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DISCRETE GROUPS AND DISCONTINUOUS ACTIONS

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Introduction. Discontinuous actions of groups play an important role in complex function theory, especially in the study of Riemann surfaces. The study of Kleinian groups, Fuchsian groups, the theory of automorphic forms are all rich areas of mathematics with many deep results. The work of Thurston on 3-manifolds has given additional focus to this very rich field of discontinuous group actions. However, there does not seem to be complete agreement regarding notation and terminology. For instance, the concept of "properly discontinuous" action is defined differently by different authors and these definitions in general are not equivalent. The present paper is a *semi-expository paper* devoted mainly to clarifying various concepts and studying the interrelationships between these concepts.

There are two definitions of "properly discontinuous actions" of a group G on a space X frequently used in literature. One of them requires the existence for each $x \in X$ of an open set U in X with $x \in U$ and $\{g \in G \mid U \cap gU \neq \phi\}$ finite. The other requires that, for any compact set K of X, the set $\{g \in G \mid K \cap gK \neq \phi\}$ be finite. Even when X is a very nice space like a manifold the two definitions are not equivalent. To clarify the difference, we introduce the concept of "strongly properly discontinuous actions" as a generalization of "properly discontinuous actions" (see Definition 5). It turns out that if G acts strongly properly discontinuously on X, then for any compact set $K \subset X$ the set $\{g \in G \mid K \cap gK \neq \phi\}$ is finite (Proposition 7). In general, even when X is a very nice space (for instance a manifold) when G acts properly discontinuously on X, the orbit space X/G need not be Hausdorff. When G acts properly discontinuously on a Hausdorff space X, the orbit space X/G is Hausdorff if and only if the action of G on X is strongly properly discontinuous (Proposition 6). Examples

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exist in literature to show (i) that a group action even on a nice space could be discontinuous without being properly discontinuous and (ii) that a group G can act properly discontinuously on X with X/G non-Hausdorff. However, these examples are complicated. In Section 2 of the present paper, we give easy examples to illustrate these phenomena (Examples 5 and 4).

There are also generalizations of proper discontinuous actions and strongly proper discontinuous actions in literature [**6, 7** and **12**]. There is again no complete agreement regarding the notation and terminology. Section 3 of the present paper is devoted to clarifying these concepts and studying the interrelationship between these concepts.

In the excellent article of Peter Scott on the geometries of 3-manifolds [**13**], the starting point is the result that for a connected Riemannian manifold M, any discrete subgroup of the group $I(M)$ of isometries of M acts strongly properly discontinuously on M. In his exposition he requires M to be complete. However, the result is valid without completeness, but connectedness of M is very crucial for the validity of the result. Actually, this result is valid more generally for the group $I(X)$ of isometries of a connected, locally compact, separable metric space X. In our present article, we present a complete proof of the known result that any discrete subgroup of $I(X)$ acts strongly properly discontinuously on X whenever X is a connected, locally compact, separable metric space (Theorem 3). Also, it is a known result that for a connected Riemannian manifold M, the isotropy group $I(M)_x$ at any $x \in M$ is compact. A similar result is valid for any connected, locally compact, separable metric space X.

In the case of a connected Riemannian manifold M one has the deeper result of Myers and Steenrod [**11**] that $I(M)$ is a Lie group. But this is not needed in establishing the link between discrete subgroups of $I(M)$ and discontinuous actions.

J. Lehner's treatise [**8**] on discontinuous groups is a comprehensive account of the deep results on discontinuous groups and automorphic forms. For the work of W. Thurston, the reader could refer to [**14**].

The authors are thankful to the referee for bringing [**6, 7** and **12**] to their notice and suggesting that the results on proper and strongly proper discontinuous actions could be extended to "proper" and "strongly proper" actions. The non-Hausdorff quotients in [**7**] are

of interest in theoretical physics.

1. Discontinuous and properly discontinuous group actions. Throughout this paper X will denote a Hausdorff space. G will denote an abstract group acting on X by means of homeomorphisms. C will denote an infinite cyclic group written multiplicatively.

Definition 1. 8The action of G will be said to be *discontinuous* at $x \in X$ if the following two conditions are satisfied:

1. The orbit Gx of x is a closed discrete subset of X and
2. The isotropy group G_x at x is finite.

For any $y = gx \in Gx$ we have $Gy = Gx$ and $G_y = gG_xg^{-1}$. Hence, G acts discontinuously at $x \Leftrightarrow G$ acts discontinuously at any $y \in Gx$. In other words, discontinuity of action of G is property of an orbit rather than of a point.

