ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 1, Winter 1997

DISTRIBUTIONAL CONTROL FOR OPERATORS ON VECTOR-VALUED L^p-SPACES

NAKHLÉ H. ASMAR AND BRIAN P. KELLY

1. Introduction. Throughout this paper, $(\Omega, \mathcal{F}, \mu)$ will denote an arbitrary measure space, and X will be an arbitrary Banach space with norm denoted $\|\cdot\|$. A function $f : \Omega \to X$ is said to be strongly measurable if there exists a sequence $\{f_n\}$ of X-valued simple functions on Ω such that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ μ almost everywhere on Ω. For each $p \in [1, \infty)$, let $L^p(\Omega, \mu, X)$ denote the set of all strongly measurable functions which satisfy $\int_\Omega \|f(\omega)\|^p \, d\mu(\omega) < \infty$. Identifying functions that are equal μ-almost everywhere, this is a Banach space with norm $\|f\|_p = (\int_\Omega \|f(\omega)\|^p \, d\mu(\omega))^{1/p}$. Similarly, the set of essentially bounded, strongly measurable functions from Ω into X, after identifying functions which are equal μ almost everywhere, becomes a Banach space with norm $\|f\|_\infty = \text{ess sup}\{\|f(\omega)\| : \omega \in \Omega\}$ and is denoted by $L^\infty(\Omega, \mu, X)$. For all $p \in [1, \infty]$, $L^p(\Omega, \mu, X)$ will also be denoted by E^p. A function $f : \Omega \to X$ is called weakly measurable if for each $x^* \in X^*$, $t \mapsto x^*(f(t))$ is a measurable scalar-valued function.

In Section 2 we define distributional control for operators and representations. We show that an operator T that is distributionally controlled can be extended to an operator $T^{(p)}$ on the norm-closure of $E^1 \cap E^\infty$ in E^p for $1 \le p \le \infty$. We obtain structural information for such operators that is motivated by the scalar-valued results in [**2**]. With these results, we construct a distributionally controlled operator on $L^1(\Omega) \cap L^\infty(\Omega)$ that dominates T. This concept is related to the linear modulus of an operator as introduced in [**3**], and the L^p-majorant for operators on vector-valued L^p-spaces as introduced in [**4**].

In the case of scalar-valued L^p-spaces, representations consisting of distributionally controlled operators have been used to derive general ergodic theorems (see [**1**] and [**2**] for illustrations). With this in mind, we have specialized some of our results to representations of locally

Received by the editors on April 1, 1994.
The work of the authors was partially funded by the University of Missouri Research Board.
This research is part of the second author's doctoral thesis.

compact abelian groups which are distributionally controlled.

In Section 3 we investigate a natural question concerning the strong continuity of distributionally controlled representations. The principle result obtained in Section 3 can be roughly stated as the following:

> *A representation which is uniformly bounded on E^{p_1} and E^{p_2}, $1 \le p_1,\, p_2 < \infty$, is strongly continuous on E^{p_1} if and only if it is strongly continuous on E^{p_2}.*

The proof we give requires representation results for elements of $(L^p(\Omega, X))^*$, $1 \le p < \infty$. Much of what we present regarding $(L^p(\Omega, X))^*$ appears in [**7**], and we include it here for completeness of the exposition.

Throughout the sequel, X^* will denote the Banach space dual of X. If $x \in X$ and $x^* \in X^*$, then $x^*(x)$ will be denoted by $\langle x, x^* \rangle$ where convenient. If Λ is any set and $f : \Lambda \to X$, the mapping $\omega \mapsto \|f(\omega)\|$ will also be denoted by $N(f)(\omega)$. When f is a real-valued function on Ω and B is a Borel subset of $\mathbf{R}$, $\{\omega \in \Omega : f(\omega) \in B\}$ will occasionally be denoted by $\{f(\omega) \in B\}$. For $A \subset \Omega$, 1_A will denote the characteristic function of A.

2. Fundamental properties of distributionally controlled operators and representations. In this section we list the fundamental properties of μ-distributionally controlled operators and representations. In several cases, the proofs follow the same lines as those found in [**1**] for the case of operators acting on scalar-valued functions. We presented the proofs in those cases where we felt the additional details were needed.

If $f : \Omega \to X$, the distribution function of f is defined for all $y > 0$ as

$$\phi(f : y) = \phi(N(f) : y) = \mu(\{\omega \in \Omega : \|f(\omega)\| > y\}).$$

The following is a simple variation on known properties of distribution functions which we state for ease of reference.

Proposition 2.1. *Let $f, f_1, f_2, f_3, \dots, f_n, \dots$ be strongly measurable X-valued functions. Then the following implications hold*:

i) *If $\|f_n(\omega)\| \nearrow \|f(\omega)\|$ μ almost everywhere on Ω, then* $\lim_{n\to\infty} \phi(f_n : y) = \phi(f : y)$ *for each $y > 0$.*

ii) *If* $f_n \to f$ *in measure we have*

a) $\phi(f : y) \leq \liminf_n \phi(f_n : y)$ *for all* $y > 0$*,*

b) $\limsup_n \phi(f_n : y') \leq \phi(f : y)$ *whenever* $0 < y < y'$*.*

Proof. This is a consequence of the version for scalar-valued functions (see [**1**, Proposition (2.1)]). Simply note that from the inequality $|\|f_n(\omega)\| - \|f(\omega)\|| \leq \|f_n(\omega) - f(\omega)\|$ it follows that $N(f_n) \to N(f)$ pointwise μ almost everywhere, respectively, in measure, whenever $f_n \to f$ pointwise μ almost everywhere, respectively in measure. □

Let Y be a vector space over **R** or **C**. Suppose G is a locally compact abelian group with the group operation being written additively. The group of bijective linear mappings of Y to itself with composition for the group operation is denoted $\mathrm{Aut}\,(Y)$. A group homomorphism $R : G \to \mathrm{Aut}\,(Y)$ is called a representation of G on Y, and it is customary to write $R(u)$ as R_u. The following definitions are motivated by [**2**, Definition 2.1] for representations acting on scalar-valued functions.

Definition 2.2. An operator $T : E^1 \cap E^\infty \to E^1 \cap E^\infty$ is said to be μ*-distributionally controlled* if there exist positive constants c and α such that for all $y > 0$ and for all $f, g \in E^1 \cap E^\infty$:

i) $\phi(Tf : y) \leq c\phi(f : \alpha y)$ and

ii) $\phi(\min\{\|Tf\|, \|Tg\|\} : y) \leq c\phi(\min\{\|f\|, \|g\|\} : \alpha y)$.

We say a representation $u \mapsto R_u$ of G on $E^1 \cap E^\infty$ is μ*-distributionally controlled* if there exist positive constants c and α such that, for all $u \in G$,

i) $\phi(R_u f : y) \leq c\phi(f : \alpha y)$, and

ii) $\phi(\min\{\|R_u f\|, \|R_u g\|\} : y) \leq c\phi(\min\{\|f\|, \|g\|\} : \alpha y)$

for all $y > 0$ and for all $f, g \in E^1 \cap E^\infty$.

The following theorem shows that, for $p \in [1, \infty]$, a μ-distributionally controlled operator T can be extended to an operator defined on the norm-closure of $E^1 \cap E^\infty$ in E^p. The norm-closure of $E^1 \cap E^\infty$ in $L^\infty(\Omega, \mu, X)$ will be denoted by $\mathcal{N}_X$.

Theorem 2.3. *Let T be a μ-distributionally controlled operator on $E^1 \cap E^\infty$. Then the following hold where c and α are as in Definition* 2.2.

i) *Suppose $p \in [1, \infty)$. There is a unique bounded linear operator $T^{(p)}$ on E^p such that $T^{(p)} f = Tf$ for all $f \in E^1 \cap E^\infty$. Moreover, $\|T^{(p)}\| \leq c^{1/p} \alpha^{-1}$.*

ii) *There is a unique bounded linear operator $T^{(\infty)}$ on $\mathcal{N}_X$ such that $T^{(\infty)} f = Tf$ for all $f \in E^1 \cap E^\infty$. Furthermore, $\|T^{(\infty)}\| \leq \alpha^{-1}$.*

Proof. Note first that for all $p \in [1, \infty)$, $E^1 \cap E^\infty$ is dense in E^p, and for $p = \infty$, $E^1 \cap E^\infty$ is dense in $\mathcal{N}_X$. The bound in Theorem 2.3i) follows from using Definition 2.2i) together with the identity

$$\int_\Omega |g(\omega)|^p \, d\mu(\omega) = \int_0^\infty p y^{p-1} \phi(g : y) \, dy,$$

which is valid for any scalar-valued measurable function g. To prove Theorem 2.3ii), if $f \in \mathcal{N}_X$, apply Definition 2.2i) with $y = \alpha^{-1}(\|f\|_\infty + \varepsilon)$ for every $\varepsilon > 0$. □

Corollary 2.4. *Let $u \mapsto R_u$ be a μ-distributionally controlled representation of a locally compact abelian group G on $E^1 \cap E^\infty$. Then, the following statements hold where c and α are as in Definition* 2.2.

i) *For $p \in [1, \infty)$, there is a unique representation $u \mapsto R_u^{(p)}$ of G by bounded linear operators on E^p such that for all $u \in G$ and all $f \in E^1 \cap E^\infty$, $R_u^{(p)} f = R_u f$. Moreover,* $\sup\{\|R_u^{(p)}\| : u \in G\} \leq c^{1/p} \alpha^{-1}$.

ii) *There is a unique representation $u \mapsto R_u^{(\infty)}$ of G by bounded linear operators on $\mathcal{N}_X$ such that, for all $u \in G$ and all $f \in E^1 \cap E^\infty$, $R_u^{(\infty)} f = R_u f$. Also,* $\sup\{\|R_u^{(\infty)}\| : u \in G\} \leq \alpha^{-1}$.

