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AFFINE ALGEBRAIC MANIFOLDS WITHOUT DOMINANT MORPHISMS FROM EUCLIDEAN SPACES

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ABSTRACT. We develop a method which in some cases enables us to establish the absence of dominant morphisms from Euclidean spaces into affine algebraic manifolds which are hypersurfaces in $\mathbf{C}^n$. This approach shows that many of recently constructed smooth contractible hypersurfaces in $\mathbf{C}^n$, $n \geq 4$, are not isomorphic to Euclidean spaces and cannot be used as counterexamples to the Zariski cancellation conjecture, to the Abhyankar-Sathaye conjecture, and to the problem of linearizing a $\mathbf{C}^*$-action on $\mathbf{C}^3$.

1. Every smooth factorial affine surface which admits a dominant mapping from a Euclidean space is isomorphic to $\mathbf{C}^2$ [**10, 8, 13**]. This theorem is a generalization of the cancellation theorem for surfaces [**3**]. For dimensions higher than two this generalized version of the cancellation theorem does not hold even if we consider contractible manifolds. P. Russell constructed a dominant morphism from $\mathbf{C}^3$ into the hypersurface $\{(x,y,z,t) \in \mathbf{C}^4 \mid x + x^2y + z^3 + t^2 = 0\}$. This hypersurface is one among many contractible hypersurfaces which appeared recently in [**2, 4, 14, 11, 16**]. The main aim of this paper is to introduce a method which enables us to prove that many of these hypersurfaces do not have dominant morphisms from Euclidean spaces and, in particular, they are not isomorphic to a Euclidean space. It is important not only in connection with the Zariski cancellation conjecture. Every nontrivial hypersurface in these papers is the zero fiber of a polynomial whose generic fibers are not isomorphic to a Euclidean space. Had this hypersurface been isomorphic to $\mathbf{C}^3$ we would have a counterexample to the Abhyankar–Sathaye conjecture [**1, 15**]. Using our method, we show that it is not so for many of these hypersurfaces. For instance, a hyperbolic modification of a smooth contractible surface of Kodaira logarithmic dimension 1 (see [**11, 16**] for definitions) is not isomorphic to $\mathbf{C}^3$. Perhaps the most interesting construction of contractible hypersurfaces is presented in [**14**] (it is a pleasure to acknowledge that our paper was inspired by the result of

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P. Russell). This construction gives contractible threefolds which admit "hard-case" $\mathbf{C}^*$-actions, and, of course, it is important to distinguish these threefolds from $\mathbf{C}^3$ in connection with the linearizing problem. Our method enables us to do this in most cases. (It is worth mentioning that eventually we constructed another invariant which distinguished all Russell's threefolds from $\mathbf{C}^3$ [**9, 6**]. But not only is the computation of this invariant far more complicated than the method presented in this paper, it is also not an obstruction to the existence of dominant mappings.)

2. Let $x = (x_1, \dots, x_n)$ be a coordinate system in $\mathbf{C}^n$ and $\xi = (\xi_1, \dots, \xi_{n-1})$ be a coordinate system in $\mathbf{C}^{n-1}$. Consider a hypersurface $X \subset \mathbf{C}^n$ given by a polynomial equation $P(x) = 0$. Suppose that $\varphi : \mathbf{C}^{n-1} \to X$ is a morphism. Then, using the generated mapping from a ring of regular functions $\mathbf{C}[X]$ on X to $\mathbf{C}[\xi]$, we shall treat the elements of $\mathbf{C}[X]$ as polynomials in ξ. Denote by J_i the determinant of the Jacobi matrix $\{\partial x_j / \partial \xi_k; k = 1, \dots, n-1; j = 1, \dots, \hat{i}, \dots, n\}$ and by P_i the ith partial derivative of P.

