rho_i|$ and let j denote the familiar primitive recursive pairing function that maps ε^2 onto ε.

For a function f from ε into ε, we define

$$n_x = (\mu n)[f_n \geq x], \quad \text{for } x \geq 0. \tag{1.1}$$

If f is a strictly increasing function with range α, relation (1.1) implies

$$x \in \alpha \iff x = f(n_x), \quad \text{for } x \geq 0, \tag{1.2}$$

$$x \leq y \quad \text{and} \quad y \in \alpha \Longrightarrow y \geq \text{ the first number } \geq x \text{ in } f_0, f_1, \dots, \quad \text{for } x, y \geq 0. \tag{1.3}$$

Let ν be an isolated set and $N = \operatorname{Req} \nu$. Then ν and N are called *even*, if $N = 2X$, for some isol X. Similarly, ν and N are called *odd*, if $N = 2Y + 1$, for some isol Y.

2. Elementary properties.

Proposition P1. *Let f be a strictly increasing, recursive function with range α, and let ν be an infinite set. Then the following two conditions on f and ν are equivalent:*

(a) *there is an effective procedure* $\bar{\Pi} : \sigma \to \bar{\sigma}$ *with* $P_{\text{fin}}(\nu) \subset \operatorname{Dom}\bar{\Pi}$ *such that*

(2.1)
$$\sigma \in P_{\text{fin}}(\nu) \quad \textit{and} \quad |\sigma| \notin \alpha \Longrightarrow \sigma \subset_{+} \bar{\sigma} \subset \nu, \quad |\bar{\sigma}| \in \alpha \quad \textit{and} \quad |\bar{\sigma}| = f(n_{|\sigma|}),$$

(b) *there is an effective procedure* $\Pi^* : \sigma \to \sigma^*$ *with* $P_{\text{fin}}(\nu) \subset \operatorname{Dom}\Pi^*$ *such that*

(2.2)
$$\sigma \in P_{\text{fin}}(\nu) \Longrightarrow \sigma \subset \sigma^* \subset \nu \quad \textit{and} \quad |\sigma^*| \in \alpha.$$

Proof. Note that, on the right side of (2.1) the first of the two conditions "$|\bar{\sigma}| \in \alpha$" and "$|\bar{\sigma}| = f(n_{|\sigma|})$" can be deleted, since it is implied by the second one. Assume (a). Define a procedure $\Pi^* : \alpha \to \alpha^*$ by

$$\operatorname{Dom}\Pi^* = \operatorname{Dom}\bar{\Pi} \cup \Big\{\sigma \in Q \Big| \, |\sigma| \in \alpha\Big\},$$
$$\sigma^* = \begin{cases} \bar{\sigma} & \text{if } |\sigma| \notin \alpha, \\ \sigma & \text{if } |\sigma| \in \alpha. \end{cases}$$

The classes $\operatorname{Dom}\bar{\Pi}$ and $\{\sigma \in Q| \, |\sigma| \in \alpha\}$ are both r.e., hence so is $\operatorname{Dom}\Pi^*$. Suppose $\sigma \in P_{\text{fin}}(\nu)$. Then we can decide whether $|\sigma| \in \alpha$, since α is a recursive set. Hence, Π^* is also an effective procedure. We distinguish two cases.

Case 1. $|\sigma| \notin \alpha$. Then the left side of (2.1) holds, hence so does the right side. Since we defined $\sigma^* = \bar{\sigma}$, we obtain $\sigma \subset \sigma^* \subset \nu$ and $|\sigma^*| \in \alpha$.

Case 2. $|\sigma| \in \alpha$. Then we defined $\sigma^* = \sigma$, so that $\sigma \subset \sigma^* \subset \nu$ and $|\sigma^*| \in \alpha$. We proved (b).

Now assume (b). Define a procedure $\bar{\Pi}$ by $\operatorname{Dom}\bar{\Pi} = \operatorname{Dom}\Pi^*$ and let, for $\sigma \in P_{\text{fin}}(\nu)$ the set $\bar{\sigma}$ be defined as follows:

Case 1. $|\sigma| \in \alpha$. Then we define $\bar{\sigma} = \sigma^*$. Relation (2.1) is now true, since its hypothesis is false.

Case 2. $|\sigma| \notin \alpha$ and $|\sigma^*| = f(n_{|\sigma|})$. Then we define $\bar{\sigma} = \sigma^*$. Let $\sigma \in P_{\text{fin}}(\nu)$. Then (2.2) implies $\sigma \subset \bar{\sigma} \subset \nu$ and $|\bar{\sigma}| \in \alpha$. Since $|\sigma| \notin \alpha$, while $|\bar{\sigma}| \in \alpha$, we have $\sigma \neq \bar{\sigma}$. Hence we can strengthen $\sigma \subset \bar{\sigma} \subset \nu$ to $\sigma \subset_+ \bar{\sigma} \subset \nu$. It follows that $\sigma \subset_+ \bar{\sigma} \subset \nu$ and $|\bar{\sigma}| \in \alpha$. Moreover, $|\sigma^*| = f(n_{|\sigma|})$, hence $|\bar{\sigma}| = f(n_{|\sigma|})$. We proved the right side of (2.1).

Case 3. $|\sigma| \notin \alpha$ and $|\sigma^*| > f(n_{|\sigma|})$. Then $\sigma \subset \sigma^* \subset \nu$ and $|\sigma^*| \in \alpha$. Note that $|\sigma| \notin \alpha$ implies $f(n_{|\sigma|}) > |\bar{\sigma}|$, hence $|\sigma^*| > |\sigma|$. We wish to define $\bar{\sigma} = \bar{\Pi}(\sigma)$ so that

$$\sigma \subset_+ \bar{\sigma} \subset \nu \quad \text{and} \quad |\bar{\sigma}| \in \alpha \quad \text{and} \quad |\bar{\sigma}| = f(n_{|\sigma|})$$

or equivalently so that

$$\sigma \subset_+ \bar{\sigma} \subset \nu \quad \text{and} \quad |\bar{\sigma}| = f(n_{|\sigma|}). \tag{2.3}$$

We therefore define $k = f(n_{|\sigma|}) - |\sigma|$, τ is the set of the smallest k elements of $\sigma^* - \sigma$ and $\bar{\sigma} = \sigma \cup \tau$. Then $\sigma \subset \bar{\sigma}$, hence $\sigma \subset \bar{\sigma} \subset \nu$. Moreover, $|\bar{\sigma}| = |\sigma| + k = f(n_{|\sigma|})$. Also, $\sigma \subset \bar{\sigma} \subset \nu$ and $\sigma \neq \bar{\sigma}$ imply $\sigma \subset_+ \bar{\sigma} \subset \nu$. Hence $\bar{\sigma}$ satisfies (2.3) and $\bar{\Pi}$ satisfies (2.1). □

Definition D1. Let f be a strictly increasing, recursive function with range α, and let ν be an infinite set. Then the finite subsets of ν are *f-extendible*, if one of the two conditions (a) and (b) of Proposition P1 holds.

Remark R1. Let f be a strictly increasing, recursive function with range α, and let ν be an infinite set. Then the finite subsets of ν are f-extendible if and only if there is an effective procedure $\bar{\Pi} : \sigma \to \bar{\sigma}$ with $P_{\text{fin}}(\nu) \subset \text{Dom}\, \bar{\Pi}$ such that

$$\sigma \in P_{\text{fin}}(\nu) \quad \text{and} \quad |\sigma| \notin \alpha \Longrightarrow \sigma \subset_+ \bar{\sigma} \subset \nu \quad \text{and} \quad |\bar{\sigma}| = |\sigma| + 1.$$

For, given any set $\sigma \in P_{\text{fin}}(\nu)$ with $|\sigma| \notin \alpha$, such a procedure can be iterated until a set $\tau \in P_{\text{fin}}(\nu)$ with $|\tau| \in \alpha$ is obtained. It follows that one can also phrase the definition of "the finite subsets of ν are f-extendible" as follows: there is a p.r. function p such that

$$\rho_n \subset \nu \quad \text{and} \quad r_n \notin \alpha \Longrightarrow n \in \delta p \quad \text{and} \quad p_n \in \nu - \rho_n.$$

We now give some examples. First consider the identity function i on ε. Condition (a) of P1 is satisfied by $f = i$, for every infinite set ν and the effective procedure $\bar{\Pi} : \sigma \to \bar{\sigma}$, where $\bar{\sigma} = \sigma$. Thus the finite subsets of every infinite set ν are i-extendible. The strictly increasing, recursive functions f such that the finite subsets of every immune set are f-extendible will be characterized in Proposition P7 of Section 4. Two less trivial examples arise if we consider the functions $f_n = 2n$ and $f_n = 2n + 1$. We have, by Remark R1,

(i) the finite subsets of an infinite set ν are $2n$-extendible if and only if there is a p.r. function p such that

$$\rho_n \subset \nu \quad \text{and} \quad r_n \text{ odd} \Longrightarrow n \in \delta p \quad \text{and} \quad p_n \in \nu - \rho_n, \tag{2.4}$$

(ii) the finite subsets of an infinite set ν are $2n+1$-extendible if and only if there is a p.r. function q such that

$$\rho_n \subset \nu \quad \text{and} \quad r_n \text{ even} \Longrightarrow n \in \delta q \quad \text{and} \quad q_n \in \nu - \rho_n. \tag{2.5}$$

We shall see in Section 7 that, for an immune set ν,

$$\nu \text{ even} \Longleftrightarrow \nu \ 2n\text{-extendible}, \quad \nu \text{ odd} \Longleftrightarrow \nu \ 2n+1\text{-extendible}.$$

We now turn to the question: "Which infinite sets ν are such that the finite subsets of ν are f-extendible, for *every* strictly increasing, recursive function f?" We call two strictly increasing functions f and g *intertwined*, if $f_0 < g_0 < f_1 < g_1 < \cdots$ or $g_0 < f_0 < g_1 < f_1 < \cdots$.

