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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A CLASS OF SELFADJOINT SECOND ORDER LINEAR SYSTEMS

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ABSTRACT. Suppose $P(x)$ is an $N \times N$ positive definite real matrix-valued C^1-function on $0 \leq x < \infty$ such that $P'(x)$ is also positive definite for x sufficiently large. We prove that if Trace $[P(x)] \to \infty$ as $x \to \infty$, then the second order linear system $y''(x) + P(x)y(x) = 0$ has a nontrivial solution which tends to zero in norm at infinity. Do all nontrivial solutions of the system tend to zero in norm at infinity? For this question we find a criterion. And applying this criterion we prove that if $P(x) = Q^2(x)$, where $Q(x)$ is a real symmetric matrix polynomials of degree ≥ 1, and with positive definite leading coefficient, then the question has an affirmative answer.

1. Introduction. In this paper we study the asymptotic behavior of the solutions of the following self-adjoint second order linear system

$$y''(x) + P(x)y(x) = 0, \tag{1.1}$$

where $P(x)$ is an $N \times N$ positive definite matrix-valued function on $[0, \infty)$, $y(x)$ is an $\mathbf{R}^N$-valued function, 0 is the zero vector in $\mathbf{R}^N$. We are interested in the questions of finding sufficient conditions which guarantee the existence of a nontrivial solution $y_0(x)$ of (1.1) such that $\lim_{x\to\infty} \|y_0(x)\| = 0$, where $\|y_0(x)\|$ is the norm of $y_0(x)$, and of finding sufficient conditions which guarantee that the norm of any nontrivial solution of (1.1) tends to zero in norm as x approaches infinity. We notice that, for the case $N = 1$, i.e., $P(x)$ in (1.1) is a scalar function, these questions had been studied by many mathematicians, notably Milloux, Hartman, Lazer, Meir, Willett and Wong (see [**1, 3, 5, 6**] and the references in these papers and book). In [**3**], for the case $N = 1$, Hartman used the Liouville transformation to transform (1.1) to a first order differential system, then he observed the related first order system and proved the Milloux theorem which says that if the

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scalar function $P(x)$ is monotone and satisfies $P(x) \to \infty$ as $x \to \infty$, then the differential equation (1.1) possesses a nontrivial solution $y(x)$ satisfying $y(x) \to 0$ as $x \to \infty$. We note that if the scalar function $P(x)$ satisfies the condition of the Milloux theorem, then the scalar equation (1.1) is oscillatory at infinity, and hence for the scalar equation (1.1), i.e., the case $N = 1$, every nontrivial solution has infinitely many zeros in $(0, \infty)$. Based on this oscillation result, Meir, Willett and Wong [**6**] proved that if the scalar function $P(x)$ is a C^3-function which satisfies the condition of the Milloux theorem and the condition that, for some $0 < \alpha < 1$,

$$\int_{x_0}^{x} |[P^{-\alpha}(x)]'''|\, dx = o(P^{1-\alpha}(x)) \tag{1.2}$$

as $x \to \infty$, then $y(x) \to 0$ as $x \to \infty$ for every solution $y(x)$ of (1.1) in the case $N = 1$. On the other hand, we note that for the case $N > 1$, there is no Liouville transformation for (1.1). Furthermore, when $N > 1$, even if (1.1) is oscillatory at infinity, it might happen that (1.1) might possess a nontrivial solution which only possesses finitely many, or no, zeros in $(0, \infty)$ (see Kaper and Kwong's work [**4**] for oscillation theory of (1.1) in case $N > 1$). Thus, for (1.1), the case $N > 1$ is quite different from the case $N = 1$. Nevertheless, the ideas of Hartman, Meir, Willett and Wong do play important roles in this paper. Following some of their ideas and using some results of Rosenblum [**8**] on the operator equation $BX - XA = Q$ to obtain some necessary estimates, we prove the following theorems among other results:

Theorem 3.1. *Let $P(x)$ be an $N \times N$ positive definite matrix-valued C^1-function on $[0, \infty)$ such that $P'(x)$ is positive definite for $x > x_0$, where $x_0 > 0$ and* Trace $P(x) \to \infty$ *as $x \to \infty$. Then the equation* (1.1) *has a nontrivial solution $y_0(x)$ such that $\|y_0(x)\| \to 0$ as $x \to \infty$.*