Lemma. *Suppose that for every point $x^0 \in X$ there exists i_0 such that $P_{i_0}(x^0) \neq 0$, i.e., X is smooth. Then for every $i = 1, \dots, n$ a polynomial $J_i \in \mathbf{C}[\xi]$ is divisible by P_i (which is also treated as an element of $\mathbf{C}[\xi]$).*

Proof. Consider $i = 1$. Note that the determinant of the Jacobi matrix

$$J_{\xi_1, \dots, \xi_{n-1}}(x_2, \dots, x_{i-1}, P, x_{i+1}, \dots, x_n)$$

is

$$P_i J_1 + (-1)^i P_1 J_i = 0.$$

Suppose that $x^0 = \varphi(\xi^0)$ and $P_1(x^0) = 0$. Choose i so that $P_i(x^0) \neq 0$. Then one can see that J_1 has a zero at ξ^0 at least of the same multiplicity as $P_1 \circ \varphi$. Hence J_1 is divisible by $P_1 \circ \varphi$. $\square$

3. Let d_i be the degree of the polynomial $x_i \in \mathbf{C}[\xi]$, and let D_i be the degree of P_i as a polynomial in ξ. Note that

$$\begin{aligned}\deg J_i &\le \sum_{j=1}^{n}(d_j-1)-(d_i-1)\\ &=\sum_{j=1}^{n} d_j - d_i-(n-1).\end{aligned}$$

This inequality and Lemma 2 imply

Theorem. *Under the assumptions of Lemma* 2 *we have*

$$\sum_{j=1}^{n} d_j \ge D_i + d_i + (n-1). \tag{3.1}$$

In particular, for every subset $I \subset \{1, 2, \dots, n\}$ *of size* $|I|$ *we have*

$$|I| \sum_{j=1}^{n} d_j \ge \sum_{i\in I}(D_i + d_i) + |I|(n-1). \tag{3.2}$$

Note that in the case when $I = \{1, \dots, n\}$ the last equality may be written in the form

$$(n-1)\sum_{j=1}^{n} d_j \ge n(n-1) + \sum_{i=1}^{n} D_i. \tag{3.3}$$

Remark. These inequalities make little sense when $J_i \equiv 0$ and $P_i \equiv 0$ since the degree of the zero polynomial is $-\infty$. Hence it is worth considering the above theorem under the assumption that φ is dominant.

Definition. A smooth hypersurface $X = \{x \in \mathbf{C}^n \mid P(x) = 0\}$ is called *simple nondominated* if one of the inequalities (3.2) cannot hold for natural $d_1, \dots, d_n$.

Corollary. *There is no dominant morphism from* $\mathbf{C}^{n-1}$ *into a simple nondominated hypersurface* $X \subset \mathbf{C}^n$.

4. Let us consider several examples. Let P be given by

$$\text{(a)} \quad x_1 + x_1^{d-1}x_2 + x_2^{d-a}x_3^a + x_4^d$$

where $d-1 > a > 1$, and a is relatively prime with d and $d-1$ (so $a > 2$) or;

$$\text{(b)} \quad x_1 + x_1^2 x_2^a + x_3^b + x_4^c.$$

where $a \geq 2, b > 3, c > 3$.

It is worth mentioning that the hypersurfaces $X = \{x \in \mathbf{C}^4 \mid P(x) = 0\}$ corresponding to the case (a) present contractible hypersurfaces from [**2**]. Anyway, in both cases (a) and (b) X is simple nondominated. Indeed, in case (a),

$$\begin{aligned} D_1 &= (d-2)d_1 + d_2 \\ D_3 &= (d-a)d_2 + (a-1)d_3 \\ D_4 &= (d-1)d_4. \end{aligned}$$

Hence for $I = \{1, 3, 4\}$, inequality (3.2) implies

$$\begin{aligned} 2d_1 + 3d_2 + 2d_3 + 2d_4 \geq {} & (d-2)d_1 + (d-a+1)d_2 \\ & + (a-1)d_3 + (d-1)d_4 + 9. \end{aligned}$$

The last inequality has no solutions with natural d_1, d_2, d_3 and d_4 under the given restrictions on a and b.

Similarly, in case (b) one may check that inequality (3.3) does not hold for natural d_1, d_2, d_3, d_4. Hypersurfaces given by the zero loci of forms (b) include some Russell's threefolds. For more accurate computation of this type for Russell's threefolds see [**7**]. It is worth mentioning that the condition on a, b, c is essential in this case. For instance the hypersurface $x_1 + x_1^2 x_2 + x_3^3 + x_4^2 = 0$ admits a dominant morphism from $\mathbf{C}^3$ (Russell) and should be fought by other means [**9**].