Proposition 1. *Let G act discontinuously at x. Then, for any sequence $\{g_n\}_{n\geq 1}$ of distinct elements in G, the sequence $\{g_nx\}_{n\geq 1}$ cannot converge in X.*

Proof. Suppose, on the contrary, $\lim_{n\to\infty} g_nx = a$ in X. Then $a \in Gx$ since Gx is closed. Let $a = gx$. Since Gx is discrete, there exists an open set U of X with $U \cap Gx = \{gx\}$. Since $\lim_{n\to\infty} g_nx = gx$, there exists an N_0 such that $g_nx \in U$ for $n \geq N_0$. But then $g_nx \in U \cap Gx = \{gx\}$ for $n \geq N_0$ yielding $g^{-1}g_n \in G_x$ for $n \geq N_0$. This contradicts the finiteness of G_x. □

When X satisfies the first axiom of countability, we have the following converse to Proposition 1.

Proposition 2. *Let X satisfy the first axiom of countability. Assume that, for any infinite sequence $\{g_n\}_{n\geq 1}$ of distinct elements in G, $\{g_nx\}_{n\geq 1}$ does not converge in X. Then G acts discontinuously at x.*

Proof. Suppose Gx is not closed in X. Let $a \in \overline{Gx} \backslash Gx$. Since X satisfies the first axiom of countability, there exists a sequence of elements of the form $g_n x$ with $g_n \in G$ and satisfying $\lim_{n\to\infty} g_n x = a$. Since $a \notin Gx$, we see that $g_n x \neq a$ for any n. Since $g_n x$ converges to a, it follows that $\{g_n x \mid n \geq 1\}$ is an infinite set. In particular, $\{g_n \mid n \geq 1\}$ is an infinite set. By passing to a subsequence, if necessary, we may assume that $\{g_n\}_{n\geq 1}$ are all distinct. But then $\lim_{n\to\infty} g_n x = a$ contradicts the hypothesis.

Now suppose Gx is not discrete. Then there exists an element $hx \in Gx$ with the property that every open set U in X with $hx \in U$ will satisfy $U \cap Gx \supsetneq \{hx\}$. We can pick a countable family of open sets $V_1 \supset V_2 \supset V_3 \supset \cdots$ in X with $\{hx\} = \cap_{n\geq 1} V_n$ and elements $g_n x \in V_n \backslash \{hx\}$. Then $\lim_{n\to\infty} g_n x = hx$. Since $g_n x \neq hx$ for each n, as earlier this leads to a contradiction.

Suppose G_x is infinite. Then we can pick an infinite sequence $\{g_n\}_{n\geq 1}$ of distinct elements in G_x. Then $g_n x = x$ for all n, hence $\lim_{n\to\infty} g_n x = x$, a contradiction. □

Definition 2. We say that G acts *discontinuously* on X if its action is discontinuous at every $x \in X$.

Remarks. 1. The orbit space X/G will always be endowed with the quotient topology. $\eta : X \to X/G$ will denote the canonical quotient map. As $\eta^{-1}(\eta(U)) = \cup_{g\in G} gU$, for any open set U of X we see that $\eta^{-1}(\eta(U))$ is open in X. Hence, $\eta : X \to X/G$ is an open map. Also, X/G is a T_1 space if and only if each orbit is closed. In particular, if G acts discontinuously on X, then X/G is a T_1 space.

2. Let G be a Hausdorff topological group acting as a topological transformation group in the sense that the map $\mu : G \times X \to X$ given by $\mu(g, x) = gx$ is continuous. Suppose there exists an element $x_0 \in X$ with Gx_0 discrete and G_{x_0} finite. Then G itself is discrete. In fact, the map $\beta : G \to X$ given by $\beta(g) = gx_0$ is continuous. Since Gx_0 is discrete there exists an open set U in X with $U \cap Gx_0 = \{x_0\}$. Clearly $\beta^{-1}(U) = G_{x_0}$. Hence G_{x_0} is open in G. Let $G_{x_0} = \{e, g_1, \ldots, g_r\}$. Then $\{e\} = G_{x_0} \backslash \{g_1, \ldots, g_r\}$ is open in G. Hence G is discrete.

In particular, any Hausdorff topological transformation group G

whose action is discontinuous at at least one point $x_0 \in X$ has to be a discrete group.