Theorem 4.1. *Suppose $P(x)$ is an $N \times N$ positive definite matrix-valued C^3-function on $[0, \infty)$, $\rho_N(x)$ is the smallest characteristic value of $P(x)$. If $P'(x)$ is positive definite for $x > x_0$, where $x_0 > 0$, $\rho_N(x) \to \infty$ as $x \to \infty$, with*

(i) $\|P'(x)\| = o(\rho_N(x))$,

(ii) $\int_{x_0}^{x} \|[P^{-1/2}(t)]'''\| \, dt = o(\rho_N^{1/2}(x))$,

(iii) $\int_{x_0}^{x} \|[P^{1/2}(t)]' - P^{-1/2}(t)[P^{1/2}(t)]' P^{1/2}(t)\| \, dt = o(\rho_N^{1/2}(x))$

as $x \to \infty$, then the solutions of (1.1) *tend to zero in norm as $x \to \infty$.*

Theorem 5.1. *Suppose $Q(x) = \sum_{j=0}^{n} x^j A_j$ is an $N \times N$ real symmetric matrix-valued function with positive definite leading coefficient A_n and $P(x) = Q^2(x)$. Let $y(x)$ be a nontrivial solution of* (1.1). *Then $\|y(x)\|$ tends to zero as x approaches infinity.*

The paper is arranged as follows. In Section 2 we state and prove some technical lemmas which shall be used later in this paper. In Section 4 we also prove a differential criterion for a positive definite matrix-valued function $P(x)$ to be a *commuting* function, i.e., $P(x)$ and $P(y)$ commute for any x and y in the domain of P; this result (Theorem 4.3) is of independent interest. The main results are proved in Sections 3, 4 and 5.

2. Some lemmas. Given a vector $y \in \mathbf{R}^N$, an $N \times N$ matrix A, and an $N \times N$ matrix-valued function $P(x)$, we shall use the notation $\|y\|$, $\|A\|$, $P'(x)$, $P''(x)$ and $P'''(x)$ to denote the norm of the vector y, the operator norm of A, the first, the second, and the third derivative of $P(x)$ with respect to x, respectively. In this section we state and prove some technical lemmas which shall be used later.

In this paper all matrix-valued functions shall be assumed to have real entries.

Lemma 2.1 [**3**]. *Let $A(t)$ be an $N \times N$ real matrix-valued continuous function on $[0, +\infty)$. Suppose that $\lim_{t\to\infty} \|z(t)\|$ exists for all solutions $z(t)$ of the first order system*

$$z'(t) = A(t)z(t), \tag{2.1}$$

where $z(t)$ are $\mathbf{R}^N$-valued functions. Then (2.1) *has a nontrivial solution $z_0(t)$ such that $\lim_{t\to\infty} \|z(t)\| = 0$ if and only if the following condition holds*:

$$\lim_{t\to\infty} \left[\int_{t_0}^{t} \text{Trace}\,(A(s))\, ds \right] = -\infty \tag{2.2}$$

for some $t_0 > 0$.

Lemma 2.2. *Suppose* $P(x)$ *is an* $N \times N$ *positive definite matrix-valued* C^1*-function on* $[0, \infty)$ *and* $y(x)$ *is an* $\mathbf{R}^N$*-valued nontrivial function which satisfies the following second order linear system*

$$y''(x) + P(x)y(x) = 0 \tag{2.3}$$

for $x > 0$. *Let*

$$Q_1[y](x) = \langle P^{-1}(x)y'(x), y'(x)\rangle + \|y(x)\|^2,$$
$$Q_2[y](x) = \|y'(x)\|^2 + \langle P(x)y(x), y(x)\rangle.$$

If $P(x)$ *is a* C^1*-function such that* $P'(x)$ *is positive definite for all* $x > 0$, *then* $Q_1[y](x)$ *is a decreasing function and* $Q_2[y](x)$ *is an increasing function.*

Proof. Since $P'(x)$ is positive definite,

$$\begin{aligned}\frac{d}{dx}Q_1[y](x) &= \langle (P^{-1})'y', y'\rangle + 2\langle P^{-1}y'', y'\rangle + 2\langle y, y'\rangle \\ &= -\langle P^{-1}P'P^{-1}y', y'\rangle - 2\langle y, y'\rangle + 2\langle y, y'\rangle \\ &= -\langle P'P^{-1}y', P^{-1}y'\rangle \le 0.\end{aligned}$$

Thus $Q_1[y](x)$ is decreasing. The assertion for $Q_2[y](x)$ is proved by a similar argument. □