5. The method used in the above examples may be formulated in the form of

Lemma. *Let P, P_i be as in Section 2, and let I be a nonempty subset of $\{1, \dots, n\}$ of size $|I|$. Suppose that for every $i \in I$ the leading part of $P_i(x)$ is a monomial $c_i x^{m^i}$ where $x^{m^i} = x_1^{m_1^i} \dots x_n^{m_n^i}$. Suppose also that for every $j = 1, \dots, n$ we have $\deg_{x_j} P_i = m_j^i$. Let $\sum_{i \in I} m_j^i \geq |I|$ when $j \notin I$, and let $\sum_{i\in I} m_j^i \geq |I| - 1$ otherwise. Then the hypersurface $X = \{P(x) = 0\}$ is simple nondominated.*

Proof. Assume that there exists a dominant morphism $\varphi : \mathbf{C}^{n-1} \to X$. Then we may treat P_i as an element of $\mathbf{C}[\xi_1, \dots, \xi_{n-1}]$. It is easy to see that its degree D_i coincides with the degree of $x^{m^i} \circ \varphi$ which is $\sum_{j=1}^n m_j^i d_j$ where d_j is the same as in Section 3. Hence (3.2) cannot hold for nonnegative d_i. □

6. Before we apply our technique to hyperbolic modifications we have to consider some properties of smooth contractible surfaces with Kodaira logarithmic dimension 1. Let n and m be coprime natural numbers such that $n > m > 1$. Let $h_{n,m}(x_1, x_2) = (x_1 + 1)^n - (x_2 + 1)^m$. Put $f_{n,m}(x_1, x_2, x_3) = h_{n,m}(x_1 x_3, x_2 x_3)/x_3$. Then $f_{n,m}$ is a polynomial. Consider the hypersurface $V_{(n,m)} = \{(x_1, x_2, x_3) \in \mathbf{C}^3 | f_{n,m}(x_1, x_2, x_3) = 1\}$. It is a smooth contractible hypersurface of Kodaira logarithmic dimension 1 [**12**]. It contains the only line $L_{n,m}$ which coincides with the zero locus of the function x_3 on $V_{(n,m)}$. The following fact may be extracted from [**12**, pp. 150–151].

Lemma. *The surface $V(n, m)$ is isomorphic to the complement of the proper transform of the curve $\{(y_1, y_2) \in \mathbf{C}^2 | \, {y_1}^n - {y_2}^m = 0\}$ in the blow-up of $\mathbf{C}^2$ at the point $(1, 1)$. For every smooth contractible surface W of Kodaira logarithmic dimension 1, there exists a unique pair (n, m) such that*

(1) *either W is isomorphic to $V(n, m)$ or*

(2) *W can be obtained by the following procedure. Let $\rho = \rho_j \circ \dots \circ \rho_1 : \overline{W} \to V(n, m)$ be a blow-up of $V(n, m)$ at a point $q_1 \in L_{n,m}$ and infinitely near points such that the center q_i of each blow-up ρ_i lies on the exceptional divisor of ρ_{i-1} for $i \geq 2$. Then W coincides with the complement of the proper transform in $\overline{W}$ of the curve $(\rho_{j-1} \circ \dots \circ \rho_1)^{-1}(L_{(n,m)})$ under the blow-up ρ_j.*

7. Let j be a nonnegative integer and let P be of the form

$$(7.1)\quad P(x_1,x_2,x_3) = [(x_3{}^{j+1}x_1 + g_1(x_3))^n - (x_3{}^{j+1}x_2 + g_2(x_3))^m - x_3]x_3^{-j-1}$$

where $g_1, g_2 \in \mathbf{C}[x_3]$, $g_1(0) = g_2(0) = 1$, $\deg g_1, \deg g_2 \leq j$, and g_1, g_2 are chosen so that P is a polynomial (note that g_1 may be chosen arbitrary and the last condition determines g_2 uniquely).

Let us denote by $\bar{k}(W)$ Kodaira logarithmic dimension of a surface W.

Theorem. *The hypersurface $P^{-1}(0)$ is smooth contractible with Kodaira logarithmic dimension* 1. *Moreover, every smooth contractible surface W with $\bar{k}(W) = 1$ may be represented in this form and when $j > 0$ it has the same meaning as in Lemma* 6 (2).