Proof. Let $p \in [1, \infty]$. From Theorem 2.3, all that remains is to show that $u \mapsto R_u^{(p)}$ is a homomorphism. Note that all $u, v \in G$ and all $f \in E^1 \cap E^\infty$, $R_{u+v}^{(p)} f = R_{u+v} f = R_u(R_v f) = R_u^{(p)}(R_v^{(p)} f)$. Since $E^1 \cap E^\infty$ is dense in the domain of $R_u^{(p)}$ and $R_u^{(p)}$ is continuous, it follows that $u \mapsto R_u^{(p)}$ is a group homomorphism of G. □

Measurable subsets of Ω, A and B are said to be *almost disjoint* if $\mu(A \cap B) = 0$. For mappings f and g from Ω to X, we say the functions have *almost disjoint supports* whenever $\|f(\cdot)\|\|g(\cdot)\| = 0$ μ almost everywhere. Suppose that $1 \leq p \leq \infty$ and that F_0 is a subspace of E^p. An operator $T : F_0 \to F_0$ is said to be *separation preserving* if for each pair $f, g \in F_0$ having almost disjoint supports, then Tf and Tg have almost disjoint supports. We say a representation $u \mapsto S_u$ of G is separation preserving if, for each $u \in G$, S_u is a separation preserving operator.

Remark 2.5. If T is a μ-distributionally controlled operator on $E^1 \cap E^\infty$, it follows from Definition 2.2ii) that $T^{(p)}$ will be a separation preserving operator for each $p \in [1, \infty]$. In the presence of Definition 2.2i), a converse of sorts holds.

Proposition 2.6. *An operator T is μ-distributionally controlled if and only if T satisfies Definition* 2.2i) *and is separation preserving on $E^1 \cap E^\infty$.*

Proof. In light of *Remark* 2.5, it remains to show that if T is separation preserving and satisfies Definition 2.2i) with constants c and α, then T satisfies Definition 2.2ii). Suppose $f_1, f_2 \in E^1 \cap E^\infty$, and consider the following disjoint subsets of Ω: $A_1 = \{\omega \in \Omega : \|f_1(\omega)\| < \|f_2(\omega)\|\}$ and $A_2 = \{\omega \in \Omega : \|f_1(\omega)\| \geq \|f_2(\omega)\|\}$. For $j = 1, 2$ and $k = 1, 2$, define $f_{j,k} = f_j 1_{A_k}$. With this notation, we have that for almost all $\omega \in \Omega$,

$$N(f_{1,1} + f_{2,2})(w) = \min(\|f_1(\omega)\|, \|f_2(\omega)\|). \tag{2.1}$$

Also, for $j = 1, 2$, we have $f_j = f_{j,1} + f_{j,2}$. The support of $f_{1,1}$ is almost disjoint from the supports of $f_{1,2}$ and $f_{2,2}$ by construction. Similarly, the support of $f_{2,1}$ is almost disjoint from the supports of $f_{1,2}$ and $f_{2,2}$. Therefore, defining $B_1 = (\operatorname{supp} Tf_{1,1}) \cup (\operatorname{supp} Tf_{2,1})$ and $B_2 = (\operatorname{supp} Tf_{1,2}) \cup (\operatorname{supp} Tf_{2,2})$, we have B_1 and B_2 are almost disjoint since T is separation preserving. Note that, for $\omega \notin B_1 \cup B_2$, $\min(\|Tf_1(\omega)\|, \|Tf_2(\omega)\|) = 0$. For $j, k = 1, 2$, we have $(Tf_j)1_{B_k} =$

$(Tf_{j,1} + Tf_{j,2})1_{B_k} = Tf_{j,k}$ μ almost everywhere by definition. Hence,

$$
\begin{aligned}
\min(\|Tf_1(\cdot)\|, \|Tf_2(\cdot)\|) &= \min(\|Tf_1(\cdot)\|, \|Tf_2(\cdot)\|)1_{B_1}(\cdot) \\
&\quad + \min(\|Tf_1(\cdot)\|, \|Tf_2(\cdot)\|)1_{B_2}(\cdot) \\
&= \min(\|Tf_{1,1}(\cdot)\|, \|Tf_{2,1}(\cdot)\|) \\
&\quad + \min(\|Tf_{1,2}(\cdot)\|, \|Tf_{2,2}(\cdot)\|) \\
&\le \|Tf_{1,1}(\cdot)\| + \|Tf_{2,2}(\cdot)\|.
\end{aligned} \tag{2.2}
$$

Note that $\|Tf_{1,1}(\cdot)\| + \|Tf_{2,2}(\cdot)\| = \|T(f_{1,1} + f_{2,2})(\cdot)\|$ since T is linear and separation preserving. Thus, (2.2) implies that

$$\phi(\min(\|Tf_1\|, \|Tf_2\|) : y) \le \phi(T(f_{1,1} + f_{2,2}) : y). \tag{2.3}$$

From (2.3) and the hypothesis that T satisfies Definition 2.2i), it now follows that

$$\phi(\min(\|Tf_1\|, \|Tf_2\|) : y) \le c\phi(f_{1,1} + f_{2,2} : \alpha y). \tag{2.4}$$

Comparing (2.1) and (2.4), we see that T satisfies Definition 2.2ii). □

Suppose T is a μ-distributionally controlled operator on $E^1 \cap E^\infty$. For each $p \in [1, \infty]$, if $T^{(p)}$ denotes the corresponding extension appearing in Theorem 2.3, we will use F^p to denote the domain of $T^{(p)}$. Thus, for $p \in [1, \infty)$, $F^p = E^p$ and $F^\infty = \mathcal{N}_X$. The proof of the following proposition is similar to that of [**1**, Proposition 2.10]. We omit the details.

Proposition 2.7. *Fix $p \in [1, \infty]$, and suppose that T is a μ-distributionally controlled operator. With $T^{(p)}$ and F^p as in the preceding paragraph, for all $y > 0$ and for all $f \in F^p$,*

$$\phi(T^{(p)} f : y) \le c\phi(f : \alpha y).$$

We now state one of the central results of this section. In the case that T is an invertible operator such that T and its inverse are μ-distributionally controlled, the following theorem provides an alternative formulation of μ-distributional control. The proof is carried out using arguments very close to the proof of [**2**, Theorem 2.6] so we will omit the details. However, we do supply a sketch of the proof because two estimates which appear in the proof will be crucial for the sequel.

Theorem 2.8. *Let $p \in [1,\infty)$, and suppose that S is an invertible operator on E^p. Then there exists an invertible μ-distributionally controlled operator T on $E^1 \cap E^\infty$ with T^{-1} μ-distributionally controlled such that $S = T^{(p)}$ and $S^{-1} = (T^{-1})^{(p)}$ if and only if S and S^{-1} are separation preserving operators and there exist constants $C_p, C_\infty > 0$ such that for each $j \in \{-1,1\}$,*

i) $\|S^j f\|_p \le C_p \|f\|_p$ *for all* $f \in E^p$ *and*

ii) $\|S^j f\|_\infty \le C_\infty \|f\|_\infty$ *for all* $f \in E^p \cap E^\infty$.

Sketch of Proof. Suppose that T and T^{-1} are each μ-distributionally controlled operators satisfying Definition 2.2 with constants c and α. Applying Theorem 2.3 and Propostion 2.7 yields that Theorem 2.8i) and ii) hold with $C_p = c^{1/p}\alpha^{-1}$ and $C_\infty = \alpha^{-1}$.