Lemma 2.3. *Let* $P(x)$ *be an* $N \times N$ *positive definite matrix-valued* C^1*-function such that* $P'(x)$ *is also positive definite. Let* $P^{1/2}(x)$ *denote the positive square root of* $P(x)$. *Then we have*

$$\mathrm{Trace}\,[[P^{-1/2}(x)]'P^{1/2}(x)] \le -\frac{\{\mathrm{Trace}\,[P(x)]\}'}{2\mathrm{Trace}\,[P(x)]}. \tag{2.4}$$

Proof. Since $P^{-1/2}P^{1/2} = I$, $P^{1/2}P^{1/2} = P$, we have

$$(P^{-1/2})'P^{1/2} + P^{-1/2}(P^{1/2})' = 0,$$
$$(P^{1/2})'P^{1/2} + P^{1/2}(P^{1/2})' = P'.$$

Hence we have

$$\text{Trace}\,[(P^{-1/2})'P^{1/2}] = -\text{Trace}\,[P^{-1/2}(P^{1/2})'],$$

and

$$\begin{aligned}\text{Trace}\,[P'P^{-1}] &= \text{Trace}\,[P^{-1/2}P'P^{-1/2}]\\ &= \text{Trace}\,[P^{-1/2}\{(P^{1/2})'P^{1/2} + P^{1/2}(P^{1/2})'\}P^{-1/2}]\\ &= \text{Trace}\,[P^{-1/2}(P^{1/2})' + (P^{1/2})'P^{-1/2}]\\ &= 2\text{Trace}\,[P^{-1/2}(P^{1/2})'].\end{aligned}$$

Thus we have

$$\text{Trace}\,[(P^{-1/2})'P^{1/2}] = -2^{-1}\text{Trace}\,[P'(x)P^{-1}(x)]. \tag{2.5}$$

Now let $\rho_1(x)$ be the largest characteristic value of $P(x)$. Then we have

$$\begin{gathered}\rho_1(x) \le \text{Trace}\,[P(x)],\\ [P'(x)]^{1/2}P^{-1}(x)[P'(x)]^{1/2} \ge \rho_1^{-1}(x)P'(x),\end{gathered}$$

and hence the following inequality holds:

$$\text{Trace}\,[P^{-1}(x)P'(x)] \ge \{\text{Trace}\,[P(x)]\}^{-1}\{\text{Trace}\,[P(x)]\}'. \tag{2.6}$$

Then (2.4) follows from (2.5) and (2.6). □

Lemma 2.4 [**8**, Theorem 4.3]. *Let A and B be two $N \times N$ matrices. If there exist two real numbers a and b such that $a > b$, $B + B^* \le bI$, $A + A^* \ge aI$, where I is the $N \times N$ identity matrix, then for any $N \times N$ matrix Q, the matrix equation*

$$BX - XA = Q \tag{2.6.1}$$

has a unique $N \times N$ matrix solution X which can be represented as follows

$$X = -\int_0^\infty \exp(tB)Q\exp(-tA)\,dt. \tag{2.6.2}$$

Furthermore,

$$\|X\| \leq \|Q\|/[2(a-b)]. \tag{2.6.3}$$

Lemma 2.5. *If $P(x)$ is an $N \times N$ positive definite matrix-valued C^1-function defined on a compact interval J, $\rho_N(x)$ is the smallest characteristic value of $P(x)$, then*

$$[P^{1/2}(x)]' = \int_0^\infty \exp[-tP^{1/2}(x)]P'(x)\exp[-tP^{1/2}(x)]\,dt, \tag{2.7.1}$$

and

$$\|[P^{1/2}(x)]'\| \leq \|P'(x)\|/[8\rho_N^{1/2}(x)]. \tag{2.7.2}$$

Proof. Since $P^{1/2}P^{1/2} = P$, we have

$$P^{1/2}(P^{1/2})' + (P^{1/2})'P^{1/2} = P',$$

which implies

$$(-P^{1/2})(P^{1/2})' - (P^{1/2})'P^{1/2} = -P'. \tag{2.8}$$

Since $P^{1/2}(x) \geq \rho_N^{1/2}(x)I$, $-P^{1/2}(x) \leq -\rho_N^{1/2}(x)I$, (2.7.1) and (2.7.2) follow from (2.8), (2.6.2) and (2.6.3) with $B = -P^{1/2}$, $A = P^{1/2}$ and $Q = -P'$ in (2.6.1). □

3. Milloux theorem for second order linear systems. In this section we shall use Lemma 2.1 to prove the following Milloux theorem for second order linear differential systems.