Proof. For $j = 0$ the statement follows from Lemma 6.1 (1). Put $\varphi_0(x_1,x_2,x_3) = (x_3x_1 + 1, x_3x_2 + 1)$ for $(x_1,x_2,x_3) \in V(n,m)$. Then $\varphi_0(V_{n,m})$ is the union of $\mathbf{C}^2 - \{(y_1,y_2)|y_1^n - y_2^m = 0\}$ and the point $w = (1,1)$. Note that $\varphi_0^{-1}(w)$ is the line $L_{n,m} = \{x_3 = nx_1 - mx_2 - 1 = 0\}$ in $V_{n,m}$.

Suppose that the theorem holds for j, and show that it is true for $j+1$. Let W and ρ_j be as in Lemma 6 (1). Then by assumption W coincides with $P^{-1}(0)$ where P is of the form (7.1). Put

$$\varphi(x_1,x_2,x_3) = (x_3^{j+1}x_1 + g_1(x_3), x_3^{j+1}x_2 + g_2(x_3))$$

for $(x_1,x_2,x_3) \in W$. By induction we may suppose that $\varphi(W)$ is again the union of $\mathbf{C}^2 - \{(y_1,y_2)|y_1^n - y_2^m = 0\}$ and w, and $\varphi^{-1}(w)$ is the line $L_W = \{x_3 = 0\} \cap W$ in W which is the exceptional divisor of ρ_j.

Let $\tilde{W}$ be a smooth contractible surface with $\bar{k} = 1$ which requires $j+1$ blow-ups described in Lemma 6 (2). Then we may suppose that $\tilde{W}$ is obtained from W by blowing up a point $\tilde{w} \in L_W$ and deleting the proper transform of L_W. Let $\tilde{w} = (a_1,a_2,0)$. Replace x_1 by $x_1 + a_1$ and x_2 by $x_2 + a_2$. Then we have to blow the origin up and delete the proper transform of the intersection of the polynomial zero fiber with the plane $x_3 = 0$. If the zero fiber of a polynomial $Q(x_1,x_2,x_3)$

contains the origin, then after this procedure the proper transform of this fiber is the zero fiber of $Q(x_1x_3, x_2x_3, x_3)/x_3$. If we apply this argument to P with the shifted x_1 and x_2, one can see that $\tilde{W}$ is given by the zero fiber of the polynomial

$$(7.2) \qquad [(x_3^{j+2}x_1 + \tilde{g}_1(x_3))^n - (x_3^{j+2}x_2 + \tilde{g}_2(x_3))^m - x_3]x^{-j-2}.$$

where $\tilde{g}_1, \tilde{g}_2$ satisfy all desired conditions. More precisely they are of the form $\tilde{g}_k(x_3) = a_k x_3^{j+1} + g_k(x_3)$. Put

$$\tilde{\varphi}(x_1, x_2, x_3) = (x_3^{j+2}x_1 + \tilde{g}_1(x_3), x_3^{j+2}x_2 + \tilde{g}_2(x_3))$$

for $(x_1, x_2, x_3) \in \tilde{W}$. Then one can see that $\tilde{\varphi}^{-1}(w)$ is again the line $\{x_3 = 0\} \cap \tilde{W}$ which is the exceptional divisor for ρ_{j+1}. This completes the induction.

It is a simple observation that for every polynomial of the from (7.2) its zero fiber is a smooth contractible hypersurface with Kodaira logarithmic dimension 1. Indeed, every pair of polynomials $\tilde{g}_1, \tilde{g}_2$ such that form (7.2) is a polynomial, $\tilde{g}_1(0) = \tilde{g}_2(0) = 1$, and $\deg \tilde{g}_1$, $\deg \tilde{g}_2 \leq j+1$ can be rewritten as $\tilde{g}_1(x_3) = a_1 x_3^{j+1} + g_1(x_3)$, $\tilde{g}_2(x_3) = a_2 x_3^{j+1} + g_2(x_3)$ where g_1, g_2 satisfy the assumption of the theorem and the point $(a_1, a_2, 0)$ belongs to the line $\varphi^{-1}(w)$ (where φ is defined via g_1, g_2 as above). □

8. Now we can obtain a result which was mentioned in the introduction.

Corollary. *Every smooth contractible surface with Kodaira logarithmic dimension* 1 *is simple non-dominated.*