Suppose that S and S^{-1} are separation preserving operators satisfying Theorem 2.8i) and ii). Let $x \in X$ with $\|x\| = 1$. We obtain the following estimates whenever $\delta \in \mathcal{F}$ with $0 < \mu(\delta) < \infty$ and $\beta = \operatorname{supp} S(x1_\delta)$ by using arguments analogous to those used to prove the estimates in Theorem 2.9 and Lemma 2.11 appearing in [**2**].

$$C_\infty^{-1} 1_\beta(\omega) \le \|S(x1_\delta)(\omega)\| \le C_\infty 1_\beta(\omega), \quad \mu \text{ a.e. on } \Omega. \tag{2.5}$$

$$(C_p C_\infty)^{-p}\mu(\beta) \le \mu(\delta) \le (C_p C_\infty)^p \mu(\beta). \tag{2.6}$$

It is important to note that (2.5) and (2.6) hold with the same constants for all choices of $x \in X$ with $\|x\| = 1$. So, as in the proof of [**2**, Theorem 2.6], it can be shown that $T = S|_{E^1 \cap E^\infty}$ satisfies Definition 2.2i) with constants $c = (C_p C_\infty)^p$ and $\alpha = C_\infty^{-1}$. Proposition 2.6 completes the proof since T is separation preserving. □

The following theorem is the analog of Theorem 2.8 for the case of representations.

Theorem 2.9. *Let $p \in [1,\infty)$, and suppose that $u \mapsto S_u$ is a representation of G on E^p. Then there is a μ-distributionally controlled representation R of G such that $S = R^{(p)}$ if and only if S consists of separation preserving operators and there exist constants C_p, $C_\infty > 0$ such that for all $u \in G$,*

i) $\|S_u f\|_p \le C_p \|f\|_p$ *for all* $f \in E^p$;

ii) $\|S_u f\|_\infty \le C_\infty \|f\|_\infty$ *for all* $f \in E^p \cap E^\infty$.

Remark 2.10. Suppose T and T^{-1} are both μ-distributionally controlled operators on $E^1 \cap E^\infty$. For each $p \in [1,\infty)$, we construct an operator on $L^p(\Omega)$, $T_0^{(p)}$, satisfying the relation $N(T^{(p)}f)(\omega) \le T_0^{(p)}(N(f))(\omega)$ μ almost everywhere for all $f \in E^p$. When R is a μ-distributionally controlled representation of G on $E^1 \cap E^\infty$, we construct a representation of G on $L^p(\Omega)$, $\mu \mapsto \hat{R}_u^{(p)}$, satisfying the relation $N(R_u^{(p)} f)(\omega) \le \alpha^{-1} \hat{R}_u^{(p)}(N(f))(\omega)$ μ almost everywhere for all $f \in E^p$. Our construction is motivated by [**8**]. We first need the following lemma.

Lemma 2.11. *Let $p \in [1,\infty)$. Suppose T is an invertible operator on E^p and that T^{-1} is separation preserving. If $f, g \in E^p$ are such that* $\operatorname{supp} f \subset \operatorname{supp} g$ *μ almost everywhere, then* $\operatorname{supp} Tf \subset \operatorname{supp} Tg$ *μ almost everywhere.*

Proof. Suppose that $f, g \in E^p$ satisfy $\operatorname{supp} f \subset \operatorname{supp} g$ μ almost everywhere. Let $\sigma = \operatorname{supp} Tg$, and write $T(f) = 1_\sigma Tf + 1_{\sigma^c} Tf$. Clearly, the support of $1_{\sigma^c} Tf$ is almost disjoint from $\operatorname{supp} 1_\sigma Tf$ and $\operatorname{supp} Tg$. Because T^{-1} is separation preserving, we get the following equalities.

$$\mu(\operatorname{supp} T^{-1}(1_{\sigma^c} Tf) \cap \operatorname{supp} T^{-1}(1_\sigma Tf)) = 0. \tag{2.7}$$

$$\mu(\operatorname{supp} T^{-1}(1_{\sigma^c} Tf) \cap \operatorname{supp} g) = 0. \tag{2.8}$$

Since $\operatorname{supp} f \subset \operatorname{supp} g$, it follows from (2.8) that

$$\mu(\operatorname{supp} T^{-1}(1_{\sigma^c} Tf) \cap \operatorname{supp} f) = 0. \tag{2.9}$$

From (2.7) and (2.9) we obtain the equality

$$\mu(\operatorname{supp} T^{-1}(1_{\sigma^c} Tf) \cap \operatorname{supp} (f - T^{-1}(1_\sigma Tf))) = 0. \tag{2.10}$$

The linearity of T^{-1} implies that

$$f - T^{-1}(1_\sigma Tf) = T^{-1}(1_{\sigma^c} Tf), \quad \mu \text{ a.e.} \tag{2.11}$$

Therefore, (2.10) and (2.11) imply that $\mu(\operatorname{supp} T^{-1}(1_{\sigma^c} Tf)) = 0$, and it follows that $f = T^{-1}(1_\sigma Tf)$ and $Tf = 1_\sigma Tf$ μ almost everywhere. This gives the desired result. □

Remark 2.12. In order to construct the desired operator $T_0^{(p)}$, we first consider a set mapping induced by T. Fix $x_0 \in X$, $\|x_0\| = 1$. If $\delta \subset \Omega$ with $\mu(\delta) < \infty$, $x_0 1_\delta \in E^1 \cap E^\infty$, so $T(x_0 1_\delta)$ is well-defined. So, in turn,

$$\Phi_T(\delta) \equiv \operatorname{supp} T(x_0 1_\delta) \tag{2.12}$$

is well-defined. Suppose that $\delta \subset \Omega$, $\mu(\delta) < \infty$. Applying Lemma 2.11 twice shows that the following equality holds up to a set of measure 0 for every $x \in X$:

$$\Phi_T(\delta) = \operatorname{supp} T(x 1_\delta). \tag{2.13}$$

Now recall that, from the proof of Theorem 2.8, $T^{(1)}$ satisfies Theorem 2.8i) with $C_1 = c\alpha^{-1}$ and Theorem 2.8ii) with $C_\infty = \alpha^{-1}$. Because of this, (2.5) and (2.13) imply that, for all $x \in X$ with $\|x\| = 1$, and for all $\delta \subset \Omega$, $\mu(\delta) < \infty$,

$$\alpha 1_{\Phi_T(\delta)}(\omega) \le \|T(x1_\delta)(\omega)\| \le \alpha^{-1} 1_{\Phi_T(\delta)}(\omega), \quad \mu \text{ a.e. on } \Omega. \tag{2.14}$$

Similarly, (2.6) and (2.13) imply that, for all $\delta \subset \Omega$, $\mu(\delta) < \infty$,

$$c^{-1}\alpha^2 \mu(\Phi_T(\delta)) \le \mu(\delta) \le c\alpha^{-2}\mu(\Phi_T(\delta)). \tag{2.15}$$

The following lemma shows that Φ_T preserves two basic set operations.

Lemma 2.13. *Suppose that T is a μ-distributionally controlled operator on $E^1 \cap E^\infty$ which is invertible with T^{-1} μ-distributionally controlled. For all $\delta \in \mathcal{F}$ such that $\mu(\delta) < \infty$, let $\Phi_T(\delta)$ be as in* (2.12)*. Suppose that, for $j = 1, \dots, n$, $\delta_j \in \mathcal{F}$ and $\mu(\delta_j) < \infty$. Then the* *equalities* $\Phi_T(\cup_{j=1}^n \delta_j) = \cup_{j=1}^n \Phi_T(\delta_j)$ *and* $\Phi_T(\cap_{j=1}^n \delta_j) = (\cap_{j=1}^n \Phi_T(\delta_j)$ *each hold up to sets of measure* 0.

Proof. It suffices to prove the lemma with $n = 2$. Suppose that, for $j = 1, 2$, $\delta_j \in \mathcal{F}$ and $\mu(\delta_j) < \infty$. For the remainder of this proof, any statements of inclusion or equality between sets are to be understood as holding up to sets of measure 0. By Lemma 2.11 and the definition of Φ_T, we have that

$$\Phi_T(\delta_1) \cup \Phi_T(\delta_2) \subset \Phi_T(\delta_1 \cup \delta_2). \tag{2.16}$$

From (2.12) and the fact that T is linear and separation preserving, we obtain

$$\begin{aligned} \Phi_T(\delta_1 \cup \delta_2) &= \operatorname{supp} T(x_0 1_{\delta_1}) \cup \operatorname{supp} T(x_0 1_{\delta_2 \backslash \delta_1}) \\ &= \Phi_T(\delta_1) \cup \Phi_T(\delta_2 \backslash \delta_1) \end{aligned} \tag{2.17}$$

where x_0 is as in (2.12). Since $\delta_2 \backslash \delta_1 \subset \delta_2$, (2.17) and Lemma 2.11 imply that

$$\Phi_T(\delta_1 \cup \delta_2) \subset \Phi_T(\delta_1) \cup \Phi_T(\delta_2). \tag{2.18}$$

Comparing (2.16) and (2.18) we get the desired equality: $\Phi_T(\delta_1 \cup \delta_2) = \Phi_T(\delta_1) \cup \Phi_T(\delta_2)$.