If X is locally compact and locally connected, a result of R. Arens [**1**] asserts that the group $H(X)$ of homeomorphisms of X is a Hausdorff topological transformation group on X when we put the compact open topology on $H(X)$. In particular, a subgroup G of $H(X)$ acting discontinuously at at least one point of X has to be a discrete subgroup of $H(X)$. The following examples illustrate that a group of homeomorphisms which is discrete in the compact open topology need not act discontinuously at any point.

Examples. 1. Let $M^n = \mathbf{R}_1^n \dot{\cup} \mathbf{R}_2^n$ be the disjoint union of two copies of $\mathbf{R}^n$ with the usual differentiable structure, where $n \geq 1$. Let $C \times C$ denote the direct product of two infinite cyclic groups with generators τ and θ. Let $0 \neq a_j \in \mathbf{R}_j^n$, $j = 1, 2$. Define an action of $C \times C$ on M^n by $(\tau^k, 1)(x) = x + ka_1$, $(\tau^k, 1)(y) = y$, $(1, \theta^l)(x) = x$ and $(1, \theta^l)(y) = y + la_2$ for all $x \in \mathbf{R}_1^n$, $y \in \mathbf{R}_2^n$ and k, l in $\mathbf{Z}$. For this action, every orbit is closed and discrete in M^n. The isotropy group at any $x \in \mathbf{R}_1^n$ is $1 \times C$ and the isotropy group at any $y \in \mathbf{R}_2^n$ is $C \times 1$. Hence, the action is nowhere discontinuous. For any k, l in $\mathbf{Z}$, we can find open balls B_j in $\mathbf{R}_j^n$ such that $ka_1 \in B_1$, $k'a_1 \notin B_1$ for any $k \neq k'$, $la_2 \in B_2$ and $l'a_2 \notin B_2$ for any $l \neq l'$. The only element of $C \times C$ carrying 0 of $\mathbf{R}_1^n$ into B_1 and 0 of $\mathbf{R}_2^n$ into B_2 simultaneously is (τ^k, θ^l). This shows that $C \times C$ is discrete in the compact open topology. Here $C \times C$ acts by C^∞ diffeomorphisms on M^n.

2. Let $A = \{(m, 0) \in \mathbf{R}^2 \mid m \in \mathbf{Z}\}$, $B = \{(m, 2) \in \mathbf{R}^2 \mid m \in \mathbf{Z}\}$ and X the cone over $A \cup B$ in $\mathbf{R}^2$ with vertex $(0, 1)$. The points in X will be of the form $(m(1-t), t)$ for $0 \leq t \leq 1$ and $m \in \mathbf{Z}$ or $(m(1-s), 2-s)$ for $0 \leq s \leq 1$ and $m \in \mathbf{Z}$. Let the group $C \times C$ act on X by $(\tau^k, 1)(m(1-t), t) = ((m+k)(1-t), t), (1, \theta^l)(m(1-t), t) = (m(1-t), t), (\tau^k, 1)(m(1-s), 2-s) = (m(1-s), 2-s), (1, \theta^l)(m(1-s), 2-s) = ((m+l)(1-s), 2-s)$ for any k, l in $\mathbf{Z}$. For this action of $C \times C$ on X each orbit is closed and discrete. However, at no point of X is the action discontinuous. The isotropy group at $(m(1-t), t)$ is $1 \times C$ for $0 \leq t < 1$, the isotropy group at $(m(1-s), 2-s)$ is $C \times 1$ for $0 \leq s < 1$, and the isotropy group at $(0, 1)$ is the whole of $C \times C$. We omit the proof that $C \times C$ is discrete in the compact open topology. Note that X is a connected, second countable, metric space. However,

X is not locally compact.

3. Let the infinite cyclic group C act on $\mathbf{R}^n$, $n \geq 1$, via $\tau^k x = 2^k x$ for all $k \in \mathbf{Z}$ and $x \in \mathbf{R}^n$. Then, for any $0 \neq x \in \mathbf{R}^n$, the orbit of x is not closed in $\mathbf{R}^n$ since 0 is in the closure of the orbit but not on the orbit. The isotropy group at 0 is the whole of C. Hence, this action is nowhere discontinuous. Again, it is easy to see that C is discrete in the compact open topology.

Definition 3. The action of G is said to be properly discontinuous at $x \in X$ if there exists an open set U in X with $x \in U$ and $\{g \in G \mid U \cap gU \neq \phi\}$ finite.