Proposition P2. *The following three conditions on an infinite set ν are mutually equivalent:*

(a) *the finite subsets of ν are f-extendible for every strictly increasing recursive function f,*

(b) *there are two intertwined strictly increasing recursive functions f and g such that the finite subsets of ν are both f-extendible and g-extendible,*

(c) ν *is recursively infinite.*

Proof. The conditional (a) $\Rightarrow$ (b) is trivial. Suppose (b) is true. We may assume without loss of generality that $f_0 < g_0 < f_1 < g_1 < \cdots$. Let σ_0 be a finite subset of ν with $|\sigma_0| = f_0$. Then $|\sigma_0| \notin \rho g$, and we can compute a finite set σ_1 with $\sigma_0 \subset_+ \sigma_1 \subset \nu$ and $|\sigma_1| = g_1$. Then $|\sigma_1| \notin \rho f$, and we can compute a set σ_2 with $\sigma_1 \subset_+ \sigma_2 \subset \nu$ and $|\sigma_2| = f_2$. Continuing this procedure we generate the infinite r.e. subset $\sigma_0 \cup \sigma_1 \cup \cdots$ of ν, hence (c) holds. Now assume (c). Let β be any infinite, r.e. subset of ν, say $\beta = (b_0, b_1, \dots)$, where b_n is a one-to-one, recursive function. Let f be any strictly increasing, recursive function and α its range. Suppose a finite subset σ of ν is given with $|\sigma| \notin \alpha$. Then

$$n_{|\sigma|} = (\mu n)[f_n \geq |\sigma|] = (\mu n)[f_n > |\sigma|],$$

since $|\sigma| \notin \alpha$. Put $k = f(n_{|\sigma|}) - |\sigma|$, then $k > 0$. Let $b_{i(1)}, \dots, b_{i(k)}$ be the first k numbers in the sequence $b_0, b_1, \dots$ which do not belong to σ. Then the set $\sigma^* = \sigma \cup \{b_{i(1)}, \dots, b_{i(k)}\}$ is such that $\sigma \subset_+ \sigma^* \subset \nu$, $|\sigma^*| \in \alpha$ and $|\sigma^*| = f(n_{|\sigma|})$. Since we can compute the set σ^* from the set σ, we conclude that (a) holds. $\square$

Remark R2. Suppose we wish to prove for some strictly increasing, recursive function f with range α and for some infinite set ν that the finite subsets of ν are f-extendible and that we use the characterization of f-extendibility described in part (a) of P1. Then we need to show that there is an effective procedure $\bar{\Pi} : \sigma \to \bar{\sigma}$ with $P_{\text{fin}}(\nu) \subset \text{Dom}\,\bar{\Pi}$ such that for $\sigma \in P_{\text{fin}}(\nu)$ and $|\sigma| \notin \alpha$,

(i) $\sigma \subset \bar{\sigma} \subset \nu$,

(ii) $0 \leq |\sigma| < f_0 \Rightarrow |\bar{\sigma}| = f_0$,

(iii) $f_n < |\sigma| < f_{n+1} \Rightarrow |\bar{\sigma}| = f_{n+1}$, for $n \geq 0$.

Then we may delete the proof of (ii) without loss of generality. For we may consider as given some finite subset σ_0 of ν with $|\sigma_0| = f_0$. Given any $\sigma \in P_{\text{fin}}(\nu)$ with $|\sigma| < f_0$, we now define $k = f_0 - |\sigma|$ and τ is the set of the smallest k elements of $\sigma_0 - \sigma$, $\bar{\sigma} = \sigma \cup \tau$. Then $\sigma \subset \bar{\sigma} \subset \nu$ and $|\bar{\sigma}| = |\sigma| + k = f_0$, where $\bar{\sigma}$ can be computed from σ.

The next proposition tells us that the notion of f-extendibility can also be introduced for RETs. We shall delete the proof, since it is routine.

Proposition P3. *Let f be a strictly increasing, recursive function with range α, and let ν and μ be infinite sets with $\nu \simeq \mu$. Then the finite subsets of ν are f-extendible if and only if the finite subsets of μ are f-extendible.*

Definition D2. Let f be a strictly increasing, recursive function, and let ν be an infinite set. Then the RET $N = \mathrm{Req}\,\nu$ is *f-extendible*, if the finite subsets of ν are f-extendible.

We see by Proposition P2 that the property of f-extendibility of an infinite set ν is only of interest if the set ν is immune. Let Ω denote the collection of all RETs and Λ the collection of all isols. In view of Propositions P2 and P3, every RET in $\Omega - \Lambda$ is f-extendible for every strictly increasing, recursive function f. The notion of f-extendibility of RETs is therefore only of interest for RETs in $\Lambda - \varepsilon$, i.e., for infinite isols.

We now turn to a crucial property of the relation of f-extendibility. Call the strictly increasing, recursive functions f and g *almost equal*, if $f_n = g_n$, for almost all numbers n.

Proposition P4. *Let f and g be strictly increasing, recursive functions with ranges α and β, respectively. Suppose that the functions f and g are almost equal. Then, for every infinite set ν,*

$(*)$ *the finite subsets of ν are f-extendible*
$\iff$ *the finite subsets of ν are g-extendible.*

Proof. Assume the hypothesis. Put $p = (\mu n)\ (\forall x)\ [x \geq n \Rightarrow f_x = g_x]$; then p exists and $x \geq p$ implies $f_x = g_x$. Put $m = f_p = g_p$, then

$$x \in \{f_p, f_{p+1}, \dots\} \iff x \in \{g_p, g_{p+1}, \dots\}, \quad \text{for } x \geq 0, \tag{2.6}$$

hence

$$x \in \alpha \iff x \in \beta, \quad \text{for } x \geq m. \tag{2.7}$$

Let ν be an infinite set. We wish to prove $(*)$, and it suffices to prove the $\Rightarrow$ part. Assume that the finite subsets of ν are f-extendible. Then there is an effective procedure $\Pi^* : \sigma \to \sigma^*$ with $P_{\text{fin}}(\nu) \subset \text{Dom}\,\Pi^*$ such that

$$\alpha \in P_{\text{fin}}(\nu) \Longrightarrow \sigma \subset \sigma^* \subset \nu \quad \text{and} \quad |\sigma^*| \in \alpha. \tag{2.8}$$

Choose a finite subset τ of ν with $|\tau| = m$. Henceforth this set τ remains fixed, and we may consider it as known. We now define a procedure $\bar{\Pi} : \sigma \to \bar{\sigma}$ by

$$\text{Dom}\,\bar{\Pi} = \{\sigma \in \text{Dom}\,\Pi^* \mid \sigma \cup \tau \in \text{Dom}\,\Pi^*\}, \qquad \bar{\sigma} = (\sigma \cup \tau)^*.$$

Then $P_{\text{fin}}(\nu) \subset \text{Dom}\,\bar{\Pi}$, and since the procedure Π^* is effective, so is the procedure $\bar{\Pi}$. We show that, under the hypothesis $\sigma \in P_{\text{fin}}(\nu)$,

(a) $\sigma \subset \bar{\sigma} \subset \nu$,

(b) $|\bar{\sigma}| \in \alpha$,

(c) $|\bar{\sigma}| \in \beta$.

Re (a). $\sigma, \tau \in P_{\text{fin}}(\nu) \Rightarrow \sigma \cup \tau \in P_{\text{fin}}(\nu) \Rightarrow \sigma \cup \tau \subset (\sigma \cup \tau)^* \subset \nu \Rightarrow \sigma \cup \tau \subset \bar{\sigma} \subset \nu \Rightarrow \sigma \subset \bar{\sigma} \subset \nu$.

Re (b). By (2.8) we have $|\xi^*| \in \alpha$ for $\xi \in P_{\text{fin}}(\nu)$. Taking $\xi = \sigma \cup \tau$, we get

$$\sigma \cup \tau \in P_{\text{fin}}(\nu) \Longrightarrow |(\sigma \cup \tau)^*| \in \alpha \Longrightarrow |\bar{\sigma}| \in \alpha.$$

Re (c). $\tau \subset \sigma \cup \tau$ and $\sigma \cup \tau \subset (\sigma \cup \tau)^*$ implies $\tau \subset (\sigma \cup \tau)^*$, hence $\tau \subset \bar{\sigma}$, $|\tau| \leq |\bar{\sigma}|$ and $m \leq |\bar{\sigma}|$.