Since Φ_T preserves finite unions, we have that $\Phi_T(\delta_1) = \Phi_T(\delta_1 \cap \delta_2) \cup \Phi_T(\delta_1 \backslash \delta_2)$ and $\Phi_T(\delta_2) = \Phi_T(\delta_1 \cap \delta_2) \cup \Phi_T(\delta_2 \backslash \delta_1)$. Since T is separation preserving, it is now easy to check that $\Phi_T(\delta_1) \cap \Phi_T(\delta_2) = \Phi_T(\delta_1 \cap \delta_2)$. □

Remark 2.14. Suppose that f is a scalar-valued simple function, $f = \sum_{k=1}^n \gamma_k 1_{\delta_k}$, where $\gamma_1, \dots, \gamma_n$ are scalars and $\delta_1, \dots, \delta_n$ are pairwise disjoint measurable subsets of Ω having finite measure. Define $\hat{T} f$ as follows:

$$\hat{T} f = \sum_{k=1}^n \gamma_k 1_{\Phi_T(\delta_k)}. \tag{2.19}$$

Using (2.15) and Lemma 2.13, it can be shown that, for each scalar-valued simple function f,

$$\phi(\hat{T} f : y) \le c\alpha^{-2} \phi(f : y). \tag{2.20}$$

The next proposition shows that $\hat{T}f$ can be extended to all of $f \in L^1(\Omega) \cap L^\infty(\Omega)$ while still satisfying (2.20).

Proposition 2.15. *Suppose that T and its inverse are μ-distributionally controlled operators on $E^1 \cap E^\infty$. Let $\hat{T}$ be as defined in* (2.19)*. Suppose that $f \in L^1(\Omega) \cap L^\infty(\Omega)$, and let $\{f_n\}$ denote a sequence of simple functions converging to f in measure. Then $\{\hat{T}f_n\}$ is a sequence that is Cauchy in measure, and so converges in measure to a function $\hat{T}f$ that depends only on f and not on the particular sequence $\{f_n\}$. Moreover, if c and α are the constants for T appearing in* (2.2)*, then for all $f \in L^1(\Omega) \cap L^\infty(\Omega)$,*

$$\phi(\hat{T}f : y) \leq c\alpha^{-2}\phi(f : y). \tag{2.21}$$

Proof. Since $f \in L^1(\Omega)$, there exists a sequence $\{f_n\}$ of simple functions which converge to f in $L^1(\Omega)$-norm and consequently in measure. Thus, $\{f_n\}$ is Cauchy in measure. With this and (2.20), it can be shown that the sequence $\{\hat{T}f_n\}$ is Cauchy in measure, and therefore will converge to a measurable function $\hat{T}f$ in measure.

From (2.20), it is clear that if $\{f_n\}$ is a sequence of simple functions converging to 0 in measure, then $\{\hat{T}f_n\}$ converges to 0 in measure. Therefore, it follows that $\hat{T}f$ is well-defined up to sets of measure 0 and does not depend on the particular sequence $\{f_n\}$.

It can be shown that, for every $f \in L^1(\Omega) \cap L^\infty(\Omega)$ and for every $y > 0$, (2.21) holds. This is done using (2.20) and Proposition 2.1ii) in an argument similar to the proof of [**1**, Proposition 2.10]. □

Proposition 2.16. *Suppose that T is an invertible operator on $E^1 \cap E^\infty$ such that T and its inverse are μ-distributionally controlled. Then the operator $\hat{T}$ defined as in* (2.19) *and Proposition* 2.15 *will be μ-distributionally controlled and satisfy Definition* 2.2i) *with constants $c' = c\alpha^{-2}$ and $\alpha' = 1$.*

Proof. By Proposition 2.6 it remains to show that $\hat{T}$ is separation preserving and maps $L^1(\Omega) \cap L^\infty(\Omega)$ into itself since $\hat{T}$ satisfies (2.21). From the construction of $\hat{T}$ and the fact that T is separation preserving, it is clear that if f and g are simple scalar-valued functions with almost disjoint supports, $\hat{T}f$ and $\hat{T}g$ will have almost disjoint supports. It will then follow that $\hat{T}$ is separation preserving on $L^1(\Omega) \cap L^\infty(\Omega)$. It can be checked that $\hat{T}$ maps into $L^1(\Omega) \cap L^\infty(\Omega)$ using (2.21). $\square$

Remark 2.17. For each $p \in [1, \infty)$, $\hat{T}$ can be extended to a bounded operator on $L^p(\Omega)$ denoted $\hat{T}^{(p)}$ such that $\|\hat{T}^{(p)}\| \leq (c\alpha^{-2})^{1/p}$ by Theorem 2.3. It is obvious that, for each $p \in [1, \infty)$, $\hat{T}^{(p)}$ is a positive operator on $L^p(\Omega)$.

The operator $T_0^{(p)} = \alpha^{-1}\hat{T}^{(p)}$ resembles the linear modulus for an operator acting on a space of scalar-valued functions as introduced in [**3**]. There, for an operator S on $L^1(\Omega)$, the linear modulus S_0 is defined as a positive operator on $L^1(\Omega)$ such that

(i) $\|S\| = \|S_0\|$;

(ii) for all $f \in L^1(\Omega)$, $|Sf| \leq S_0(|f|)$;

(iii) for all $f \in L^1(\Omega)^+$, $S_0(f) = \sup\{|S(g)| : g \in L^1(\Omega),\ |g| \leq f\}$.

The operator $T_0^{(p)}$ is a positive operator, and the corresponding version of (ii) will be established in the following proposition.

Proposition 2.18. *Suppose T and T^{-1} are μ-distributionally controlled operators. Let $T^{(p)}$ be the operator on E^p given by Theorem* 2.3, *and suppose that $\hat{T}^{(p)}$ is the operator on $L^p(\Omega)$ obtained in Remark* 2.17. *Then, for all $f \in E^p$, and for almost all $\omega \in \Omega$, we have*

$$N(T^{(p)}f)(\omega) \leq \alpha^{-1}\hat{T}^{(p)}(N(f))(\omega). \tag{2.22}$$

Proof. Let f be a simple function in E^p, $f = \sum_{k=1}^n y_k 1_{\delta_k}$ where, for $k = 1, \dots, n$, $y_1, \dots, y_n$ are nonzero elements of X, and $\delta_1, \dots, \delta_n$ are pairwise disjoint measurable subsets of Ω with $\mu(\delta_k) < \infty$. For $k = 1, \dots, n$, let $x_k = y_k/\|y_k\|$. Then (2.14) and the disjointness of the δ_k's imply that μ almost everywhere

$$\begin{aligned} N(T^{(p)}f)(\cdot) &= \sum_{k=1}^n \|y_k\| \|T^{(p)}(x_k 1_{\delta_k})(\cdot)\| \\ &\leq \alpha^{-1} \sum_{k=1}^n \|y_k\| 1_{\Phi_T(\delta_k)}(\cdot). \end{aligned}$$

But $N(f) = \sum_{k=1}^{n} \|y_k\| 1_{\delta_k}$ and $\hat{T}^{(p)}(N(f)) = \sum_{k=1}^{n} \|y_k\| 1_{\Phi_T(\delta_k)}$ by definition. Thus, (2.22) holds for simple functions in E^p. The proof for an arbitrary $f \in E^p$ follows by using a sequence of simple functions converging to f pointwise μ almost everywhere. □

Remark 2.19. (i) Suppose that $(\Omega, \mathcal{F}, \mu)$ is a probability space and that S is an operator on $L^p(\Omega, X)$. In [**4**] there are several results showing that with additional hypotheses on S, X and p, there exists an operator S_0 on $L^p(\Omega)$ such that $\|S(f)(\omega)\| \leq S_0(N(f)(\omega))$. Therein, the operator S_0 is referred to as an L^p-majorant for S. Proposition 2.18 shows that, in a similar manner, the operator $T_0^{(p)} = \alpha^{-1}\hat{T}^{(p)}$ can be referred to as an L^p-majorant for $T^{(p)}$. As mentioned in Remark 2.17, this operator also resembles the linear modulus for an operator acting on a space of scalar-valued functions. The operator $T_0^{(p)}$ obtained here has been constructed with much weaker conditions on $(\Omega, \mathcal{F}, \mu)$ and X, but we require stronger conditions on the operator, namely the operator must be invertible.