When U satisfies the condition stated in Definition 3, for any $h \in G$, hU is an open set in X with $hx \in hU$ and $\{\alpha \in G \mid hU \cap \alpha hU \neq \phi\} = \{\alpha \in G \mid U \cap h^{-1}\alpha hU \neq \phi\} = hEh^{-1}$ where $E = \{g \in G \mid U \cap gU \neq \phi\}$. Thus, the set $\{\alpha \in G \mid hU \cap \alpha hU \neq \phi\}$ is finite. It follows that G acts properly discontinuously at x if and only if it acts properly discontinuously at hx for any $h \in G$. Since $G_x \subset E$ we see that G_x is finite. The set $\{g \in G \mid gx \in U\}$ is a subset of E. If $\Delta = E \backslash G_x$, then $V = U - \{gx \mid g \in \Delta\}$ is an open set in X with $V \cap Gx = \{x\}$. For any $h \in G$, the set hV is an open set in X with $hV \cap Gx = \{hx\}$. Thus Gx is discrete.

In case G is a topological transformation group of X acting properly discontinuously at some $x \in X$, then from Remark (2) it follows that G is discrete.

Let G act properly discontinuously at x, and let Gy be an orbit of y with $x \notin Gy$. The Hausdorffness of X yields open sets W and V in X with $y \in W$, $x \in V$ and $W \cap V = \phi$. We may choose V to satisfy the condition that the set $E = \{g \in G \mid V \cap gV \neq \phi\}$ is finite. We claim that the set $D = \{h \in G \mid hy \in V\}$ has to be finite. If $D \neq \phi$, let $h_0 \in D$. From $hy = hh_0^{-1}h_0 y$ we see that $\{hh_0^{-1} \mid h \in D\} \subset E$. Hence D is finite. The set $U = V \backslash \{hy \mid h \in D\}$ is an open set in X with $x \in U$ and $U \cap Gy = \phi$. Hence, $\eta(U)$ is an open set in X/G with $\eta(x) \in \eta(U)$ and $\eta(y) \notin \eta(U)$.

Remark 3. In Example 3, the action of C on $\mathbf{R}^n$ is properly discontinuous at any $x \neq 0$ in $\mathbf{R}^n$. To see this, let $\|x\| = r$ and $\varepsilon = r/4$.

We claim that $\theta^k(B_\varepsilon(x)) \cap B_\varepsilon(x) = \phi$ for every $k \neq 0$ in $\mathbf{Z}$. In fact, for any $y \in B_\varepsilon(x)$, we have

$$\begin{aligned}\|\tau^k y - x\| &= \|2^k y - x\| \\ &\geq \|2^k x - x\| - \|2^k y - 2^k x\| \\ &\geq |2^k - 1|\|x\| - 2^k(r/4).\end{aligned}$$

For $k \geq 1$, we have

$$\begin{aligned}\|\tau^k y - x\| &\geq (2^k - 1)r - 2^{k-2}r \\ &\geq 2^{k-2}(4r - r) - r \\ &\geq 2^{k-2}3r - r \\ &\geq 3r/2 - r \\ &\geq r/2.\end{aligned}$$

For $k < 0$, we have

$$\begin{aligned}\|\tau^k y - x\| &\geq (1 - 2^k)r - 2^k(r/4) \\ &\geq r - r/2^l - r/2^{l+2} \quad \text{where } l = -k \geq 1 \\ &\geq r - 5r/2^{l+2} \\ &\geq r - 5r/8 = 3r/8.\end{aligned}$$

This proves that $\tau^k(B_\varepsilon(x)) \cap B_\varepsilon(x) = \phi$ for all $k \neq 0$. Thus, in this example, C acts properly discontinuously at every $x \neq 0$ in $\mathbf{R}^n$ but the action is nowhere discontinuous.

Any open set containing 0 in $\mathbf{R}^n$ will meet the orbit of any $x \neq 0$. Hence, the only open set containing $\eta(0)$ in $\mathbf{R}^n/C$ is the whole space $\mathbf{R}^n/C$.

Definition 4. G is said to act *properly discontinuously* on X if G acts properly discontinuously at all the points of X.