Put $x = |\bar{\sigma}|$, then $x \geq m$. Moreover, $x \in \alpha$ by (b), hence $x \in \beta$ by (2.7), i.e., $|\bar{\sigma}| \in \beta$. Since $\sigma \in P_{\text{fin}}(\nu)$ implies (a) and (c), we conclude that the finite subsets of ν are also g-extendible. □

3. Combinatorial operators.

Proposition P5. *Let f be a strictly increasing recursive combinatorial function, Φ any recursive, combinatorial operator which induces f, and let ν be any immune set. Then the set $\Phi(\nu)$ is also immune and its finite subsets are f-extendible.*

Proof. Assume the hypothesis. Let Φ^* be the normal, recursive, combinatorial operator which is equivalent to Φ. By [**3**, p. 35] there is a p.r. bijection g from $\Phi(\varepsilon)$ onto $\Phi^*(\varepsilon)$ such that $g\Phi(\sigma) = \Phi^*(\sigma)$, for $\sigma \subset \varepsilon$. Thus $\Phi(\nu) \simeq \Phi^*(\nu)$ and, according to P3, the finite subsets of $\Phi(\nu)$ are f-extendible if and only if the finite subsets of $\Phi^*(\nu)$ are f-extendible. We may therefore assume without loss of generality that Φ is the *normal*, recursive, combinatorial operator which induces f, i.e., that

$$\Phi(\nu) = \{j(x, y) \in \varepsilon \mid \rho_x \subset \nu, \quad y < c_{r(x)}\},$$

$$\text{where } f_n = \sum_{i=0}^{n} c_i \binom{n}{i}.$$

Let the set ν be immune. Then ν is isolated, hence so is $\Phi(\nu)$. Since f is strictly increasing, we have $f_0 < f_1$, i.e., $c_0 < c_0 + c_1$, hence $c_1 > 0$. Then

$$\{j(x, 0) \in \varepsilon \mid \rho_x \subset \nu,\ r_x = 1,\ 0 < c_{r(x)}\} \subset \Phi(\nu),$$

where the set on the left is infinite, since it is equivalent to ν. Thus $\Phi(\nu)$ is infinite, hence immune. Put $\alpha = \rho f$ and $\mu = \Phi(\nu)$. We claim that there is an effective procedure $\Pi : \sigma \to \sigma^*$ with $P_{\text{fin}}(\mu) \subset \operatorname{Dom} \Pi$ such that

$$\sigma \in P_{\text{fin}}(\mu) \Longrightarrow \sigma \subset \sigma^* \subset \mu \quad \text{and} \quad |\sigma^*| \in \alpha.$$

We define $\operatorname{Dom} \Pi = P_{\text{fin}}(\varepsilon)$. Hence $\operatorname{Dom} \Pi$ is the r.e. class Q of all finite sets. For $\sigma \in \operatorname{Dom} \Pi$, we define

$$\sigma_0 = \{x | (\exists y)[j(x, y) \in \sigma]\},\ \sigma_1 = \bigcup \{\rho_x \mid x \in \sigma_0\},$$
$$\sigma^* = \Phi(\sigma_1).$$

Given any set $\sigma \in \operatorname{Dom} \Pi$, we can compute σ_0, σ_1 and σ^*, hence Π is an effective procedure. Now assume $\sigma \in P_{\text{fin}}(\mu)$. Then we claim that

(a) $\sigma \subset \sigma^*$,

(b) $\sigma^* \subset \mu$,

(c) $|\sigma^*| \in \alpha$.

Re (a). By the definitions of σ_0, σ_1 and σ^*, we have

$$(3.1) \qquad j(x, y) \in \sigma \Longrightarrow (x \in \sigma_0,\ \rho_x \subset \sigma_1) \Longrightarrow \rho_x \subset \sigma_1.$$

Moreover, $\sigma \subset \mu$, where $\mu = \Phi(\nu) = \{j(x,y) \in \varepsilon \mid \rho_x \subset \nu, \quad y < c_{r(x)}\}$, hence $j(x,y) \in \sigma \Rightarrow y < c_{r(x)}$. It now follows from (3.1) that

$$j(x,y) \in \sigma \Longrightarrow (\rho_x \subset \sigma_1,\ y < c_{r(x)}) \Longrightarrow j(x,y) \in \Phi(\sigma_1) \Longrightarrow j(x,y) \in \sigma^*,$$

so that $\sigma \subset \sigma^*$.

Re (b). We wish to prove $\sigma^* \subset \mu$, i.e., $\Phi(\sigma_1) \subset \Phi(\nu)$. This would follow from $\sigma_1 \subset \nu$. We have

$$\sigma_1 = \bigcup\{\rho_x \mid x \in \sigma_0\} = \bigcup\{\rho_x \mid (\exists y)[j(x,y) \in \sigma]\}. \tag{3.2}$$

However, $\sigma \subset \Phi(\nu)$, hence

$$(\exists y)[j(x,y) \in \sigma] \Longrightarrow (\exists y)[j(x,y) \in \Phi(\nu)] \Longrightarrow \rho_x \subset \nu.$$

Thus, relation (3.2) implies

$$\sigma_1 \subset \bigcup\{\rho_x \mid (\exists y)[j(x,y) \in \Phi(\nu)]\}, \quad \text{i.e., } \sigma_1 \subset \nu.$$

Re (c). $\sigma^* = \Phi(\sigma_1) \Longrightarrow |\sigma^*| = |\Phi(\sigma_1)| \Longrightarrow |\sigma^*| = f(|\sigma_1|) \Longrightarrow |\sigma^*| \in \alpha$. □

Corollary. *Let ν be an immune set and $N = \mathrm{Req}\,\nu$. Then*

(a) *N even implies that N is $2n$-extendible,*

(b) *N odd implies that N is $2n+1$-extendible.*

Proof. Let Φ be a recursive, combinatorial operator which induces the strictly increasing, recursive, combinatorial function $f_n = 2n$. Every even immune set ν has the form $\Phi(\xi)$, for some immune set ξ. Thus the finite subsets of ν are $2n$-extendible, hence so is N. Part (b) can be proved in a similar manner using the function $f_n = 2n+1$. □

We shall see in Section 7 that the converses of (a) and (b) also hold.

4. Regressive sets. We assume that the reader is familiar with the notions of a regressive function, a regressive set and a regressive isol [**4**, Section 3].

Definition D3. A strictly increasing, recursive function f is *trivial*, if $f_{n+1} = f_n + 1$, for almost all n, or equivalently, if there is a number k such that $f_n = n + k$, for almost all n.

Definition D4. Let $N = \mathrm{Req}\,\nu$, where ν is an isolated set. Then the set ν and the isol N are *multiple-free*, if

$$2X \leq N \Longrightarrow X \text{ finite}, \quad \text{for } X \in \Lambda.$$

Proposition P6. *Let ν be an immune, regressive set and f be a strictly increasing, recursive function. If the finite subsets of ν are f-extendible and the function f is nontrivial, then ν is not multiple-free.*

Proof. Assume the hypothesis. Let u_n be a regressive function ranging over ν. Now suppose that the finite subsets of ν are f-extendible and the function f is nontrivial. Since f is strictly increasing, we have $f_n < f_{n+1}$, i.e., $f_n + 1 \leq f_{n+1}$, for all n. If $f_n + 1 = f_{n+1}$, for almost all n, the function f would be trivial. Thus $f_n + 1 < f_{n+1}$, i.e., $f_n + 2 \leq f_{n+1}$, for infinitely many n. Hence there is a strictly increasing, recursive function s_k such that $f(s_0) + 2 \leq f(s_0 + 1)$, $f(s_1) + 2 \leq f(s_1 + 1), \ldots$, i.e.,

$$f(s_k) + 2 \leq f(s_k + 1), \quad \text{for } k \geq 0. \tag{4.1}$$

Define

$$\mu_k = \{u_0, \ldots, u_{fs(k)}\}, \quad \text{for } k \geq 0, \tag{4.2}$$

so that

$$|\mu_k| = f(s_k) + 1, \quad \text{for } k \geq 0. \tag{4.3}$$

Since the finite subsets of ν are f-extendible, we can, given any finite subset σ of ν, effectively extend σ to a finite set σ^* such that $\sigma \subset \sigma^* \subset \nu$ and $|\sigma^*| \in \alpha$, hence such that

$$\sigma \subset \sigma^* \subset \nu \quad \text{and} \quad |\sigma^*| \geq \text{the first element } \geq |\sigma| \text{ in } f_0, f_1, \ldots,$$

by formula (1.3). Thus we can effectively extend the finite set $\sigma = \mu_k$ to a finite set $\sigma^* = \xi_k$ such that $|\xi_k| \in \alpha$ and

$$\mu_k \subset \xi_k \subset \nu$$

and

$$|\xi_k| \geq \text{the first element} \geq |\mu_k| \text{ in } f_0, f_1, \dots .$$

However, $|\mu_k| = f(s_k) + 1$ by (4.3) and the number $f(s_k) + 1$ does not occur in the sequence $f_0, f_1, \dots$ by (4.1). This implies

$$\text{(4.4)} \qquad \mu_k \subset \xi_k \subset \nu, \qquad |\xi_k| \geq f(s_k) + 2.$$

We define

$$\delta_0 = (u_{fs(0)}, u_{fs(1)}, \dots), \qquad \delta_1 = (u_{fs(0)+1}, u_{fs(1)+1}, \dots)$$

and we claim that

$$\text{(4.5)} \qquad u_{fs(k)} \simeq u_{fs(k)+1},$$

$$\text{(4.6)} \qquad \delta_0 \mid \delta_1,$$

$$\text{(4.7)} \qquad \delta_0 \cup \delta_1 \mid \nu - (\delta_0 \cup \delta_1),$$

$$\text{(4.8)} \qquad \nu \text{ is not multiple-free.}$$

Re (4.5). Let h be the mapping which maps $u_{fs(k)}$ onto $u_{fs(k)+1}$, for $k \geq 0$, then h is one-to-one. It therefore suffices to show that both h and h^{-1} have p.r. extensions. The function h^{-1} has a p.r. extension, since u_n is a regressive function ranging over ν. Now assume that the number $u_{fs(k)}$ is given, then we can compute u_k, hence also a finite set ξ_k such that (4.4) holds. Since $|\xi_k| \geq p + 1$, for $p = fs(k) + 1$, we can from ξ_k compute the element $u_p = u_{fs(k)+1}$ in the enumeration $u_0, u_1, \dots,$ of ν. Thus the mapping h also has a p.r. extension.

Re (4.6). The functions s_k and f are both strictly increasing, hence

$$s_k < s_{k+1} \Longrightarrow s_k + 1 \leq s_{k+1} \Longrightarrow f(s_k + 1) \leq f(s_{k+1}),$$
$$\text{for } k \geq 0.$$

Combining this with (4.1) we get, for $k \geq 0$,

$$f(s_k) + 2 \leq f(s_k + 1) \leq f(s_{k+1}),$$

hence $f(s_k)+1 < f(s_{k+1})$ and $f(s_0) < f(s_0)+1 < f(s_1) < f(s_1)+1 < f(s_2) < \cdots$. This proves that the sets δ_0 and δ_1 are disjoint. The function $f(s_n)$ is also strictly increasing and recursive. Thus, given any number $u_n \in \delta_0 \cup \delta_1$, we can compute n and decide whether

$$n \in (fs(0), fs(1), \dots)$$

or

$$n \in (fs(0)+1, fs(1)+1, \dots),$$

i.e., determine whether $u_n \in \delta_0$ or $u_n \in \delta_1$. Hence $\delta_0 \mid \delta_1$.