(ii) We wish to give an example of the above construction. Fix $q \in (1, \infty)$, and let $X = L^q(\mathbf{T})$. For each $p \in [1, \infty)$, define $E^p = L^p(\mathbf{R}, X)$, where $\mathbf{R}$ has Lebesgue measure. For $x \in X = L^q(\mathbf{T})$, we use $\tilde{x}$ to denote the harmonic conjugate of x and $m(x)$ to denote $\hat{x}(0)$. Define the operator $T^{(p)} : E^p \to E^p$ by $T^{(p)} f(\omega) = \widetilde{f(2\omega)} + m(f(2w))$ almost everywhere on $\mathbf{R}$ for each $f \in E^p$. As a consequence of M. Riesz's theorem, the mapping $x \mapsto \tilde{x}$ is a bounded operator mapping $L^q(\mathbf{T})$ into itself with a norm we will denote A_q. We can also see that $T^{(p)}$ is a bijective operator with $(T^{(p)})^{-1} f(\omega) = -\widetilde{f(\omega/2)} + m(f(w/2))$. Clearly, $\|T^{(p)}\|$, $\|(T^{(p)})^{-1}\| \leq 2A_q + 1$.

One can verify that T is separation preserving while satisfying Theorem 2.8i) and ii) with constants $C_p = 2A_q + 1$ and $C_\infty = A_q$, respectively. Hence, the operator $T^{(p)}|_{E^1 \cap E^\infty}$ is distributionally controlled. The corresponding operator $\hat{T}$ on $L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ transforms simple scalar-valued functions in the following manner: $\hat{T}(\sum_{k=1}^{n} \gamma_k 1_{A_k}) = \sum_{k=1}^{n} \gamma_k 1_{2A_k}$. Furthermore, note that the powers of $T^{(p)}$ are not uniformly bounded; thus, this operator cannot be found within a distributionally controlled group representation.

Although the form of the set mapping Φ is obvious in this case, this example illustrates how the results proved in [**4**] do not circumscribe what we have obtained here. The example $T^{(p)}$ of the preceding paragraph is never an isometry of E^p, while the theorems in [**4**] treated isometries. Also, for spaces such as $L^p(\mathbf{R}, L^q(\mathbf{T}))$, the restriction $p \neq q$ is needed in [**4**]. We do not need to make such a restriction.

(iii) Suppose $u \mapsto R_u$ is a μ-distributionally controlled representation of a locally compact group G into $E^1 \cap E^\infty$ with c and α for the constants appearing in Definition 2.2. Constructing $\hat{R}_u$ as above for each $u \in G$, it follows that $\hat{R}_u$ satisfies Definition 2.2 with constants $c' = c\alpha^{-2}$ and $\alpha' = 1$ for all $u \in G$. We will show that $u \mapsto \hat{R}_u$ is a representation. It is for this reason that we distinguish the operators $\hat{R}_u$, $u \in G$. In the case of a representation, the set function Φ_{R_u} will be denoted simply as Φ_u for each $u \in G$.

Proposition 2.20. *Let R be a μ-distributionally controlled representation of G. For each $u \in G$, let $\hat{R}_u$ be defined as in* (2.19) *and Proposition* 2.15. *Then $u \mapsto \hat{R}_u$ is a μ-distributionally controlled representation of G on $L^1(\Omega) \cap L^\infty(\Omega)$.*

Proof. From Proposition 2.16, all that remains is to show that $u \mapsto \hat{R}_u$ is a group homomorphism since Definition 2.2i) holds with constants independent of u. It will suffice to prove that $\hat{R}_{u+v} f = \hat{R}_u(\hat{R}_v f)$ for $f = 1_\delta$ where $\delta \in \mathcal{F}$ such that $\mu(\delta) < \infty$. This is equivalent to proving that $\Phi_{u+v}(\delta) = \Phi_u(\Phi_v(\delta))$. Now $\Phi_{u+v}(\delta) = \operatorname{supp} R_{u+v}(x_0 1_\delta) = \operatorname{supp} R_u(R_v(x_0 1_\delta))$. From Lemma 2.11, it follows that $\operatorname{supp} R_u(R_v(x_0 1_\delta)) = \operatorname{supp} R_u(x_0 1_{\Phi_v(\delta)})$ up to a set of measure 0. But by the definition of Φ_u, this is $\Phi_u(\Phi_v(\delta))$. □

Remark 2.21. We now give an example of the above construction involving a distributionally controlled representation. Fix $q \in (1, \infty)$. For each $p \in [1, \infty)$, let X, E^p and A_q be as in Remark 2.19ii). Let $T : E^p \to E^p$ be given by $Tf(\omega) = \widetilde{f(\omega)} + \operatorname{im}(f(w))$ almost everywhere on $\mathbf{R}$ for each $f \in E^p$. We can then see that T is a bijective operator with $T^{-1} f(\omega) = -\widetilde{f(\omega)} - \operatorname{im}(f(w))$. Clearly, $\|T\|$, $\|T^{-1}\| \leq A_q + 1$.

One can check that $T^2 = -I$ and $T^4 = I$ where I denotes the identity operator on $L^p(\mathbf{R}, X)$. Let $\mathbf{Z}$ act on $E^p = L^p(\mathbf{R}, X)$ by $R_n^{(p)} = T^n$. Clearly, $R^{(p)}$ is separation preserving and satisfies Theorem 2.9i) and

ii) with constants $C_p = A_q + 1$ and $C_\infty = A_q$, respectively. Hence, the representation is distributionally controlled. The corresponding representation $\hat{R}$ on $L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ is merely the identity operator in this case.

3. Strong continuity for distributionally controlled representations. For this section, consider the Banach space X as a vector space over $\mathbf{R}$. Suppose G is a locally compact abelian group with Haar measure λ, and let $u \mapsto R_u$ be a μ-distributionally controlled representation of G on $E^1 \cap E^\infty$. For all $p \in [1, \infty)$, the corresponding representation of G on E^p given by Theorem 2.3 will be denoted as $u \mapsto R_u^{(p)}$ or simply $R^{(p)}$.

The main result of this section is the proposition that if, for some $p_0 \in [1, \infty)$, $R^{(p_0)}$ is strongly continuous, then for all $p \in [1, \infty)$, $R^{(p)}$ is strongly continuous. This question is motivated by the scalar-valued case treated in [**1**] and arises naturally when considering operators of the form $H_k f \equiv \int_G k(u) R_{-u}^{(p)} f \, d\lambda(u)$ where $k \in L^1(G, \lambda)$. The proof herein requires results appearing in [**7**] regarding the Banach space dual of vector-valued L^p-spaces. Also, some aspects of the proof involve an approach which differs from the methods used in the scalar-valued case; this approach uses a fundamental result from [**9**].

Up to Theorem 3.4, the discussion which follows is preliminary material necessary for a statement of the results from [**7**] used herein. Much of the terminology and notation is adopted from [**7**]. The reader who is familiar with the representations of $(L^p(\Omega, X))^*$ may omit Definition 3.1 through *Remark* 3.6 and proceed to Lemma 3.7.

The integration theory in [**7**] is based upon an upper integral $I(\cdot)$ defined for all nonnegative extended real-valued functions and a suitable collection of functions $\mathcal{R}$ which receives special consideration. Letting $\overline{\mathbf{R}}_+$ denote $[0, \infty]$, the precise conditions required for $I(\cdot)$ and $\mathcal{R}$ follow.

Definition 3.1. If Λ is a set, a mapping $I : (\overline{\mathbf{R}}_+)^\Lambda \to \overline{\mathbf{R}}_+$ is called an *upper integral on* Λ if the following hold for $f, g_1, g_2, \ldots \in (\overline{\mathbf{R}}_+)^\Lambda$:

i) $I(0) = 0$:

ii) $I(\lambda f) = \lambda I(f)$ for all $\lambda > 0$;

iii) if $f \leq \sum_{n=1}^\infty g_n$, then $I(f) \leq \sum_{n=1}^\infty I(g_n)$;

iv) if $\{g_n\}$ is an increasing sequence, then $I(\sup_n g_n) = \sup_n I(g_n)$.

Suppose I satisfies Definition 3.1. Let $\mathcal{R}$ denote a vector space of real-valued functions on Λ satisfying the following conditions.

i) for every $f \in \mathcal{R}$, $I(|f|) < \infty$;

ii) for each continuous $\rho : \mathbf{R} \to \mathbf{R}$ such that $\rho(0) = 0$, $\rho \circ f \in \mathcal{R}$ for every $f \in \mathcal{R}$;

(iii) if $f, g \in \mathcal{R}$ are such that $f \geq 0$ and $g \geq 0$, then $I(f+g) = I(f) + I(g)$.

In this paper, if I is an upper integral on the set Λ and $\mathcal{R}$ is a vector space satisfying conditions (i)–(iii), the triple $(\Lambda, \mathcal{R}, I)$ will be called an *upper integral space*.

For $p \in [1, \infty)$, let $I_p(f) \equiv I(|f|^p)^{1/p}$ for all real-valued functions on Λ. The collection of all $f : \Lambda \to \mathbf{R}$ for which there exists a sequence $\{f_n\} \subset \mathcal{R}$ such that $\lim_n I_p(f - f_n) = 0$ will be denoted as $\mathcal{L}^p(\Lambda, I, \mathcal{R})$. Then $L^p(\Lambda, I, \mathcal{R})$ is the space of equivalence classes obtained by identifying pairs of functions $f, g \in \mathcal{L}^p(\Lambda, I, \mathcal{R})$ whenever $I(|f - g|) = 0$. By the remarks following Definition I.2.2 in [**7**], $L^p(\Lambda, I, \mathcal{R})$ is a Banach space with norm $I_p(\cdot)$.