Theorem 3.1. *Let $P(x)$ be an $N \times N$ positive definite matrix-valued C^1-function on $[0,\infty)$ such that $P'(x)$ is positive definite for $x > x_0$, where $x_0 > 0$, and* Trace $[P(x)] \to \infty$ *as $x \to \infty$. Then the equation (1.1) has a nontrivial solution $y_0(x)$ such that* $\lim_{x\to\infty} \|y_0(x)\| = 0$.

Proof. Let

$$z(x) = \operatorname{col}(y(x), P^{-1/2}(x)y'(x)), \tag{3.1}$$

where $y(x)$ is any solution of (1.1). Then $z(x)$ satisfies the following first order linear system

$$z'(x) = \begin{bmatrix} 0 & P^{1/2} \\ -P^{1/2} & [P^{-1/2}(x)]'P^{1/2}(x) \end{bmatrix} z(x). \tag{3.2}$$

Write (3.2) as $z'(x) = A(x)z(x)$. Note that, by Lemma 2.2,

$$\lim_{x\to\infty} \|z(x)\|^2$$

exists for any solution of (3.2). Thus, it follows from Lemma 2.1 that if the following condition

$$\lim_{x\to\infty} \int_{x_0}^{x} \operatorname{Trace}[(P^{-1/2}(t))'P^{1/2}(t)]\,dt = -\infty \tag{3.3}$$

holds, then (3.2) possesses a nontrivial solution $z_0(x)$ such that $\|z_0(x)\| \to 0$ as $x \to \infty$. We can write $z_0(x) = \operatorname{col}(y_0(x), P^{-1/2}(x)y_0'(x))$. Then $y_0(x)$ is a solution of (1.1) with $\|y_0(x)\| \to 0$ as $x \to \infty$. On the other hand, by Lemma 2.3 we have

$$\int_{x_0}^{x} \operatorname{Trace}[(P^{-1/2}(t))'P^{1/2}(t)]\,dt \le -2^{-1}\log\{\operatorname{Trace}[P(x)]\}|_{x_0}^{x}.$$

Thus the condition that $\operatorname{Trace}[P(x)] \to \infty$ as $x \to \infty$ implies (3.3) holds, and the proof is complete. □

4. The asymptotic behavior of the solutions of (1.1). Suppose $P(x)$ is an $N \times N$ positive definite matrix-valued C^3-function on $[0,\infty)$. Let $\rho_1(x) \ge \rho_2(x) \ge \cdots \ge \rho_N(x)$ be the characteristic values of $P(x)$. We shall assume that $P'(x)$ is positive definite. Then $\rho_j(x)$ are increasing. We shall also assume that $\rho_N(x) \to \infty$ as $x \to \infty$.

Let $z(x)$ be as in (3.1),

$$R(x) = \operatorname{diag}[P^{1/2}(x), P^{1/2}(x)].$$

Then by (1.1) we have

$$\langle R(x)z(x), z(x)\rangle' = \{\langle (P^{1/2})'y, y\rangle - \langle P^{-1/2}(P^{1/2})'P^{1/2}y, y\rangle\} + [\langle (P^{-1/2})'y', y\rangle]' - \langle (P^{-1/2})''y', y\rangle,$$

which implies that

$$\begin{aligned}
&[\langle Rz, z\rangle - \langle (P^{-1/2})'y', y\rangle]' \\
&\qquad = \{\langle (P^{1/2})'y, y\rangle - \langle P^{-1/2}(P^{1/2})'P^{1/2}y, y\rangle\} \\
&\qquad\quad - \langle (P^{-1/2})''y', y\rangle \\
&\qquad = \{\langle (P^{1/2})'y, y\rangle - \langle P^{-1/2}(P^{1/2})'P^{1/2}y, y\rangle\} \\
&\qquad\quad - 2^{-1}[\langle (P^{-1/2})''y, y\rangle]' + 2^{-1}\langle (P^{-1/2})'''y, y\rangle.
\end{aligned}$$

Thus we have

$$\begin{aligned}
(4.1)\quad &[\langle Rz, z\rangle - \langle (P^{-1/2})'y', y\rangle + 2^{-1}\langle (P^{-1/2})''y, y\rangle]' \\
&\qquad = \{\langle (P^{1/2})'y, y\rangle - \langle P^{-1/2}(P^{1/2})'P^{1/2}y, y\rangle\} \\
&\qquad\quad + 2^{-1}\langle (P^{-1/2})'''y, y\rangle.
\end{aligned}$$