Re (4.7). For an element $u_n \in \nu$,

$$u_n \in \delta_0 \cup \delta_1 \Longrightarrow n \in (fs(0), fs(1), \dots) \cup (fs(0)+1, fs(1)+1, \dots),$$

where the sets on the right are recursive and disjoint. Given any element x of ν, we can compute the number n with $x = u_n$, hence deciding whether $x \in \delta_0 \cup \delta_1$ or $x \in \nu - (\delta_0 \cup \delta_1)$. This proves (4.7).

Re (4.8). Let $N = \operatorname{Req} \nu$ and $D = \operatorname{Req} \delta_0$. Then $\operatorname{Req}(\delta_0 \cup \delta_1) = 2D$ by (4.5) and (4.6), while $2D \leq N$ by (4.7). The isol D is infinite, hence the relation $2D \leq N$ implies that N is not multiple-free. □

Proposition P7. *A strictly increasing, recursive function f has the property that the finite subsets of every immune set are f-extendible if and only if f is trivial.*

Proof. (a) Let f be a strictly increasing, recursive function which is trivial, and let ν be any immune set. Put $\alpha = \rho f$. We now show that the finite subsets of ν are f-extendible. Since f is trivial, there are numbers k and s such that $f(n) = n + k$, for $n \geq s$. Thus, $(s+k, s+k+1, \dots)$ is a subset of α and $|\beta| \geq s+k \Rightarrow |\beta| \in \alpha$, for every finite set β. Taking the contrapositive we get

$$|\beta| \notin \alpha \Longrightarrow |\beta| < s + k. \tag{4.9}$$

Let a set $\sigma \in P_{\text{fin}}(\nu)$ be given with $|\sigma| \notin \alpha$. Then we have, by (4.9),

$$|\sigma| < s + k \quad \text{and} \quad s + k = f(s). \tag{4.10}$$

In view of Remark R2, we may assume without loss of generality that $f_m < |\sigma| < f_{m+1}$, for some number m. Then

$$f_{m+1} = \text{the first number} > |\sigma| \text{ in the sequence } f_0, f_1, \dots, \tag{4.11}$$

hence

$$f_{m+1} = f(n_{|\sigma|}). \tag{4.12}$$

However, the number $f(s) = s + k$ is greater than $|\sigma|$ by (4.10), hence

$$f(n_{|\sigma|}) \leq s + k, \quad \text{where } s + k = f(s). \tag{4.13}$$

Since $|\sigma| \notin \alpha$, we have $|\sigma| < f(n_{|\sigma|})$. Put $e = f(n_{|\sigma|}) - |\sigma|$, then e is positive. We may consider as known a finite subset ξ of ν with $|\xi| = f(s) = s + k$. Define τ as the set of the smallest e numbers in $\xi - \sigma$ and $\sigma^* = \sigma \cup \tau$. The sets σ and τ are disjoint subsets of ν, so that $\sigma \subset_+ \sigma^* \subset \nu$, since e is positive, hence τ is nonempty. Finally,

$$|\sigma^*| = |\sigma| + |\tau| = |\sigma| + e = |\sigma| + f(n_{|\sigma|}) - |\sigma| = f(n_{|\sigma|}).$$

Since σ^* can be computed from σ, it follows that the finite subsets of ν are f-extendible.

(b) We shall show that, for a strictly increasing, recursive function f,

$$(*) \qquad f \text{ nontrivial} \Longrightarrow \begin{array}{l}\text{there is an immune set } \nu \text{ whose} \\ \text{finite subsets are not } f\text{-extendible.}\end{array}$$

For assume the hypothesis of $(*)$. According to [**1**, Theorem 4.1] there is an immune, regressive set ν which is multiple-free. If the finite subsets of ν were f-extendible, it would follow by Proposition P6 that ν is not multiple-free. Thus the finite subsets of ν are not f-extendible. This completes the proof. □

We know by Proposition P5 that if f is a strictly increasing, recursive function which is also combinatorial, there is an immune set μ such that the finite subsets of μ are f-extendible. This raises the question whether the hypothesis that f be combinatorial can be dropped. According to the next proposition, this is the case.

Proposition P8. *Let f be any strictly increasing, recursive function. Then there is an immune, regressive set μ such that the finite subsets of μ are f-extendible.*

Proof. Assume the hypothesis. We shall use the notations

$$\nu_k = \{x \in \varepsilon \mid x < k\}, \qquad j(x, \sigma) = \{j(x, y) \mid y \in \sigma\},$$
$$\text{for } k,\ x \geq 0,\ \sigma \subset \varepsilon.$$

Thus $|\nu_k| = k$ and ν_k is empty if and only if $k = 0$. Let t_n be a regressive function with an immune range. Define

$$a_0 = f_0, \qquad a_{n+1} = f_{n+1} - f_n,$$
$$\eta_n = j(t_n, \nu_{a(n)}), \quad \text{for } n \geq 0,$$

then $|\eta_n| = |\nu_{a(n)}| = a_n$ so that η_0 is empty if and only if $f_0 = 0$, while η_n is nonempty for $n \geq 1$. Define $\mu = \eta_0 \cup \eta_1 \cup \cdots$; then we claim that μ satisfies the requirements. For, first of all, μ is immune and regressive. Now suppose that a finite subset σ of μ is given. Then we may assume without loss of generality that σ is nonempty. Define

$$m = \max\{n \in \varepsilon \mid \sigma \cap \eta_n \text{ is nonempty}\},$$
$$\sigma^* = \bigcup_{n=0}^{m} \eta_n.$$

Since the function t_n is regressive and the function a_n is recursive, we can compute m and σ^* from σ. Clearly, $\sigma \subset \sigma^* \subset \mu$. Moreover,

$$|\sigma^*| = \sum_{n=0}^{m} |\eta_n| = a_0 + a_1 + \cdots + a_n$$
$$= f_0 + (f_1 - f_0) + \cdots + (f_m - f_{m-1}) = f_m,$$

hence $|\sigma^*| \in \rho f$. Thus the finite subsets of μ are f-extendible. □

5. A fundamental lemma. Let f be a strictly increasing, recursive function with range α, and let ν be an infinite set. Suppose that the finite subsets of ν are f-extendible. Then there exists an effective procedure Π with $P_{\text{fin}}(\nu) \subset \text{Dom}\,\Pi$ such that

$$(*) \qquad \tau \in P_{\text{fin}}(\nu) \Longrightarrow \tau \subset \Pi(\tau) \subset \nu \quad \text{and} \quad |\Pi(\tau)| \in \alpha.$$

We shall show that, in the special case that ν is immune, there also exists an effective procedure Δ which shares relation $(*)$ with Π, but which has several additional properties, in particular, $\text{Ran}\,\Delta \subset \text{Dom}\,\Delta$, $\Delta^2 = \Delta$ and

$$\sigma \subset \tau \quad \text{and} \quad \tau \in \text{Dom}\,\Delta \Longrightarrow \sigma \in \text{Dom}\,\Delta \quad \text{and} \quad \Delta(\sigma) \subset \Delta(\tau).$$

If S is a class of finite sets, δS denotes the class of all (finite) subsets of sets in S; thus, $S \subset \delta S$.

Lemma FL. *Let f be a strictly increasing, recursive function with range α, and let ν be an immune set. Suppose that the finite subsets of ν are f-extendible. Then there also exists an effective procedure Δ with $P_{\text{fin}}(\nu) \subset \text{Dom}\,\Delta$ such that*

(a) $\tau \in P_{\text{fin}}(\nu) \Rightarrow \tau \subset \Delta(\tau) \subset \nu$ *and* $|\Delta(\tau)| \in \alpha$,

(b) $\tau \in \text{Dom}\,\Delta \Rightarrow \tau \subset \Delta(\tau)$ *and* $|\Delta(\tau)| \in \alpha$,

(c) $\sigma \subset \tau$ *and* $\tau \in \text{Dom}\,\Delta \Rightarrow \sigma \in \text{Dom}\,\Delta$ *and* $\Delta(\sigma) \subset \Delta(\tau)$,

(d) $\tau \in \text{Dom}\,\Delta \Rightarrow \Delta(\tau) \in \text{Dom}\,\Delta$ *and* $\Delta^2(\tau) = \Delta(\tau)$,

(e) $\text{Ran}\,\Delta = \{\tau \in \text{Dom}\,\Delta | \Delta(\tau) = \tau\}$, *hence* $\text{Ran}\,\Delta \subset \text{Dom}\,\Delta$,

(f) *the class* $E = \text{Ran}\,\Delta$ *is closed under intersection and* $\delta E = \text{Dom}\,\Delta$.