Suppose that $(\Omega, \mathcal{F}, \mu)$ is a complete measure space, and let $\mathcal{R}_\mu$ denote the set of μ-integrable simple functions. For $f : \Omega \to \overline{\mathbf{R}}_+$, let $\mathcal{R}_f$ denote the set of sequences $\{g_n\}_{n=1}^\infty \subset \mathcal{R}_\mu$ such that each $g_n \geq 0$ and $\sum_n g_n \geq f$. Define $I_\mu : (\overline{\mathbf{R}}_+)^\Omega \to \overline{\mathbf{R}}_+$ by

$$I_\mu(f) = \inf\left\{ \sum_n \int_\Omega g_n \, d\mu : \{g_n\} \in \mathcal{R}_f \right\},$$

with the convention that $\inf \varnothing = \infty$. It can be shown that $(\Omega, \mathcal{R}_\mu, I_\mu)$ is an upper integral space. If $g \in \mathcal{R}_\mu$ and $g \geq 0$, it is clear that $I_\mu(g) = \int_\Omega g \, d\mu$.

It can be shown that, for each $p \in [1, \infty)$, the spaces $L^p(\Omega, I_\mu, \mathcal{R}_\mu)$ and $L^p(\Omega, \mathcal{F}, \mu)$ are identical. Furthermore, the upper integral I_μ has the additional property of being regular in the following sense. For every $f : \Omega \to \overline{\mathbf{R}}_+$, let $\mathcal{D}_f$ denote the collection of $g \in L^1(\Omega, \mathcal{F}, \mu)$ such that $g \geq f$. Again, with $\inf \varnothing = \infty$, it can then be shown that

$$I_\mu(f) = \inf\left\{ \int_\Omega g \, d\mu : g \in \mathcal{D}_f \right\}. \tag{3.1}$$

Suppose that $(\Omega_0, \mathcal{F}_0, \mu_0)$ is a σ-finite measure space. There are several concepts defined in [**7**] for a general upper integral space $(\Lambda, \mathcal{R}, I)$ for which there are equivalent formulations in terms of the measure space $(\Omega_0, \mathcal{F}_0, \mu_0)$ when the upper integral space is in fact $(\Omega_0, \mathcal{R}_{\mu_0}, I_{\mu_0})$. The set A is called I_{μ_0}-negligible if $I_{\mu_0}(1_A) = 0$; this occurs if and only if $\mu_0(A) = 0$. The set A is called $(I_{\mu_0}, \mathcal{R}_{\mu_0})$-integrable if $1_A \in \mathcal{L}^1(\Omega_0, I_{\mu_0}, \mathcal{R}_{\mu_0})$, which is equivalent to stating that $A \in \mathcal{F}_0$ and $\mu_0(A) < \infty$. The set A is called $(I_{\mu_0}, \mathcal{R}_{\mu_0})$-measurable if $A \cap B$ is $(I_{\mu_0}, \mathcal{R}_{\mu_0})$-integrable for each $(I_{\mu_0}, \mathcal{R}_{\mu_0})$-integrable set B. Since the measure space is σ-finite, this is equivalent to stating that $A \in \mathcal{F}_0$. In the sequel, when such notions arise, they will be expressed in the terminology of the measure space $(\Omega_0, \mathcal{F}_0, \mu_0)$.

The algebra of all bounded $f : \Omega_0 \to \mathbf{R}$ for which there exists $\{h_n\} \subset \mathcal{R}_{\mu_0}$ such that $\lim_n h_n(x) = f(x)$ μ_0 almost everywhere on Ω_0 will be denoted $M^\infty(\Omega_0, I_{\mu_0}, \mathcal{R}_{\mu_0})$. For every $f \in M^\infty(\Omega_0, I_{\mu_0}, \mathcal{R}_{\mu_0})$, define $I_\infty(f) = \inf\{\alpha : \{t : |f(t)| > \alpha\}$ is I-negligible$\}$. Then $I_\infty(\cdot)$ is a semi-norm on $M^\infty(\Omega_0, I_{\mu_0}, \mathcal{R}_{\mu_0})$. Since the measure space is σ-finite, $L^\infty(\Omega_0, I_{\mu_0}, \mathcal{R}_{\mu_0})$ can be characterized as the space of equivalence classes obtained by identifying pairs of functions $f, g \in M^\infty(\Omega_0, I_{\mu_0}, \mathcal{R}_{\mu_0})$ whenever $f = g$ μ_0 almost everywhere.

For a general upper integral space $(\Lambda, \mathcal{R}, I)$, the mapping $\bar{I} : (\overline{\mathbf{R}}_+)^\Lambda \to \overline{\mathbf{R}}_+$ defined by $\bar{I}(f) = \sup\{I(gf) : g \in \mathcal{R}$ and $0 \leq g \leq 1\}$ is another upper integral on Λ. The upper integral space $(\Lambda, I, \mathcal{R})$ is said to be strictly localizable if the following hold:

(i) $I(f) = \bar{I}(f)$ for all $f \in (\overline{\mathbf{R}}_+)^\Lambda$;

(ii) there exists $\mathcal{B}$ a collection of $(I, \mathcal{R})$-measurable subsets of Λ such that, for every $(I, \mathcal{R})$-integrable set A, there exists $\mathcal{B}_A \subset \mathcal{B}$ such that $\mathcal{B}_A$ is a countable set and $A \backslash (\cup_{B \in \mathcal{B}_A} B)$ is an I-negligible set.

It is clear that $(\Omega_0, I_{\mu_0}, \mathcal{R}_{\mu_0})$ is strictly localizable since the measure space is σ-finite. For the remainder of this paper, the upper integral I_{μ_0} will be denoted simply as I and $\mathcal{R}_{\mu_0}$ will be denoted by $\mathcal{R}$.

If f is a vector-valued function on Ω_0, define $I_p(f) = I_p(N(f))$ for each $p \in [1, \infty)$. The space $\mathcal{L}^p(\Omega_0, I, \mathcal{R}, X)$ is essentially the collection of X-valued functions which are limits of sequences in $\mathcal{R}_X = \{\sum_{k=1}^n f_k x_k : x_k \in X, f_k \in \mathcal{R} \text{ for } k = 1, \dots, n\}$ where the limit is taken in the $I_p(\cdot)$ seminorm. The space $L^p(\Omega_0, I, \mathcal{R}, X)$ is obtained by identifying pairs of functions in $\mathcal{L}^p(\Omega_0, I, \mathcal{R}, X)$ which are equal μ_0 almost everywhere. Theorem 6 in Chapter VI of [**7**], implies that the space $L^p(\Omega_0, I, \mathcal{R}, X)$ is identical to $L^p(\Omega_0, \mu_0, X)$.

A function $f : \Omega_0 \to X^*$ is called weak*-measurable if, for each $x \in X$, $t \mapsto \langle x, f(t)\rangle$ defines a $\mathcal{F}_0$-measurable scalar-valued function. Suppose $p \in [1, \infty)$. Let $\mathcal{L}^p_{X^*}(\Omega_0, I, X)$ denote the collection of $f : \Omega_0 \to X^*$ which are weak*-measurable functions and satisfy $I_p(f) < \infty$. Identifying pairs of functions f, g whenever $\langle x, f(t)\rangle = \langle x, g(t)\rangle$ μ_0 almost everywhere for all $x \in X$, a space of equivalence classes is obtained which will be denoted by $L^p_{X^*}(\Omega_0, I, X)$ or simply $L^p_{X^*}(\Omega_0, X)$. Using f to denote the equivalence class containing f as well as f itself, Proposition 3 in Chapter VI of [**7**] shows that this is a Banach space with norm $\|f\|_{L^p_{X^*}(\Omega_0,X)} = I_p(f)$.

Since $(\Omega_0, \mathcal{F}_0, \mu_0)$ is σ-finite, $M^\infty_{X^*}(X; \Omega_0, I, \mathcal{R})$ can be considered as the set of weak*-measurable functions, $f : \Omega_0 \to X^*$, such that for each $x \in X$, $\langle x, f(t)\rangle \in L^\infty(\Omega_0)$. Pairs of functions f, g are identified whenever $\langle x, f(t)\rangle = \langle x, g(t)\rangle$ μ_0 almost everywhere for all $x \in X$ to obtain a space of equivalence classes denoted $L^\infty_{X^*}(\Omega_0, I, X)$ or simply $L^\infty_{X^*}(\Omega_0, X)$. The corollary to Proposition 1 in Chapter VI of [**7**] shows that $\|f\|_{L^\infty_{X^*}(\Omega_0,X)} = I_\infty(f)$ is a norm making this a Banach space.