Proof. Let P be of the form (7.1) and P_1, P_2 be its partial derivatives with respect to x_1 and x_2. Then,

$$P_1(x_1, x_2, x_3) = n(x_3^{j+1}x_1 + g_1(x_3))^{n-1}$$

and

$$P_2(x_1, x_2, x_3) = m(x_2 x_3^{j+1} + g_2(x_3))^{m-1}.$$

Note that $nx_1^{n-1}x_3^{(n-1)(j+1)}$ and $mx_2^{m-1}x_3^{(m-1)(j+1)}$ are leading monomials for P_1 and P_2, respectively. These monomials and $I = \{1,2\}$ satisfy the assumption of Lemma 5. Therefore, the hypersurface $P = 0$ is simple nondominated. □

9. In Section 6 we used the following procedures. We blow up a surface W at a point w from a line L_W on this surface and then remove the proper transform of L_W. As a result we get a new surface $\tilde{W}$. This procedure is called a half-point attachment (see [**3**]). Note that L_W may be treated as a hypersurface in W. One can generalize this procedure by considering a blow-up of the algebraic manifold X at a smooth locus C which belongs to a hypersurface E in X and then removing the proper transform of E. We call this procedure a half locus attachment over the divisor E with locus C. The result of this attachment is an algebraic manifold $\tilde{X}$. This procedure is important since when X is affine, X, C, and E are contractible, and E is smooth then $\tilde{X}$ is again affine contractible [**5**] (the assumption on smoothness of E may be weakened).

Let us consider the case when $X = \mathbf{C}^n$, $C = \{x_1 = \cdots = x_\ell = 0\}$, and E is the zero fiber of a polynomial P. One can easily check that in this case $\tilde{X}$ coincides with the nonzero fiber of the polynomial

$$\tilde{P}(x_1, \ldots, x_{n+1}) = P(x_1x_{n+1}, \ldots, x_\ell x_{n+1}, x_{\ell+1}, \ldots, x_n)/x_{n+1}$$

in $\mathbf{C}^{n+1}$.

Lemma. *Let P, P_i, x^{m_i}, I satisfy the assumption of Lemma 5. Let $C = \{x_1 = \cdots = x_\ell = 0\}$ be contained in the zero fiber of P. Suppose that $\tilde{X} = \{\tilde{P}(x_1, \ldots, x_{n+1}) = 0\}$ is the result of the half locus attachment over $P^{-1}(0)$ with locus C. Let*

$$\sum_{j=1}^{\ell} \sum_{i \in I} m_j^i \geq 2|I|.$$

Then $\tilde{X}$ is simple non-dominated.

Proof. Let $\tilde{P}_i$ be the ith partial derivative of $\tilde{P}_i$. Then for $i \in I$ the leading part of $\tilde{P}_i$ is the monomial $c_i x^{m^i} x_{n+1}^{m_{n+1}^i}$ where $m_{n+1}^i =$

$\sum_{j=1}^{\ell} m_j^i - 1$, by construction. The assumption of this lemma implies that $\sum_{i \in I} m_{n+1}^i \geq |I|$. Hence $\tilde{P}, \tilde{P}_i, I$ satisfy Lemma 5 and, therefore, $\tilde{X}$ is simple non-dominated. □

10. When in the previous lemma $\ell = n$, i.e., C is the origin, $\tilde{P}$ is called a hyperbolic modification of P (see [**11, 16**]). If the hypersurface $P^{-1}(0)$ is smooth contractible then the remarkable fact from [**11, 16**] says that every fiber of $\tilde{P}$ is diffeomorphic to a Euclidean space. Moreover, the Kodaira logarithmic dimension of $\tilde{P}^{-1}(0)$ is not less than the Kodaira logarithmic dimension of $P^{-1}(0)$. The main example in [**11**] and [**16**] is $P(x_1, x_2, x_3) = [(x_1x_3+1)^n - (x_2x_3+1)^m]x_3^{-1} - 1$, i.e., $P^{-1}(0) = V(n,m)$ in terms of Section 6. Since $\bar{k}(V(n,m)) = 1$, the fiber $\tilde{P}^{-1}(0)$ is not isomorphic to $\mathbf{C}^3$, but nothing is said there about nonzero fibers of $\tilde{P}$. Meanwhile an application of Lemma 9 immediately shows that all fibers of $\tilde{P}$ are simple non-dominated. The situation is similar if P is of form (7.1). It remains the same due to Lemma 9 if, instead of one hyperbolic modification of P of form (7.1), we consider a sequence of hyperbolic modifications beginning from this polynomial.

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