Now, integrating (4.1), we have

(4.2)

$$\begin{aligned}
&\langle R(x)z(x), z(x)\rangle \\
&\quad = C_{x_0} + \langle (P^{-1/2}(x))'y'(x), y(x)\rangle - 2^{-1}\langle (P^{-1/2}(x))''y(x), y(x)\rangle \\
&\quad\quad + 2^{-1}\int_{x_0}^{x} \langle (P^{-1/2}(t))'''y(t), y(t)\rangle\, dt \\
&\quad\quad + \int_{x_0}^{x} \langle \{(P^{1/2}(t))' - P^{-1/2}(t)(P^{1/2}(t))'P^{1/2}(t)\}y(t), y(t)\rangle\, dt.
\end{aligned}$$

Suppose there is a nontrivial solution $y(x)$ of (1.1) such that

$$Q_1[y](x) = \|y(x)\|^2 + \langle P^{-1}(x)y'(x), y'(x)\rangle$$

decreases to $l > 0$ as $x \to \infty$. Then as $l > 0$, for any $\varepsilon > 0$, there exists $x_0 > 0$ such that

$$(4.3)\qquad l \le Q_1[y](x) < l(1+\varepsilon)$$

for all $x \geq x_0$. Note that we have

$$\begin{aligned}
(4.4)\quad |\langle (P^{-1/2}(x))' y'(x), y(x)\rangle| &= |\langle -P^{-1/2}(x)(P^{1/2}(x))' P^{-1/2} y'(x), y(x)\rangle| \\
&\leq \|P^{-1/2}(x)(P^{1/2}(x))'\| \, \|P^{-1/2}(x) y'(x)\| \, \|y(x)\| \\
&\leq 2^{-1}\|P^{-1/2}(x)(P^{1/2}(x))'\| Q_1[y](x) \\
&< 2^{-1}\|P^{-1/2}(x)(P^{1/2}(x))'\| l(1+\varepsilon)
\end{aligned}$$

for $x \geq x_0$, and

$$(4.5)\qquad (P^{-1/2}(x))'' = (P^{-1/2})''(x_0) + \int_{x_0}^{x} (P^{-1/2}(t))''' \, dt.$$

By (4.2), (4.3), (4.4) and (4.5) we have, as $\langle Rz, z\rangle \geq \rho_N^{1/2}\|z\|^2 = \rho_N^{1/2} Q_1[y]$, that

(4.6)

$$\begin{aligned}
l\rho_N^{1/2}(x) \leq K_0 &+ 2^{-1}\|(P^{-1/2})''(x_0)\| l(1+\varepsilon) \\
&+ 2^{-1}\|P^{-1/2}(x)(P^{1/2}(x))'\| l(1+\varepsilon) \\
&+ \left(\int_{x_0}^{x} \|(P^{-1/2}(t))'''\| \, dt\right) l(1+\varepsilon) \\
&+ \left[\int_{x_0}^{x} \|(P^{1/2}(t))' - P^{-1/2}(t)(P^{1/2}(t))' P^{1/2}(t)\| \, dt\right] l(1+\varepsilon),
\end{aligned}$$

where K_0 is a constant. By (4.6), we have the following result.

Theorem 4.1. *Suppose that $P(x)$ is an $N \times N$ positive definite matrix-valued C^3-function, $P'(x)$ is positive definite for $x \geq x_0$ and $\rho_N(x)$ is the smallest characteristic value of $P(x)$ such that $\rho_N(x) \to \infty$ as $x \to \infty$. If $P(x)$ satisfies the following conditions*:

(i) $\|P'(x)\| = o(\rho_N(x))$ *as* $x \to \infty$,

(ii) $\int_{x_0}^{x} \|(P^{-1/2}(t))'''\| \, dt = o(\rho_N^{1/2}(x))$ *as* $x \to \infty$,

(iii) $\int_{x_0}^{x} \|(P^{1/2}(t))' - P^{-1/2}(t)(P^{1/2}(t))' P^{1/2}(t)\| \, dt = o(\rho_N^{1/2}(x))$ *as* $x \to \infty$, *then the solutions of* (1.1) *tend to zero in norm at infinity.*