Proof. Assume the hypothesis. Then there exists an effective procedure Π with $P_{\text{fin}}(\nu) \subset \text{Dom}\,\Pi$ such that

$$(5.1) \qquad \tau \in P_{\text{fin}}(\nu) \Longrightarrow \tau \subset \Pi(\tau) \subset \nu \quad \text{and} \quad |\Pi(\tau)| \in \alpha.$$

Define $\text{Dom}_0\Pi$ as the class of all $\tau \in \text{Dom}\,\Pi$ such that

$$(5.2) \quad \forall \sigma, \sigma \subset \tau \Longrightarrow \sigma \in \text{Dom}\,\Pi \quad \text{and} \quad \sigma \subset \Pi(\sigma) \quad \text{and} \quad |\Pi(\sigma)| \in \alpha.$$

Then

$$P_{\text{fin}}(\nu) \subset \operatorname{Dom}_0\Pi \subset \operatorname{Dom}\Pi \quad \text{and} \quad \operatorname{Dom}_0\Pi \text{ is r.e.}$$

and

FL(1). $\tau \in \operatorname{Dom}_0\Pi$ and $\sigma \subset \tau$ implies $\sigma \in \operatorname{Dom}_0\Pi$, by the definition of $\operatorname{Dom}_0\Pi$. We now define

$$\operatorname{Dom}\Pi_0 = \operatorname{Dom}_0\Pi, \qquad \Pi_0(\tau) = \bigcup\{\Pi(\xi) \mid \xi \subset \tau\}.$$

For every set $\tau \in \operatorname{Dom}_0\Pi$, the set $\Pi_0(\tau)$ is well-defined, since each of the sets $\Pi(\xi)$, for $\xi \subset \tau$ is defined by the definition of $\operatorname{Dom}_0\Pi$.

FL(2). $\tau \in \operatorname{Dom}_0\Pi$ implies $\tau \subset \Pi_0(\tau)$.

Proof. By the definitions of $\operatorname{Dom}_0\Pi$ and Π_0. □

FL(3). *Let* $\tau \in \operatorname{Dom}_0\Pi$ *and* $\sigma \subset \tau$. *Then*

(a) $\sigma \in \operatorname{Dom}_0\Pi$, (b) $\Pi(\sigma) \subset \Pi_0(\tau)$, (c) $\Pi_0(\sigma) \subset \Pi_0(\tau)$.

Proof. Assume the hypothesis. Then (a) holds by FL(1) and (b) by the definition of $\Pi_0(\tau)$. Also,

$$\sigma \subset \tau \Longrightarrow \{\Pi(\xi) \mid \xi \subset \sigma\} \subset \{\Pi(\xi) \mid \xi \subset \tau\} \Longrightarrow \Pi_0(\sigma) \subset \Pi_0(\tau). \qquad \square$$

We define $\Pi_0^1 = \Pi_0$ and $\Pi_0^{n+1} = \Pi_0\Pi_0^n$ for $n \geq 1$, and we call a set $\tau \in \operatorname{Dom}_0\Pi$ *terminal*, if there is a positive number k such that

$$(5.3) \qquad \tau, \Pi_0(\tau), \dots, \Pi_0^k(\tau) \in \operatorname{Dom}_0\Pi \quad \text{and} \quad \Pi_0^k(\tau) = \Pi_0^{k+1}(\tau).$$

FL(4). *Assume* $\sigma \subset \tau$, $\tau \in \operatorname{Dom}_0\Pi$ *and* $\Pi_0(\tau) \in \operatorname{Dom}_0\Pi$. *Then*

(a) $\sigma \in \operatorname{Dom}_0\Pi$, (b) $\Pi_0(\sigma) \in \operatorname{Dom}_0\Pi$.

Proof. Under the hypothesis, (a) holds by FL(1). We now prove (b). By FL(3c) we have $\Pi_0(\sigma) \subset \Pi_0(\tau)$. By hypothesis $\Pi_0(\tau) \in \operatorname{Dom}_0\Pi$, hence $\Pi_0(\sigma) \in \operatorname{Dom}_0\Pi$ by FL(1). □

FL(5). *Let* $\tau \in \operatorname{Dom}_0\Pi$ *and* τ *be terminal. Then*

$\sigma \subset \tau$ implies $\sigma \in \mathrm{Dom}\,_0\Pi$ and σ is terminal.

Proof. Assume the hypothesis and $\sigma \subset \tau$. Then $\sigma \in \mathrm{Dom}\,_0\Pi$ by FL(3a). There is a number k such that (5.3) holds, since τ is terminal. In view of $\sigma \subset \tau$ and $\tau \in \mathrm{Dom}\,_0\Pi$, we know that $\Pi_0(\sigma) \subset \Pi_0(\tau)$ by FL(3c). Moreover, $\Pi_0(\tau) \in \mathrm{Dom}\,_0\Pi$ by (5.3), hence in view of FL(3a),

$$\Pi_0(\sigma) \subset \Pi_0(\tau)$$

and

$$\Pi_0(\tau) \in \mathrm{Dom}\,_0\Pi \Longrightarrow \Pi_0(\sigma) \in \mathrm{Dom}\,_0\Pi$$

using FL(3c). By the same reasoning one obtains $\Pi_0^2(\sigma) \subset \Pi_0^2(\tau)$ and $\Pi_0^2(\sigma)$, $\Pi_0^2(\tau) \in \mathrm{Dom}\,_0\Pi$. It now follows from (5.3) that

$$\Pi_0^t(\sigma) \subset \Pi_0^k(\tau) \quad \text{and} \quad \Pi_0^t(\sigma) \in \mathrm{Dom}\,_0\Pi, \quad \text{for } t \geq 1. \tag{5.4}$$

Using (5.4) and FL(2), we conclude that

$$\sigma \subset \Pi_0(\sigma) \subset \Pi_0^2(\sigma) \subset \cdots \subset \Pi^k(\tau). \tag{5.5}$$

Since $\Pi_0^k(\tau)$ is a finite set, there is a number $s \geq 0$ such that $\Pi_0^s(\sigma) = \Pi_0^{s+1}(\sigma) = \cdots$. Combining the last relation with (5.5) and the fact that $\sigma \in \mathrm{Dom}\,_0\Pi$, we conclude that the set σ is also terminal. This completes the proof of FL(5). □

FL(6). *Every finite subset of ν is terminal.*

Proof. Every finite subset of a set in $P_{\text{fin}}(\nu)$ also belongs to $P_{\text{fin}}(\nu)$, hence every finite subset of ν belongs to $\mathrm{Dom}\,_0\Pi$. We recall that

$$\Pi_0(\delta) = \bigcup\{\Pi(\xi) \mid \xi \subset \delta\}, \quad \text{for } \delta \in \mathrm{Dom}\,_0\Pi.$$

Thus,

$$\delta \in P_{\text{fin}}(\nu) \Longrightarrow \Pi_0(\delta) \in P_{\text{fin}}(\nu)$$

and

$$\Pi_0(\delta) \in \mathrm{Dom}\,{}_0\Pi.$$

Now assume that $\tau \in P_{\mathrm{fin}}(\nu)$. Then τ, $\Pi_0(\tau), \Pi_0^2(\tau), \ldots$ all belong to $P_{\mathrm{fin}}(\nu)$, hence also to $\mathrm{Dom}\,{}_0\Pi$. Moreover, by FL(2) we have

$$\tau \subset \Pi_0(\tau) \subset \Pi_0^2(\tau) \subset \cdots .$$

Since Π is an effective procedure and each $\Pi_0^s(\tau)$ for $s \geq 1$ is a finite subset of the immune set ν which can be computed from τ, there is a number k such that $\Pi_0^k(\tau) = \Pi_0^{k+1}(\tau) = \cdots$. Thus τ is a terminal set. □

We define

$$\mathrm{Dom}\,\Delta = \{\tau \in \mathrm{Dom}\,{}_0\Pi \mid \tau \text{ is terminal}\},$$
$$\Delta(\tau) = \Pi_0^k(\tau), \quad \text{where } \Pi_0^k(\tau) = \Pi_0^{k+1}(\tau), \text{ for } \tau \in \mathrm{Dom}\,\Delta.$$

FL(7). Δ *is an effective procedure and* $P_{\mathrm{fin}}(\nu) \subset \mathrm{Dom}\,\Delta$.

Proof. We know that Π is an effective procedure and that $\mathrm{Dom}\,{}_0\Pi$ is an r.e. class. This implies that the subclass $\mathrm{Dom}\,\Delta$ of $\mathrm{Dom}\,{}_0\Pi$ is r.e. Also, given any set $\tau \in \mathrm{Dom}\,\Delta$, we can compute the smallest number k such that $\Pi_0^k(\tau) = \Pi_0^{k+1}(\tau)$, hence $\Pi_0^k(\tau) = \Delta(\tau)$. Thus Δ is an effective procedure. Since every set in $P_{\mathrm{fin}}(\nu)$ is terminal, we have $P_{\mathrm{fin}}(\nu) \subset \mathrm{Dom}\,\Delta$. □

FL(8). $\sigma \subset \tau$ *and* $\tau \in \mathrm{Dom}\,\Delta$ *implies* $\sigma \in \mathrm{Dom}\,\Delta$ *and* $\Delta(\sigma) \subset \Delta(\tau)$.

Proof. Assume the hypothesis. Then $\tau \in \mathrm{Dom}\,{}_0\Pi$ and τ is terminal. Also, we see by FL(5) that $\sigma \subset \tau$ implies $\sigma \in \mathrm{Dom}\,\Delta$. To prove that $\Delta(\sigma) \subset \Delta(\tau)$, we note that in the proof of FL(5) relation (5.5) implies

$$\sigma \subset \Pi_0(\sigma) \subset \Pi_0^2(\sigma) \subset \cdots \subset \Delta(\tau).$$

Since σ is terminal, it follows that $\Delta(\sigma) \subset \Delta(\tau)$. □

FL(9). $\tau \in \operatorname{Dom}\Delta$ *implies* $\Delta(\tau) \in \operatorname{Dom}\Delta$ *and* $\Delta^2(\tau) = \Delta(\tau)$.