Let G act properly discontinuously on X. The comments preceding Remark 3 show that every orbit of G is discrete and that the isotropy group G_x at any $x \in X$ is finite. Also, for any x and y lying on distinct orbits we get an open set in X/G containing $\eta(x)$ but not containing $\eta(y)$. This means that X/G is a T_1 space, equivalently, every orbit of G is closed in X. It follows that G acts discontinuously at every point. In Section 2, we will see that the converse to this is not true in general. However, when X is a metric space and G acts by isometries on X, the converse is true (Proposition 4). Moreover, when G is a discontinuous group of isometries of X, the quotient space X/G is metrizable. Before dealing with isometric actions, we record another fact concerning properly discontinuous actions.

Proposition 3. *Let G act properly discontinuously at $x \in X$. Then there exists an open set V in X with $x \in V$, $gV = V$ for all $g \in G_x$ and $V \cap gV = \phi$ for all $g \in G \backslash G_x$.*

Proof. By the very definition of proper discontinuity at x, there exists an open set U of X with $x \in U$ and $E = \{g \in G \mid U \cap gU \neq \phi\}$ finite. Clearly, $G_x \subset E$. Let $\Delta = E \backslash G_x$. For any $h \in \Delta$ we have $hx \neq x$. There exist open sets W_h and N_h in X with $x \in W_h$, $hx \in N_h$ and $W_h \cap N_h = \phi$. Then $W = U \cap (\cap_{h \in \Delta} W_h) \cap (\cap_{h \in \Delta} h^{-1}(N_h))$ is an open set in X with $x \in W$. Moreover, $h(W) \subset N_h$, hence $W \cap h(W) \subset W_h \cap N_h = \phi$ for any $h \in \Delta$. Let $V = \cap_{g \in G_x} gW$. Then V is open in X, $x \in V$ and $V \cap hV \subset W \cap h(W) = \phi$ for all $h \in \Delta$. From $V \subset U$ we see that $V \cap gV = \phi$ for $g \notin E$. It follows that $V \cap gV = \phi$ for $g \notin G_x$. Moreover, for any $h \in G_x$ we have $hV = h(\cap_{g \in G_x} gW) = \cap_{g \in G_x} hgW$. For any $h \in G_x$, the set $\{hg \mid g \in G_x\}$ is the same as $\{g \mid g \in G_x\}$. Hence, $hV = \cap_{g \in G_x} gW = V$. $\square$

Proposition 4. *Let G be a group acting by isometries on a metric space X. Suppose Gx_0 is discrete and G_{x_0} is finite where x_0 is some element of X. Then G acts properly discontinuously at x_0.*

Proof. Since Gx_0 is discrete, there exists an $\varepsilon > 0$ with $B_\varepsilon(x_0) \cap Gx_0 = \{x_0\}$. Let $V = B_{\varepsilon/2}(x_0)$. Suppose $g \in G$ satisfies $V \cap gV \neq \phi$. Let $v \in V$ satisfy $gv \in V$. Then

$$\begin{aligned} d(gx_0, x_0) &\leq d(gx_0, gv) + d(gv, x_0) \\ &\leq d(x_0, v) + d(gv, x_0) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence, $gx_0 \in Gx_0 \cap B_\varepsilon(x_0) = \{x_0\}$. Thus, $V \cap gV \neq \phi$ implies $g \in G_{x_0}$. It follows that $\{g \in G \mid V \cap gV \neq \phi\} = G_{x_0}$ which is finite. □

Corollary 1. *Let G be a group acting as a group of isometries on a metric space X. If G acts discontinuously at $x_0 \in X$, then it acts properly discontinuously at x_0.*

In particular, G acts discontinuously on X if and only if it acts properly discontinuously on X.

Observe that the quotient space X/G is Hausdorff if and only if for any two elements x, y in X with $Gx \cap Gy = \phi$ we can find open sets U, V in X with $x \in U$, $y \in V$ and $U \cap GV = \phi$. In this case automatically $GU \cap GV = \phi$.

Proposition 5. *Let G be a group acting by isometries on a metric space X. Suppose every orbit is closed. Then X/G is metrizable.*