Proof. It suffices to prove that, under the assumptions of this theorem, if $y(x)$ is a nontrivial solution of (1.1), then we have $Q_1[y](x) \to 0$ as $x \to \infty$. Suppose there were a nontrivial solution $y(x)$ of (1.1) such that

$$\lim_{x\to\infty} Q_1[y](x) = l > 0.$$

Then, for $\varepsilon > 0$, there exists $x_0 > 0$ so that for $x \geq x_0$ (4.6) holds. By Lemma 2.5 the condition (i) implies that

$$\lim_{x\to\infty} \rho_N^{-1/2}(x)\|P^{-1/2}(x)(P^{1/2}(x))'\| = 0.$$

Thus, by (i), (ii) and (iii), (4.6) implies that $l \leq 0$, which is absurd. □

Remarks. (1) Readers will probably think that if we assume that $P^{1/2}$ and $(P^{1/2})'$ commute, then (iii) will be unnecessary. But the author wants to point out, and will show later in Theorem 4.3, that under the latter assumption, and assuming further that $P(x)$ is real analytic and has distinct characteristic values, the function $P(x)$ can be diagonalized simultaneously for all x, and then the equation (1.1) becomes N second order scalar differential equations, and this will reduce the problem to the case treated by Meir, Willett and Wong in [**6**].

(2) Let $P(x) = [a_{ij}(x)]$, $a_{ij} = a_{ji}$. By the Gerschgorin theorem in linear algebra, if we let

$$\eta(x) = \min\left\{a_{jj}(x) - \sum_{\substack{k=1\\k\neq j}}^{N} |a_{jk}(x)| : j = 1, \dots, N\right\},$$

then $\rho_N(x) \geq \eta(x)$. Thus Theorem 4.1 has the following corollary, which is easier to use in practice.

Corollary 4.1.1. *Suppose $Q(x)$ is an $N \times N$ positive definite matrix-valued C^3-function on $[0,\infty)$, $P(x) = Q^2(x)$, $\eta(x)$ as above, $P'(x)$ is positive definite, $\eta(x) > 0$ and tends to infinity as $x \to \infty$. If*

(i) $\|P'(x)\| = o(\eta(x))$ *as* $x \to \infty$,

(ii) $\int_{x_0}^{x} \|(Q^{-1}(t))'''\| \, dt = o(\eta^{1/2}(x))$ *as* $x \to \infty$,

(iii) $\int_{x_0}^{x} \eta^{-1/2}(t) \|Q(t)Q'(t) - Q'(t)Q(t)\| dt = o(\eta(x))$ *as* $x \to \infty$,

then every nontrivial solution of (1.1) *tends to zero in norm as* $x \to \infty$.

Lemma 4.2. *Suppose that* $P(x)$ *is an* $N \times N$ *positive definite matrix-valued* C^1*-function defined on a compact interval* J. *Then* $[P^{1/2}(x)]'$ *and* $P^{1/2}(x)$ *commute for all* x *in* J *if and only if* $P'(x)$ *and* $P(x)$ *commute for all* x *in* J.

Proof. Since $P(x) > 0$ for all x in the compact set J, there exist $b > a > 0$ such that $\sigma(P(x)) \subset [a, b]$ for all x in J, where $\sigma(P(x))$ denotes the spectrum of $P(x)$. Now let $Q(x)$ denote $P^{1/2}(x)$. If $Q(x)$ and $Q'(x)$ commute, then

$$\begin{aligned} P(x)P'(x) &= Q^2(x)[Q'(x)Q(x) + Q(x)Q'(x)] \\ &= [Q'(x)Q(x) + Q(x)Q'(x)]Q^2(x) \\ &= P'(x)P(x). \end{aligned}$$

Conversely, if $P'(x)$ and $P(x)$ commute, then for any continuous function $f(t)$ on $[a, b]$ we have

$$P'(x)f(P(x)) = f(P(x))P'(x)$$

for all x in J. Since $a > 0$, $f(t) = t^{1/2} \in C[a, b]$. Thus, we have

$$P'(x)P^{1/2}(x) = P^{1/2}(x)P'(x)$$

and hence

$$[(P^{1/2})'P^{1/2} + P^{1/2}(P^{1/2})']P^{1/2} = P^{1/2}[(P^{1/2})'P^{1/2} + P^{1/2}(P^{1/2})'].$$

Notice that the latter implies that

$$(P^{1/2})'P = P(P^{1/2})'.$$

Therefore, for $f(t) = t^{1/2}$ we have

$$(P^{1/2}(x))'f(P(x)) = f(P(x))(P^{1/2}(x))',$$

which is what we want to prove. □

Theorem 4.3. *Suppose that $P(x)$ is an $N \times N$ positive definite matrix-valued real analytic function in an open interval J such that the characteristic values of $P(x)$ are all distinct for all x in J. Then $P(x)$ is a commuting function, i.e.,*

$$P(x_1)P(x_2) = P(x_2)P(x_1)$$

for all x_1 and x_2 in J if and only if $P'(x)$ and $P(x)$ commute for all x in J, or equivalently, $[P^{1/2}(x)]'$ and $P^{1/2}(x)$ commute for all x in J.