Proof. Let $\tau \in \operatorname{Dom}\Delta$. Then $\tau \in \operatorname{Dom}_0\Pi$ and τ is terminal. Let k be a number such that

$$\tau, \Pi_0(\tau), \dots, \Pi_0^k(\tau) \in \operatorname{Dom}_0\Pi$$

and

$$\Pi_0^k(\tau) = \Pi_0^{k+1}(\tau) = \cdots .$$

Then $\Delta(\tau) = \Pi_0^k(\tau)$ and hence $\Delta(\tau) \in \operatorname{Dom}_0(\Pi)$. Also,

$$\Pi_0\Delta(\tau) = \Pi_0\Pi_0^k(\tau) = \Pi_0^{k+1}(\tau) = \Pi_0^k(\tau) = \Delta(\tau)$$

and we proved that $\Delta(\tau)$ is terminal. Hence $\Delta(\tau) \in \operatorname{Dom}\Delta$. Moreover, since $\Delta(\tau) \in \operatorname{Dom}_0\Pi$ and $\Pi_0\Delta(\tau) = \Delta(\tau)$, it follows that $\Delta(\Delta(\tau)) = \Delta(\tau)$, i.e., that $\Delta^2(\tau) = \Delta(\tau)$. This completes the proof of FL(9). □

FL(10). *Let* $\tau \in \operatorname{Dom}\Delta$. *Then*

(a) $\tau, \Pi_0(\tau), \Pi_0^2(\tau), \ldots \in \operatorname{Dom}_0\Pi$.

(b) $\tau \subset \Pi_0(\tau) \subset \Pi_0^2(\tau) \subset \cdots$.

Proof. The hypothesis $\tau \in \operatorname{Dom}\Pi$ implies that $\tau \in \operatorname{Dom}_0\Pi$ and that τ is terminal. Then (a) holds, since τ is terminal, while (b) holds by FL(2) and FL(3). □

FL(11). *Let* $\tau \in \operatorname{Dom}\Delta$. *Then*

$$\text{(a) } \tau \subset \Delta(\tau), \quad \text{(b) } \sigma \subset \Delta(\tau) \Rightarrow \Pi_0(\sigma) \subset \Delta(\tau),$$
$$\text{(c) } \sigma \subset \Delta(\tau) \Rightarrow \Pi(\sigma) \subset \Delta(\tau), \quad \text{(d) } \Pi\Delta(\tau) = \Delta(\tau).$$

Proof. Let $\tau \in \operatorname{Dom}\Delta$. Then $\Delta(\tau) \in \operatorname{Dom}\Delta$ by FL(9). Then also $\tau \in \operatorname{Dom}_0\Pi$ and $\Delta(\tau) \in \operatorname{Dom}_0\Pi$. Suppose $\Delta(\tau) = \Pi_0^k(\tau) = \Pi_0^{k+1}(\tau)$.

Re (a). We have, by FL(10), $\tau \subset \Pi_0(\tau)$ and $\Pi_0(\tau) \subset \Pi_0^{k+1}(\tau)$. Thus $\tau \subset \Pi_0^{k+1}(\tau)$, hence $\tau \subset \Delta(\tau)$.

Re (b). $\sigma \subset \Delta(\tau)$ implies $\sigma \subset \Pi_0^k(\tau)$. However, $\sigma \subset \Pi_0^k(\tau) \Rightarrow \Pi_0(\sigma) \subset \Pi_0^{k+1}(\tau)$ by FL(3), hence $\Pi_0(\sigma) \subset \Delta(\tau)$.

Re (c). $\sigma \subset \Delta(\tau)$ implies $\Pi_0(\sigma) \subset \Delta(\tau)$ by (b). The definition of Π_0 implies that $\Pi(\sigma) \subset \Pi_0(\sigma)$. Hence $\Pi(\sigma) \subset \Delta(\tau)$.

Re (d). The definition of Π_0 implies $\Pi\Delta(\tau) \subset \Pi_0\Delta(\tau)$. Also, $\Pi_0\Delta(\tau) = \Pi_0^{k+1}(\tau) = \Delta(\tau)$. Thus $\Pi\Delta(\tau) \subset \Delta(\tau)$. Since $\Delta(\tau) \in \mathrm{Dom}_0\Pi$, it follows by (5.2) that $\Delta(\tau) \subset \Pi\Delta(\tau)$. We conclude that $\Pi\Delta(\tau) = \Delta(\tau)$. □

FL(12). $\tau \in \mathrm{Dom}\,\Delta \Rightarrow |\Delta(\tau)| \in \alpha$.

Proof. Assume $\tau \in \mathrm{Dom}\,\Delta$. Then $\Delta(\tau) \in \mathrm{Dom}\,\Pi_0$ by the definition of $\Delta(\tau)$. Moreover,

$$\sigma \in \mathrm{Dom}\,\Pi_0 \Longrightarrow \sigma \in \mathrm{Dom}_0\Pi \Longrightarrow |\Pi(\sigma)| \in \alpha$$

by the definitions of $\mathrm{Dom}\,\Pi_0$ and Π, respectively. Thus $|\Pi\Delta(\tau)| \in \alpha$. However, $\tau \in \mathrm{Dom}\,\Delta$ implies $\Pi\Delta(\tau) = \Delta(\tau)$ by FL(11d), hence $|\Delta(\tau)| \in \alpha$. □

FL(13). $\tau \in \mathrm{Dom}\,\Delta$ *and* $\sigma \subset \Delta(\tau)$ *implies* $\Delta(\sigma) \subset \Delta(\tau)$.

Proof. Assume the hypothesis. Then $\tau \in \mathrm{Dom}\,\Delta$ implies $\Delta(\tau) \in \mathrm{Dom}\,\Delta$ by FL(9), while $\sigma \subset \Delta(\tau)$ implies $\sigma \in \mathrm{Dom}\,\Delta$, by FL(8). Since both σ and $\Delta(\tau)$ belong to $\mathrm{Dom}\,\Delta$, we can apply FL(8) and FL(9). Then

$$\sigma \subset \Delta(\tau) \Longrightarrow \Delta(\sigma) \subset \Delta^2(\tau) \Rightarrow \Delta(\sigma) \subset \Delta(\tau). \qquad \square$$

FL(14). *Let* $\sigma, \tau \in \mathrm{Dom}\,\Delta$. *Then*

(a) $\Delta(\sigma) \cap \Delta(\tau) \in \mathrm{Dom}\,\Delta$,

(b) $\Delta[\Delta(\sigma) \cap \Delta(\tau)] = \Delta(\sigma) \cap \Delta(\tau)$.

Proof. Let $\sigma, \tau \in \mathrm{Dom}\,\Delta$. Part (a) then follows by FL(9) and FL(8), as

$$\begin{aligned}\sigma, \tau \in \operatorname{Dom} \Delta &\Longrightarrow \Delta(\sigma), \Delta(\tau) \in \operatorname{Dom} \Delta \\ &\Longrightarrow \underbrace{\Delta(\sigma) \cap \Delta(\tau) \in \operatorname{Dom} \Delta}_{\text{(i)}}.\end{aligned}$$

We claim

(ii) $\Delta(\sigma) \cap \Delta(\tau) \subset \Delta[\Delta(\sigma) \cap \Delta(\tau)]$,

(iii) $\Delta[\Delta(\sigma) \cap \Delta(\tau)] \subset \Delta(\sigma)$,

(iv) $\Delta[\Delta(\sigma) \cap \Delta(\tau)] \subset \Delta(\tau)$,

(v) $\Delta[\Delta(\sigma) \cap \Delta(\tau)] \subset \Delta(\sigma) \cap \Delta(\tau)$.

Re (ii). By (i) and FL(11a).

Re (iii) and (iv). By FL(13).

Re (v). By (iii) and (iv).

Part (b) now follows from (ii) and (v). □

FL(15). *Let* $\tau \in P_{\text{fin}}(\nu)$. *Then* $\tau \in \operatorname{Dom} \Delta$ *and* $\Delta(\tau) \in P_{\text{fin}}(\nu)$.

Proof. Assume $\tau \in P_{\text{fin}}(\nu)$. Then τ is terminal by FL(6), hence $\tau \in \operatorname{Dom} \Delta$. We therefore have shown that $P_{\text{fin}}(\nu) \subset \operatorname{Dom} \Delta$. Now assume $\Delta(\tau) = \Pi_0^k(\tau)$, where $k \geq 0$. Then

$$\tau \subset \nu \Longrightarrow \Pi_0(\tau) = \bigcup\{\Pi(\xi) \mid \xi \subset \tau\} \Longrightarrow \Pi_0(\tau) \subset \nu,$$

since Π maps $P_{\text{fin}}(\nu)$ into itself. Repeating this argument we see that

$$\begin{aligned}\tau \subset \nu &\Longrightarrow \Pi_0(\tau) \subset \nu \Longrightarrow \Pi_0^2(\tau) \subset \nu \\ &\Longrightarrow \cdots \Longrightarrow \Pi_0^k(\tau) \subset \nu \Longrightarrow \Delta(\tau) \subset \nu.\end{aligned}$$ □

FL(16). $\operatorname{Ran} \Delta = \{\tau \in \operatorname{Dom} \Delta \mid \Delta(\tau) = \tau\}$, *hence* $\operatorname{Ran} \Delta \subset \operatorname{Dom} \Delta$.

Proof. We claim that

(i) $\operatorname{Ran} \Delta \subset \{\tau \in \operatorname{Dom} \Delta \mid \Delta(\tau) = \tau\}$,

(ii) $\{\tau \in \operatorname{Dom} \Delta \mid \Delta(\tau) = \tau\} \subset \operatorname{Ran} \Delta$.

However, (ii) is obvious, hence we only have to prove (i). Let $\sigma \in \operatorname{Ran}\Delta$, say $\sigma = \Delta(\tau)$ and $\tau \in \operatorname{Dom}\Delta$. Then

$$\begin{aligned}\tau \in \operatorname{Dom}\Delta &\Longrightarrow \Delta(\tau) \in \operatorname{Dom}\Delta \quad \text{and} \quad \Delta^2(\tau) = \Delta(\tau) \quad \text{by FL(9)}\\ &\Longrightarrow \sigma \in \operatorname{Dom}\Delta \quad \text{and} \quad \Delta(\sigma) = \sigma.\end{aligned}$$

Thus $\sigma \in \operatorname{Ran}\Delta \Rightarrow \sigma \in \operatorname{Dom}\Delta$ and $\Delta(\sigma) = \sigma$, and this implies (i). □

Henceforth, the class $\operatorname{Ran}\Delta$ will also be denoted by E. Hence, $E = \{\tau \in \operatorname{Dom}\Delta \mid \Delta(\tau) = \tau\}$ by FL(16).