The next two results exhibit the role $L^p_{X^*}(\Omega_0, X)$ will play in this paper. They appear as corollaries to Theorems 7 and 9, respectively, in Chapter VII of [**7**], and the proofs can be found there.

Theorem 3.4. *Let X be a Banach space and suppose that $(\Omega_0, I, \mathcal{R})$ is strictly localizable. Then, there exists $\Psi_1 : (L^1(\Omega_0, \mu_0, X))^* \to L^\infty_{X^*}(\Omega_0, X)$, an isometric isomorphism of Banach spaces, such that if $\tau \in (L^1(\Omega_0, \mu_0, X))^*$ and $g = \Psi_1(\tau)$, then*

$$\tau(f) = \int_{\Omega_0} \langle f(t), g(t)\rangle \, d\mu_0(t)$$

for all $f \in L^1(\Omega_0, \mu_0, X)$.

Theorem 3.5. *Let X be a Banach space, and suppose that $(\Omega_0, I, \mathcal{R})$*

is strictly localizable. Let $p \in (1, \infty)$ with $q = p/(p-1)$. There exists $\Psi_p : (L^p(\Omega_0, \mu_0, X))^* \to L^q_{X^*}(\Omega_0, X)$, an isometric isomorphism of Banach spaces, such that if $\tau \in (L^p(\Omega_0, \mu_0, X))^*$ and $g = \Psi_p(\tau)$, then

$$\tau(f) = \int_{\Omega_0} \langle f(t), g(t) \rangle \, d\mu_0(t)$$

for all $f \in L^p(\Omega_0, \mu_0, X)$.

Remark 3.6. Note that Theorems 3.4 and 3.5 imply a version of Hölder's inequality. Suppose that $p \in [1, \infty)$ and $1/p + 1/q = 1$. Then, for each $f \in L^p(\Omega_0, \mu_0, X)$ and each $g \in L^q_{X^*}(\Omega_0, X)$,

$$\left| \int_{\Omega_0} \langle f(t), g(t) \rangle \, d\mu_0(t) \right| \le \int_{\Omega_0} \|f(t)\|_X \|g(t)\|_{X^*} \, d\mu_0(t) \le \|f\|_p I_q(g). \tag{3.2}$$

This inequality could also be obtained directly using (3.1) and the usual Hölder's inequality for $(\Omega_0, \mathcal{F}_0, \mu_0)$.

Let $\mathcal{U}$ denote the set of weak*-measurable functions $f : \Omega_0 \to X^*$ such that there exist $M \in [0, \infty)$ and $A \in \mathcal{F}_0$ with $\mu(A) < \infty$ such that $\|f(t)\|_{X^*} \le M 1_A(t)$. Clearly, for each $p \in [1, \infty]$, $\mathcal{U} \subset L^p_{X^*}(\Omega_0, X)$.

Lemma 3.7. *Let $q \in [1, \infty)$. The collection $\mathcal{U}$ is norm-dense in $L^q_{X^*}(\Omega_0, X)$.*

Proof. Suppose that $f \in L^q_{X^*}(\Omega_0, X)$. Since $I_q(f) < \infty$, by (3.1) there exists $g \in L^1(\Omega_0)$ such that $\|f(t)\|^q_{X^*} \le g(t)$ μ_0 almost everywhere on Ω_0.

Let $\varepsilon > 0$ be given. Since $g \in L^1(\Omega_0)$, there exists $A \in \mathcal{F}_0$ with $\mu_0(A) < \infty$ such that $0 \le g(t) \le M$ for some $M \in [0, \infty)$ μ_0 almost everywhere on A, and

$$\int_{\Omega_0 \setminus A} g(t) \, d\mu_0(t) < \varepsilon^q. \tag{3.3}$$

Observe that, by the choice of A, $\|f(t) 1_A(t)\|^q_{X^*} \le g(t) 1_A(t) \le M 1_A(t)$. Since A is a measurable set, $f 1_A$ is a weak*-measurable function, and thus $f 1_A \in \mathcal{U}$.

Using (3.1) and (3.3), we obtain that

$$I_q(f - f1_A) \leq \left(\int_{\Omega_0 \setminus A} g(t)\, d\mu_0(t) \right)^{1/q} \leq \varepsilon.$$

From this, the desired result follows. □

In order to use the preceding results when the measure space at hand is not σ-finite, the following lemma will be needed. In general, if $(\Omega, \mathcal{F}, \mu)$ is any measure space and Ω_0 is a measurable subset of Ω, then $\mathcal{F}_0 = \{A \cap \Omega_0 : A \in \mathcal{F}\}$ is a σ-algebra of subsets of Ω_0, and $\mu_0 = \mu|_{\mathcal{F}_0}$ defines a measure on $\mathcal{F}_0$.

Lemma 3.8. *Suppose that X is a Banach space. Let $p \in [1, \infty)$ with q satisfying $1/p + 1/q = 1$. Assume that $(\Omega, \mathcal{F}, \mu)$ is a measure space with Ω_0 a σ-finite measurable subset of Ω and with μ_0 defined as in the preceding paragraph. Then, for any $g \in L^q_{X^*}(\Omega_0, I, X)$, the mapping $f \mapsto \int_{\Omega_0} \langle f(\omega), g(\omega) \rangle \, d\mu(\omega)$ is a continuous linear functional on $L^p(\Omega, \mu, X)$.*

Proof. Consider the operator $T : L^p(\Omega, \mu, X) \to L^p(\Omega_0, \mu_0, X)$ defined by $T(f) = f1_{\Omega_0}$. Clearly, T is bounded with norm 1, and the result follows. □

Suppose Y is a Banach space. If (Z, τ) is a topological space, a function $f : Z \to Y$ is said to be strongly continuous if f is continuous into Y equipped with the norm topology. If f is a continuous mapping into Y equipped with the weak topology, f is said to be weakly continuous.

Following the terminology used in [**9**], when $u \mapsto S_u$ is a representation of a locally compact abelian group G into a Banach space Y, $y \in Y$ is called a strong continuous vector for the representation if $u \mapsto S_u y$ is a strongly continuous function on G. Similarly, $y \in Y$ is called a weak continuous vector for S if $u \mapsto S_u y$ is a weakly continuous function on G. A representation is said to be strongly continuous if every vector of Y is a strong continuous vector for the representation. We now proceed to the main result of this section.

Theorem 3.9. *Let X be a Banach space. Suppose that G is a locally compact abelian group and that $(\Omega, \mathcal{F}, \mu)$ is a measure space. Let $p_1, p_2 \in [1, \infty)$, and for $j = 1, 2$, suppose that $u \mapsto S_{j,u}$ is a representation of G on E^{p_j} such that $a_j = \sup\{\|S_{j,u}\| : u \in G\} < \infty$. Suppose also that, for all $f \in E^{p_1} \cap E^{p_2}$ and all $u \in G$, $S_{1,u}f = S_{2,u}f$. Then, the representation $u \mapsto S_{1,u}$ is strongly continuous if and only if $u \mapsto S_{2,u}$ is strongly continuous.*

Proof. Without loss of generality, we will suppose that $u \mapsto S_{2,u}$ is strongly continuous. Let $f \in E^{p_1} \cap E^{p_2}$, and fix K a compact symmetric neighborhood of the identity element for G. Let $C = K + K$, and let $\mathcal{J} = \{S_{2,u}f : u \in C\} \subset E^{p_1} \cap E^{p_2}$. The set C is compact by [**5**, Theorem II.4.4], so $\mathcal{J}$ is norm-compact in E^{p_2}. Thus, there exists a countable subset $\mathcal{J}_0 = \{S_{2,u_n}f : n \in \mathbf{N}\}$ which is E^{p_2}-dense in $\mathcal{J}$. It follows that there exists an increasing sequence of μ-measurable subsets $\{\beta_j\}_{j=1}^{\infty} \subset \Omega$, each having finite measure, such that if $\Omega_0 = \cup_{j=1}^{\infty} \beta_j$, each $g \in \mathcal{J}$ vanishes μ almost everywhere outside of Ω_0. The subspace of E^{p_1} consisting of functions which are supported on Ω_0 will be denoted by $E_0^{p_1}$. We shall consider two cases.

Case I. Suppose $p_1 = 1$. The proof is essentially the same as that appearing in [**1**, Proposition 3.2] for the context of representations acting on L^p-spaces of scalar-valued functions. We sketch the details in order to exhibit the adaptations required when the functions are vector-valued.

We first show that $u \mapsto S_{1,u}f$ is strongly measurable on C. It can be shown that $\mathcal{J}$ is a separable metric space in the E^1-metric using an argument completely analogous to that used in the proof of [**1**, Proposition 3.2]. So, by Pettis's theorem [**6**, Theorem 7.5.10], it suffices to prove that $u \mapsto S_{1,u}f$ is a weakly measurable mapping on C.