Proof. If $P(x)$ is a commuting function, then $P(x)P(y) = P(y)P(x)$ implies that

$$\frac{[P(y) - P(x)]P(x)}{y - x} = \frac{P(x)[P(y) - P(x)]}{y - x}.$$

Letting y approach x we have $P'(x)P(x) = P(x)P'(x)$. Conversely, suppose $P(x)$ and $P'(x)$ commute. Then, by Rellich's theorem [**7**, Chapter I, Theorem 1], there exists an orthogonal matrix-valued real analytic function $U(x)$ and a diagonal matrix-valued real analytic function $D(x) = \operatorname{diag}[\rho_1(x), \dots, \rho_N(x)]$ such that

$$P(x) = U^*(x)D(x)U(x), \tag{4.7}$$

$$P'(x) = U^*(x)D'(x)U(x). \tag{4.8}$$

By (4.7) and applying (4.8), we have

$$\begin{aligned} P' &= -U^*U'U^*DU + U^*D'U + U^*DU' \\ &= -U^*U'P + P' + PU^*U', \end{aligned}$$

i.e., we have

$$P[U^*U'] = [U^*U']P \tag{4.9}$$

in J. Since, for each x, the characteristic values of $P(x)$ are distinct, it follows from [**2**, Chapter 8, Corollary 1 to Theorem 2] that (4.9) implies that

$$U^*(x)U'(x) = f(P(x); x), \tag{4.10}$$

where, for each x in J, $f(t,x)$ is a polynomial in t-variable. Since the entries of $U(x)$ are reals, the coefficients of $f(t;x)$ are reals. Then it follows from (4.10) that $U^*(x)U'(x)$ is symmetric for each x in J. Therefore, we have

$$U^*(x)U'(x) = U'^*(x)U(x)$$

for all x in J. Hence $(U^*)' = U^*U'U^*$. But as $(U^*)' = (U^{-1})' = -U^*U'U^*$, we have $(U^*(x))' = 0$ for al x in J, i.e., $U(x)$ is a constant matrix. Let $U(x) = U_0$ for all x in J. Then $P(x) = U_0^*D(x)U_0$, which implies that $P(x)$ is a commuting function. □

5. Second order linear systems whose coefficients are matrix polynomials. In this section we shall apply Theorem 4.1 to study (1.1) for the case that $P(x)$ is a matrix polynomial. We shall prove the following result.

Theorem 5.1. *Suppose $A_n, A_{n-1}, \dots, A_1, A_0$ are $N \times N$ real symmetric matrices, A_n is a positive definite matrix, where $n \geq 1$. Let $Q(x) = x^n A_n + \dots + xA_1 + A_0$, $P(x) = Q^2(x)$, and $y(x)$ be a nontrivial solution of* (1.1). *Then we have* $\lim_{x\to\infty} \|y(x)\| = 0$.

Proof. Note that under the assumption of Theorem 5.1 there exists $x_0 > 0$ so that both $Q(x)$ and $Q'(x)$ are positive definite for $x \geq x_0$.

Given an $N \times N$ real symmetric matrix A we shall use the notation $\mu_1(A) \geq \mu_2(A) \geq \dots \geq \mu_N(A)$ to denote the characteristic values of A. For $P(x)$ as that in Theorem 5.1, let $\eta_N(x)$ be the smallest characteristic value of $Q(x)$, $\rho_N(x)$ be the smallest characteristic value of $P(x)$. Then for $x \geq x_0$, $\rho_N(x) = \eta_N^2(x)$. Let y_x be a characteristic vector of $Q(x)$ corresponding to $\eta_N(x)$, $\|y_x\| = 1$. Then by the minimum principle we have

$$\begin{aligned}\eta_N(x) &= x^n\langle A_n y_x, y_x\rangle + \dots + x\langle A_1 y_x, y_x\rangle + \langle A_0 y_x, y_x\rangle \\ &\geq \sum_{j=0}^{n} \mu_N(A_j)x^j.\end{aligned}$$