FL(17). *The class E is closed under intersection.*

Proof. Let $\beta, \gamma \in E$, say $\beta = \Delta(\sigma)$, $\gamma = \Delta(\tau)$, where $\sigma, \tau \in \operatorname{Dom}\Delta$. Then

$$\begin{aligned}\sigma, \tau \in \operatorname{Dom}\Delta &\Longrightarrow \Delta(\sigma), \Delta(\tau) \in \operatorname{Dom}\Delta \quad \text{by FL(9)}\\ &\Longrightarrow \Delta(\sigma) \cap \Delta(\tau) \in \operatorname{Dom}\Delta\\ &\qquad \text{and} \quad \Delta[\Delta(\sigma) \cap \Delta(\tau)] = \Delta(\sigma) \cap \Delta(\tau)\\ &\Longrightarrow \beta \cap \gamma \in \operatorname{Dom}\Delta\\ &\qquad \text{and} \quad \Delta(\beta \cap \gamma) = \beta \cap \gamma \Longrightarrow \beta \cap \gamma \in E. \qquad \square\end{aligned}$$

FL(18). $\delta E = \operatorname{Dom}\Delta$.

Proof. We claim

(i) $\delta E \subset \operatorname{Dom}\Delta$,

(ii) $\operatorname{Dom}\Delta \subset \delta E$.

Re (i). Recall that Q is the class of all finite sets. By the definition of δE,

$$\begin{aligned}\delta E &= \{\sigma \in Q \mid \sigma \subset \xi, \text{ for some } \xi \in E\}\\ &= \{\sigma \in Q \mid \sigma \subset \Delta(\tau), \text{ for some } \tau \in \operatorname{Dom}\Delta\}.\end{aligned}$$

However, by FL(9),

$$\tau \in \operatorname{Dom}\Delta \Longrightarrow \Delta(\tau) \in \operatorname{Dom}\Delta \text{ and } \sigma \subset \Delta(\tau)$$

implies $\sigma \in \operatorname{Dom}\Delta$ by FL(8). Thus, $\delta E \subset \{\sigma \in Q \mid \sigma \in \operatorname{Dom}\Delta\}$, i.e., $\delta E \subset \operatorname{Dom}\Delta$.

Re (ii). Using FL(11a), we see that

$$\begin{aligned}\sigma \in \operatorname{Dom}\Delta &\Longrightarrow \sigma \subset \Delta(\sigma)\\ &\Longrightarrow \sigma \subset \Delta(\sigma) \quad \text{and} \quad \Delta(\sigma) \in E\\ &\Longrightarrow \sigma \in \delta E. \qquad \square\end{aligned}$$

The different parts of FL have now been proved. For, first of all, Δ is an effective procedure with $P_{\text{fin}}(\nu) \subset \operatorname{Dom}\Delta$ by FL(7). Now consider (a)–(f).

Re (a). Let $\tau \in P_{\text{fin}}(\nu)$. Then $\Delta(\tau)$ is defined. Also, $\tau \subset \Delta(\tau)$ by FL(11a) and $\Delta(\tau) \subset \nu$ by FL(15). Finally, $|\Delta(\tau)| \in \alpha$ by FL(12).

Re (b). By FL(11) and FL(12).

Re (c). By FL(8).

Re (d). By FL(9).

Re (e). By FL(16).

Re (f). By FL(17) and FL(18). $\square$

6. Frames. Recall that Q is the class of all finite sets and that

$$\delta S = \{\tau \in Q \mid (\exists\sigma)[\tau \subset \sigma, \sigma \in S]\}, \quad \text{for } S \subset Q,$$

so that $S \subset \delta S$. A *frame* is a class of finite sets which is closed under intersection. Frames can therefore be finite or denumerable. They were introduced by Nerode in [**8**, Section 2]. They formed his basic tool for extending properties and relations from ε to Λ and Ω. A set β is *attainable* from a frame F, if for all sets τ,

$$\tau \in Q \quad \text{and} \quad \tau \subset \beta \Longrightarrow (\exists\delta)[\tau \subset \delta \subset \beta, \delta \in F].$$

We write $\mathcal{A}(F)$ for the class of all sets attainable from F. Every frame F has the following properties:

Q1. $F \subset \mathcal{A}(F)$,

Q2. A finite set is attainable from F if and only if it belongs to F, Q3. τ is a finite subset of some set in $\mathcal{A}(F)$ if and only if $\tau \in \delta F$.

If F is a frame, C_F is the mapping defined by

$$\operatorname{Dom} C_F = \delta F, \qquad C_F(\tau) = \bigcap \{\xi \in F \mid \tau \subset \xi\}.$$

We claim that

Q4. C_F maps δF onto F, for a frame F.

For, let $\Gamma_F(\tau) = \{\xi \in F \mid \tau \subset \xi\}$, then $C_F(\tau) = \cap\Gamma_F(\tau)$ for $\tau \in \delta F$. Then Q4 would follow from

(a) $F \subset \operatorname{Ran} C_F$,

(b) $\operatorname{Ran} C_F \subset F$.

Part (a) is true, since $C_F(\tau) = \tau$ for $\tau \in F$. Now consider (b). Let $\tau \in \delta F$. Then $\Gamma_F(\tau)$ is finite or denumerable. If $\Gamma_F(\tau)$ is finite, $\cap\Gamma_F(\tau) \in F$, since F is a frame. Now assume that $\Gamma_F(\tau)$ is denumerable, say $\Gamma_F(\tau) = (\xi_0, \xi_1, \dots)$. Define the sequence $\mu_0, \mu_1, \dots$ of sets by $\mu_0 = \xi_0$, $\mu_{n+1} = \mu_n \cap \xi_{n+1}$ for $n \geq 0$. Then $\mu_0 \supset \mu_1 \supset \cdots$. The inclusion $\mu_n \supset \mu_{n+1}$ can be proper for at most finitely many values of n, since μ_0 is a finite set. Thus there is a number k such that $\mu_n = \mu_k$ for $n \geq k$. Put $\bar{\tau} = \mu_k$, then $\bar{\tau} = \mu_0 \cap \mu_1 \cap \cdots = \xi_0 \cap \xi_1 \cap \cdots$. Thus

(i) $\tau \subset \bar{\tau}$,

(ii) $\bar{\tau} \in F$,

(iii) $\xi \in F,\ \tau \subset \xi \Rightarrow \bar{\tau} \subset \xi$.

Using (i), (ii) and (iii), we see that $C_F(\tau) = \bar{\tau}$, hence $C_F(\tau) \in F$.

By definition, for $\tau \in \delta F$, $C_F(\tau)$ is the smallest set in F which includes τ. The mapping C_F from δF onto F maps a subclass of Q into Q, hence it is a procedure. A frame F is *recursive*, if the procedure C_F is effective, i.e., if the mapping $\operatorname{can} \sigma \to \operatorname{can} C_F(\sigma)$, for $\sigma \in \delta F$, is p.r. If $S \subset Q$ we write S^* for the set of all numbers of the form $|\sigma|$ for $\sigma \in S$. A frame F is an *α-frame*, if $F^* \subset \alpha$, i.e., if α contains the cardinality of each set in F. For more information concerning frames, see Nerode [**8**, Section 2], McLaughlin [**6**, Chapter 11] and Barback and Jackson [**2**, Section 3].

Proposition P9. *Let f be a strictly increasing, recursive function with range α, and let ν be an immune set. Then the following two conditions on f and ν are equivalent*:

(a) *the finite subsets of ν are f-extendible*,

(b) *the set ν is attainable from some recursive α-frame.*

Proof. Assume (a). According to FL there is an effective procedure Δ which satisfies the six conditions FL(a)–FL(f). Consider the class $E = \operatorname{Ran}\Delta$. We claim that

(i) E is an α-frame,

(ii) $\nu \in \mathcal{A}(E)$,

(iii) E is a recursive frame.

Re (i). E consists of finite sets and is closed under intersection by FL(f). Let $\sigma \in E$, say $\sigma = \Delta(\tau)$, where $\tau \in \operatorname{Dom}\Delta$. Then $|\sigma| = |\Delta(\tau)|$ and $|\sigma| \in \alpha$ by FL(b). Hence E is an α-frame.

Re (ii). We wish to prove:

$$(*) \qquad \xi \in Q \quad \text{and} \quad \xi \subset \nu \Longrightarrow (\exists\tau)[\xi \subset \tau \subset \nu, \tau \in \operatorname{Ran}\Delta].$$

Assume the hypothesis of $(*)$. Then $\xi \in P_{\text{fin}}(\nu)$, hence $\xi \in \operatorname{Dom}\Delta$. Thus $\xi \subset \Delta(\xi) \subset \nu$ by FL(a). Hence the set $\tau = \Delta(\xi)$ satisfies $\xi \subset \tau \subset \nu$ and $\tau \in \operatorname{Ran}\Delta$. This proves the conclusion of $(*)$.

Re (iii). Suppose we could prove for $\tau \in \operatorname{Dom}\Delta$, i.e., $\tau \in \delta E$,

(iv) $\Delta(\tau) \in \{\xi \in E \mid \tau \subset \xi\}$, i.e., $\Delta(\tau) \in E$ and $\tau \subset \Delta(\tau)$.

(v) $\xi \in E$ and $\tau \subset \xi \Rightarrow \xi \in E$ and $\Delta(\tau) \subset \xi$.