Let $\theta \in (E^1)^*$, and define $\theta_0 \equiv \theta|_{E_0^1}$. By Theorem 3.4, there exists $\psi \in L^{\infty}_{X^*}(\Omega_0, X)$ such that, for all $g \in E_0^1$,

$$\theta(g) = \theta_0(g) = \int_{\Omega_0} \langle g(\omega), \psi(\omega)\rangle \, d\mu(\omega). \tag{3.4}$$

Therefore, for all $u \in C$, $\theta(S_{1,u}f) = \int_{\Omega_0} \langle S_{1,u}f(\omega), \psi(\omega)\rangle \, d\mu(\omega)$. The dominated convergence theorem implies that, for each $u \in C$, letting

$\psi_j = \psi 1_{\beta_j}$,

$$\lim_{j\to\infty} \int_{\Omega_0} \langle S_{1,u} f, \psi_j \rangle \, d\mu = \theta(S_{1,u} f). \tag{3.5}$$

It is clear that, for each $j \in \mathbf{N}$, $\psi_j \in \mathcal{U} \subset L^{q_2}_{X^*}(\Omega_0, \mu_0, X)$. Hence, by Lemma 3.8, the mapping $g \mapsto \int_{\Omega_0} \langle g, \psi_j \rangle \, d\mu$ belongs to $(E^{p_2})^*$. From this point on, the proof is similar to that used in the context of scalar-valued L^p-spaces. The crucial point being that by (3.5) and the hypotheses on $S_{2,u}$, the mapping $u \mapsto \theta(S_{1,u} f)$ is the pointwise limit of continuous functions on C. Hence, it is λ-measurable on C. Since the argument holds for every $\theta \in (E^1)^*$, $u \mapsto S_{1,u} f$ is a weakly measurable mapping on C.

Having shown that $u \mapsto S_{1,u} f$ is strongly measurable, an averaging argument similar to that employed in [**1**, Proposition 3.2] can now be applied to show that $u \mapsto S_{1,u} f$ is strongly continuous at the identity of G. Using translations in G, it then follows that $u \mapsto S_{1,u} f$ is strongly continuous on all of G. Therefore, every $f \in E^1 \cap E^{p_2}$ is a strong continuous vector for $u \mapsto S_{1,u}$. Since $E^1 \cap E^{p_2}$ is norm-dense in E^1 and the representation $u \mapsto S_{1,u}$ is uniformly bounded, routine calculations show that every $f \in E^1$ is a strongly-continuous vector for S_1.

Case II. Suppose that $1 < p_1 < \infty$, and let $q_1 = p_1/(p_1 - 1)$. Choose $\phi \in (E^{p_1})^*$, and define $\phi_0 \equiv \phi|_{E_0^{p_1}}$ so that $\phi_0 \in (E_0^{p_1})^*$. By Theorem 3.5, there exists $h \in L^{q_1}_{X^*}(\Omega_0, \mu_0, X)$ such that, for all $g \in E_0^{p_1}$,

$$\phi(g) = \phi_0(g) = \int_{\Omega_0} \langle g(\omega), h(\omega) \rangle \, d\mu(\omega). \tag{3.6}$$

Since $\mathcal{J} \subset E_0^{p_1}$, (3.6) holds for all $g \in \mathcal{J}$.

Let $\varepsilon > 0$ be given. By Lemma 3.7, there exists a sequence $\{h_n\}_{n=1}^{\infty} \subset \mathcal{U}$ such that $h_n \to h$ in $L^{q_1}_{X^*}(\Omega_0, X)$ norm as $n \to \infty$. For each $u \in C$

and for each $n \in \mathbf{N}$,

$$
\begin{aligned}
|\phi(S_{1,u}f) - \phi(f)| &= \left| \int_{\Omega_0} \langle S_{1,u}f, h\rangle \, d\mu - \int_{\Omega_0} \langle f, h\rangle \, d\mu \right| \\
&\leq \left| \int_{\Omega_0} \langle S_{1,u}f, h - h_n\rangle \, d\mu \right| \\
&\quad + \left| \int_{\Omega_0} \langle S_{1,u}f - f, h_n\rangle \, d\mu \right| \\
&\quad + \left| \int_{\Omega_0} \langle f, h_n - h\rangle \, d\mu \right|.
\end{aligned} \tag{3.7}
$$

Now (3.2) and the hypothesis that $a_1 < \infty$ together imply that (3.7) does not exceed

$$
(a_1 + 1)\|f\|_{p_1} I_{q_1}(h - h_n) + \left| \int_{\Omega_0} \langle S_{1,u}f - f, h_n\rangle \, d\mu \right|. \tag{3.8}
$$

For n_0 sufficiently large, the first term of (3.8) is less than $\varepsilon/2$. Since $f \in E^{p_1} \cap E^{p_2}$ and $S_{1,u}f = S_{2,u}f$ for all $u \in G$, we have that, for all $u \in G$,

$$
\left| \int_{\Omega_0} \langle S_{1,u}f - f, h_{n_0}\rangle \, d\mu \right| = \left| \int_{\Omega_0} \langle S_{2,u}f - f, h_{n_0}\rangle \, d\mu \right|. \tag{3.9}
$$

But $h_{n_0} \in \mathcal{U} \subset L^{q_2}_{X^*}(\Omega_0, \mu_0, X)$. Thus, by Lemma 3.8, the mapping $g \mapsto \int_{\Omega_0} \langle g(\omega), h_{n_0}(\omega)\rangle \, d\mu(\omega)$ belongs to $(E^{p_2})^*$. Since the representation $u \mapsto S_{2,u}$ is strongly continuous, there exists an open neighborhood V_1 of the identity in G such that, for all $u \in V_1$,

$$
\left| \int_{\Omega_0} \langle S_{2,u}f - f, h_{n_0}\rangle \, d\mu \right| < \varepsilon/2. \tag{3.10}
$$

Let V_2 be an nonvoid neighborhood for the identity of G such that $V_2 \subset K \cap V_1$. From (3.10) and the choice of n_0, we get that for all $u \in V_2$, $|\phi(S_{1,u}f) - \phi(f)| < \varepsilon$. Because the preceding argument holds for any $\phi \in (E^{p_1})^*$, it follows that the mapping $u \mapsto S_{1,u}f$ is weakly continuous at the identity in G. It then follows that $u \mapsto S_{1,u}f$ is weakly continuous on all of G using translations in G.

The preceding argument holds for all $f \in E^{p_1} \cap E^{p_2}$. Since $E^{p_1} \cap E^{p_2}$ is norm dense in E^{p_1}, it follows that $E^{p_1} \cap E^{p_2}$ is a weakly dense subset of E^{p_1} consisting of weak continuous vectors for $u \mapsto S_{1,u}$. By Theorem 1 in Chapter 1 of [**9**], there exists a norm-dense subset of E^{p_1} consisting of strongly-continuous vector for $u \mapsto S_{1,u}$. As in Case I, the existence of such a set and the fact that the representation is uniformly bounded imply that every $f \in E^{p_1}$ is a strongly-continuous vector for S_1. $\square$

Corollary 3.10. *Let $u \mapsto R_u$ be a μ-distributionally controlled representation of G on $E^1 \cap E^\infty$. If $u \mapsto R_u^{(p_1)}$ is strongly continuous for some $p_1 \in [1, \infty)$, then $u \mapsto R_u^{(p)}$ is strongly continuous for all $p \in [1, \infty)$.*

REFERENCES

1. N. Asmar, E. Berkson and T.A. Gillespie, *Transference of weak-type maximal inequalities by distributionally bounded representations*, Quart. J. Math. Oxford **43** (1992), 259–282.

2. ———, *Distributional control and generalized analyticity*, Integral Equations Operator Theory **14** (1991), 311–341.

3. R.V. Chacon and U. Krengel, *Linear modulus of a linear operator*, Proc. Amer. Math. Soc. **15** (1964), 553–559.

4. S. Guerre and Y. Raynaud, *Sur les isométries de $L^p(X)$ et le théorème ergodique vectoriel*, Canad. J. Math. **40** (1988), 360–391.

5. E. Hewitt and K.A. Ross, *Abstract harmonic analysis*, Vol. I, Grundlehren Math. Wiss. **115** (1963).

6. E. Hille, *Methods in classical and functional analysis*, Addison-Wesley, Reading, Massachusetts, 1972.

7. A. Ionescu-Tulcea and C. Ionescu-Tulcea, *Topics in the theory of lifting*, Springer-Verlag, Berlin, 1969.

8. J. Lamperti, *On the isometries of certain function spaces*, Pacific J. Math. **8** (1958), 459–466.

9. R.T. Moore, *Measurable, continuous and smooth vectors for semigroups and group representations*, Mem. Amer. Math. Soc. **78** (1968).

Department of Mathematics, University of Missouri-Columbia, Columbia, Missouri 65211