Let $p(x) = \sum_{j=0}^n \mu_N(A_j)x^j$, $q(x) = \sum_{j=0}^n \mu_1(|A_j|)x^j$, where $|A_j| = (A_j^*A_j)^{1/2}$, $|A_n| = A_n$. Then there exists $x_0 > 0$ such that, for

$x \geq x_0$, $Q(x)$, $Q'(x)$ and $P'(x)$ are positive definite, $p(x)$ and $q(x)$ are strictly increasing, $p(x) \to \infty$ as $x \to \infty$, $q'(x)$, $q''(x)$ and $q'''(x)$ are nonnegative and nondecreasing for $x \geq x_0$, and

$$\rho_N^{1/2}(x) = \eta_N(x) \geq p(x), \tag{5.1}$$

$$\|Q(x)\| \leq q(x), \qquad \|P(x)\| \leq q^2(x), \qquad \|Q^{-1}(x)\| \leq p^{-1}(x), \tag{5.2}$$

$$\|Q'(x)\| \leq q'(x), \qquad \|P'(x)\| \leq 2q(x)q'(x), \tag{5.3}$$

$$\|Q''(x)\| \leq q''(x), \tag{5.4}$$

$$\|Q'''(x)\| \leq q'''(x), \tag{5.5}$$

for $x \geq x_0$. Thus we have, by (5.1), (5.3) and $\deg(q') = \deg(p) - 1$, that

$$\frac{\|P'(x)\|}{\rho_N(x)} \leq \frac{2q(x)q'(x)}{p^2(x)},$$

and hence

$$\|P'(x)\| = o(\rho_N(x)) \quad \text{as } x \longrightarrow \infty. \tag{5.6}$$

Now, by computation, we have

$$\begin{aligned} (Q^{-1})''' &= 3Q^{-1}Q'Q^{-1}Q''Q^{-1} + 3Q^{-1}Q''Q^{-1}Q'Q^{-1} \\ &\quad - 6Q^{-1}Q'Q^{-1}Q'Q^{-1}Q'Q^{-1} - Q^{-1}Q'''Q^{-1}. \end{aligned} \tag{5.7}$$

Thus, by (5.2), (5.3), (5.4) and (5.5), we have

$$\begin{aligned} &\|[P^{-1/2}(x)]'''\| \\ &\qquad \leq p^{-4}(x)[6p(x)q'(x)q''(x) + 6(q'(x))^3 + p^2(x)q'''(x)] \\ &\qquad = h(x)/[p(x)]^4, \end{aligned}$$

where $h(x)$ is a polynomial of degree $\leq 3n-3$. Thus, we have

$$\int_{x_0}^{x} \|[P^{-1/2}(x)]'''(t)\|\, dt = O(x^{-(n+2)}), \tag{5.8}$$

and it follows from (5.8) that

$$\int_{x_0}^{x} \|[P^{-1/2}(t)]'''\|\, dt = o(\rho_N^{1/2}(x)) \quad \text{as } x \to \infty. \tag{5.9}$$

Finally, we note that as $Q'(x)Q(x) - Q(x)Q'(x)$ is a matrix polynomial of degree at most $2n-2$, $p(x)$ is a polynomial of degree n,

$$\begin{aligned} &\|[P^{1/2}(x)]' - P^{-1/2}(x)[P^{1/2}(x)]'P^{1/2}(x)\| \\ &\qquad \le \|Q^{-1}(x)\|\,\|Q(x)Q'(x) - Q'(x)Q(x)\| = O(x^{n-2}). \end{aligned}$$

Hence, as $\deg(p) = n$, $\rho_N^{1/2} \ge p$, we have

$$\left[\int_{x_0}^{x} \|[P^{1/2}(t)]' - P^{-1/2}(t)[P^{1/2}(t)]'P^{1/2}(t)\|\, dt\right] \rho_N^{-1/2}(x) = O(x^{-1}).$$

Thus we have

$$\int_{x_0}^{x} \|[P^{1/2}(t)]' - P^{-1/2}(t)[P^{1/2}(t)]'P^{1/2}(t)\|\, dt = o(\rho_N^{1/2}(x)) \tag{5.10}$$

as $x \to \infty$. By (5.6), (5.9) and (5.10), Theorem 4.1 tells us that Theorem 5.1 holds. □

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