Then (iv) implies $C_E(\tau) \subset \Delta(\tau)$, while (v) implies $\Delta(\tau) \subset C_E(\tau)$, so that $C_E(\tau) = \Delta(\tau)$ for $\tau \in \operatorname{Dom}\Delta$, i.e., $C_E = \Delta$.

Re (iv). Assume $\tau \in \operatorname{Dom}\Delta$. Then $\Delta(\tau) \in \operatorname{Ran}\Delta$, i.e., $\Delta(\tau) \in E$. Also, $\tau \in \operatorname{Dom}\Delta$ implies $\tau \subset \Delta(\tau)$ by FL(b).

Re (v). This is true, since $\tau \subset \xi$ implies $\Delta(\tau) \subset \Delta(\xi)$ by FL(c), while $\xi \in E$ implies $\Delta(\xi) = \xi$.

We have now proved that $C_E = \Delta$. However, Δ is an effective procedure, hence the frame E is recursive. We have proved (b).

Now assume (b), i.e., that the set ν is attainable from some recursive α-frame, say G. Since $\nu \in \mathcal{A}(G)$ we know that for all sets τ,

$$(6.1) \qquad \tau \in Q \quad \text{and} \quad \tau \subset \nu \Longrightarrow (\exists\delta)[\tau \subset \delta \subset \nu, \delta \in G].$$

Put $\Pi = C_G$ and $\tau^* = \Pi(\tau)$ for $\tau \in \operatorname{Dom}\Pi$. We claim that

$$(6.2) \qquad \Pi \text{ is an effective procedure from } \delta G \text{ onto } G,$$

$$(6.3) \qquad P_{\text{fin}}(\nu) \subset \operatorname{Dom}\Pi,$$

$$(6.4) \qquad \tau \subset \tau^* \subset \nu, \quad \text{for } \tau \in P_{\text{fin}}(\nu),$$

$$(6.5) \qquad |\tau^*| \in \alpha, \quad \text{for } \tau \in P_{\text{fin}}(\nu).$$

Re (6.2). C_G is a mapping from δG onto G by Q4. This mapping is an effective procedure, since G is a recursive frame.

Re (6.3). Let $\tau \in P_{\text{fin}}(\nu)$. Then τ is a finite subset of some set attainable from G, namely of ν. This implies $\tau \in \delta G$, hence $\tau \in \operatorname{Dom}\Pi$.

Re (6.4). Let $\tau \in P_{\text{fin}}(\nu)$. Then $\tau \in \operatorname{Dom}\Pi$, hence $\tau \in \delta G$. Thus $\tau \subset \cap\{\xi \in G \mid \tau \subset \xi\}$, i.e., $\tau \subset \tau^*$. Moreover, since $\nu \in \mathcal{A}(G)$ and $\sigma \in P_{\text{fin}}(\nu)$, there is according to (6.1) a set $\delta \in G$ with $\tau \subset \delta \subset \nu$. Thus, among the (finite) sets in G which include τ there is a subset of ν. Hence, $\cap\{\xi \in G \mid \tau \subset \xi\} \subset \nu$, i.e., $\tau^* \subset \nu$.

Re (6.5). Assume $\tau \in P_{\text{fin}}(\nu)$. Then $\tau \in \operatorname{Dom}\Pi$ by (6.3), hence $\tau \in \delta G$. Then

$$\tau \in \delta G \Longrightarrow C_G(\tau) \in G \Longrightarrow \tau^* \in G \Longrightarrow |\tau^*| \in \alpha,$$

since G is an α-frame. We have proved (6.2), (6.3), (6.4) and (6.5), hence that the finite subsets of ν are f-extendible. □

7. Representability.

Definition D5. Let f be a recursive combinatorial function, and let μ be an immune set. Then μ is *f-representable*, if there is a recursive, combinatorial operator Φ and an immune set ν such that Φ induces f and $\mu = \Phi(\nu)$.

If f is a recursive, combinatorial function, we write f_Λ for Myhill's canonical extension of f to a function from Λ into Λ; see [**4**, p. 277]. We define

$$f_\Lambda(\Lambda) = \{f_\Lambda(X) \mid X \in \Lambda\}. \tag{7.1}$$

It follows that if f is a recursive, combinatorial function, then an immune set μ is f-representable if and only if $\mathrm{Req}\,\mu \in f_\Lambda(\Lambda)$. An isol X is *attainable* from a frame F, if X contains at least one set of $\mathcal{A}(F)$. Nerode [**8**, Section 2] associated with every set α a collection α_Λ of isols, namely

$$\alpha_\Lambda = \{X \in \Lambda | X \text{ is attainable from some recursive } \alpha\text{-frame}\}. \tag{7.2}$$

For some of the basic properties of the mapping $\alpha \to \alpha_\Lambda$, see [**2**, Section 3, relations (6)–(10)]. A function f from ε into ε is said to be *linear*, if there are numbers a and b such that $f_n = an + b$; it is *eventually linear*, if there are numbers a and b such that $f_n = an + b$, for almost all n.

Proposition P10 (Nerode). *Let f be a strictly increasing, recursive, combinatorial function with range α. Then*

(a) $f_\Lambda(\Lambda) \subset \alpha_\Lambda$,

(b) *if f is eventually linear,* $f_\Lambda(\Lambda) = \alpha_\Lambda$,

(c) *if f is not eventually linear,* $f_\Lambda(\Lambda) \subset_+ \alpha_\Lambda$.

Proof. [**9**, Lemmas 5.2, 5.3, 5.4]. □

Proposition P11. *Let f be a strictly increasing, recursive, combinatorial function, and let μ be an immune set. Then*

(a) *μ f-representable implies the finite subsets of μ are f-extendible,*

(b) *if f is eventually linear, μ f-representable if and only if the finite subsets of μ are f-extendible,*

(c) *if f is not eventually linear, the converse of* (a) *is false, i.e., there is an immune set μ whose finite subsets are f-extendible, though μ is not f-representable.*

Proof. Assume the hypothesis. Let $M = \operatorname{Req}\mu$. Then μ is f-representable if and only if $M \in f_\Lambda(\Lambda)$, while by P9 the finite subsets of μ are f-extendible if and only if $M \in \alpha_\Lambda$. The three parts of P11 now follow from the three parts of P10. □

The following four statements are corollaries of P11:

(1) *the finite subsets of an immune set μ are $2n$-extendible if and only if μ is $2n$-representable (i.e., even),*

(2) *the finite subsets of an immune set μ are $2n+1$-extendible if and only if μ is $2n+1$-representable (i.e., odd),*

(3) *there is an immune set μ such that the finite subsets of μ are n^2-extendible, though μ is not n^2-representable,*

(4) *there is an immune set μ such that the finite subsets of μ are 2^n-extendible, though μ is not 2^n-representable.*

Remark R3. Let f be a strictly increasing, recursive function with range α, and let ν be an immune set. In Remark R1 we characterized the relation "the finite subsets of ν are f-extendible" as follows: there is a p.r. function p such that

$$(7.3) \qquad \rho_n \subset \nu \quad \text{and} \quad r_n \notin \alpha \Longrightarrow n \in \delta p \quad \text{and} \quad p_n \in \nu - \rho_n.$$

According to (3) there is an immune set ν which is not n^2-representable, though its finite subsets are n^2-extendible. Let $N = \operatorname{Req}\nu$. Then relation (7.3) holds for ν and $\alpha = (0, 1, 4, 9, \dots)$, though N is not a perfect square, i.e., $N \neq X^2$, for every isol X. A similar statement can be made using (4): there is an immune set ν with $\operatorname{Req}\nu = N$ such that relation (7.3) holds for ν and $\alpha = (1, 2, 4, 8, \dots)$, though $N \neq 2^X$, for every isol X.

Let us consider the following two statements from isolic arithmetic:

(7.4) if an isol is divisible by 2 and 3, it is also divisible by 6,

(7.5) if an isol is both a square and a cube, it is also a sixth power.

Re (7.4). This is true. For, let $X \in \Lambda$ and $X = 2Y = 3Z$, for $Y, Z \in \Lambda$. Then $2|3Z$, hence $2|Z$ by [**5**, Theorem 103], say $Z = 2U$. Then $X = 3Z = 6U$, so that $6|X$.

Re (7.5). This is false. See Nerode [**9**, Section 4].

We would like to find out whether the following two statements are true:

(7.6) if the finite subsets of an immune set are both $2n$-extendible and $3n$-extendible, they are also $6n$-extendible,

(7.7) if the finite subsets of an immune set are both n^2-extendible and n^3-extendible, they are also n^6-extendible.

Proposition P12. *Let s and t be strictly increasing, recursive functions with ranges σ and τ, respectively, and let ν be an immune set. Assume that the set $\alpha = \sigma \cap \tau$ is infinite and that u is the strictly increasing recursive function which ranges over α. If the finite subsets of ν are both s-extendible and t-extendible, they are also u-extendible.*

Proof. Assume the hypothesis and also that the finite subsets of ν are both s-extendible and t-extendible. Since the finite subsets of ν are s-extendible, we know by P9 that ν is attainable from some recursive σ-frame. Put $N = \mathrm{Req}\,\nu$; then N is attainable from some recursive σ-frame, hence $N \in \sigma_\Lambda$ by (7.2). Similarly, we see that $N \in \tau_\Lambda$; hence $N \in \sigma_\Lambda \cap \tau_\Lambda$. However, $\sigma_\Lambda \cap \tau_\Lambda = (\sigma \cap \tau)_\Lambda$ by [**2**, Section 3] so that $N \in \alpha_\Lambda$. Hence, the set ν is recursively equivalent to a set whose finite subsets are u-extendible. This implies by P3 that the finite subsets of ν are u-extendible. □

Corollary 1. *If the finite subsets of an immune set are both $2n$-extendible and $3n$-extendible, they are also $6n$-extendible.*

Corollary 2. *If the finite subsets of an immune set are both n^2-extendible and n^3-extendible, they are also n^6-extendible.*

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