Proof. Let $\eta : X \to X/G$ denote the canonical quotient map. Consider $\inf_{g \in G} d(x, gy)$. When $Gx = Gy$ we see that $\inf_{g \in G} d(x, gy) = 0$ since $x = gy$ for some $g \in G$. If $x \notin Gy$, since Gy is closed we see that $\inf_{g \in G} d(x, gy) > 0$. Also $\inf_{g \in G} d(hx, gy) = \inf_{g \in G} d(x, h^{-1}gy) = \inf_{g' \in G} d(x, g'y)$. Thus, $\inf_{g \in G} d(x, gy)$ depends only on $\eta(x)$ and $\eta(y)$. We set $D(\eta(x), \eta(y)) = \inf_{g \in G} d(x, gy)$. From $\inf_{g \in G} d(y, gx) = \inf_{g \in G} d(g^{-1}y, x) = \inf_{g \in G} d(x, g^{-1}y) = \inf_{h \in G} d(x, hy)$ we see that $D(\eta(x), \eta(y)) = D(\eta(y), \eta(x))$. The triangle inequality for D can easily be checked. We will show that D is a metric on the quotient space X/G. For any $x \in X$ it is clear that $\eta^{-1}(B_\varepsilon(\eta(x))) \supset B_\varepsilon(x)$. Hence, to show that D metrizes X/G, we have only to show that for any open set U in X and $x \in U$, there exists some $\delta > 0$ with $B_\delta(\eta(x)) \subset \eta(U)$. If this is not the case for each integer $n \geq 1$, we will have an element $v_n \in X \backslash GU$ such that $\inf_{g \in G} d(v_n, gx) < 1/n$. This in turn yields a $g_n \in G$ with $d(v_n, g_n x) < 2/n$ or $d(g_n^{-1} v_n, x) < 2/n$. From $v_n \in X \backslash GU$ we see that $g_n^{-1} v_n \notin U$. Since U is open in X, there exists some $r > 0$ with $B_r(x) \subset U$. Hence $d(g_n^{-1} v_n, x) \geq r$ for all n, contradicting $d(g_n^{-1} v_n, x) < 2/n$. □

Corollary 2. *Let G act on a metric space by isometries. If the* *action of G on X is discontinuous, then X/G is metrizable.*

2. Strongly properly discontinuous group actions.

Definition 5. We say that G acts *strongly properly discontinuously* on X if for any x, y in X there exist open sets U and V in X with $x \in U$, $y \in V$ and $\{g \in G \mid gU \cap V \neq \phi\}$ is finite.

In particular, when $x = y$, by taking $W = U \cap V$ we see that $\{g \in G \mid gW \cap W \neq \phi\}$ is finite. Hence, if G acts strongly properly discontinuously on X, then its action is properly discontinuous. We will see later in this section that the converse is not true in general. However, when G acts by isometries on a metric space, the converse is true.

Remark 4. Suppose G acts properly discontinuously at $x \in X$ and $y = hx$ is any other point of the orbit Gx of x. Let V be an open set in X with $x \in V$ and satisfying the condition that $E = \{g \in G \mid V \cap gV \neq \phi\}$ is finite. Let $W = hV$. Then W is open in X and $y \in W$. Moreover, the set $\{g \in G \mid gV \cap W \neq \phi\} = \{g \in G \mid gV \cap hV \neq \phi\} = \{g \in G \mid (h^{-1}gV) \cap V \neq \phi\} = \{g \in G \mid h^{-1}g \in E\} = hE$ is finite.

Proposition 6. *Suppose G acts properly discontinuously on X. Then X/G is Hausdorff if and only if the action of G on X is strongly properly discontinuous.*

Proof. Assume X/G is Hausdorff. To show that G acts strongly properly discontinuously, because of Remark 4, we need only consider the case when x and y are in different orbits. Since X/G is Hausdorff, there exist open sets U, V in X with $x \in U$, $y \in V$ and $GU \cap GV = \phi$. For this choice of U and V, we have $\{g \in G \mid gU \cap V \neq \phi\} = \phi$. Conversely, assume that G acts strongly properly discontinuously on X. Let x and y be any two elements of X with $Gx \cap Gy = \phi$. We can find open sets S and T in X with $x \in S$, $y \in T$, $S \cap T = \phi$ and $E = \{g \in G \mid S \cap gT \neq \phi\}$ finite. Let $E = \{g_1, \dots, g_r\}$. From $Gx \cap Gy = \phi$, we see that $x \neq gy$ for any $g \in G$; in particular, $x \neq g_i y$ for $1 \leq i \leq r$. We can find open sets A_i and B_i in X with $x \in A_i \subset S$,

$y \in B_i \subset T$ and $A_i \cap g_i B_i = \phi$ for $1 \leq i \leq r$. Then $U = \cap_{i=1}^r A_i$, $V = \cap_{i=1}^r B_i$ are open sets in X with $x \in U$, $y \in V$ and $U \cap gV = \phi$ for all $g \in G$. This means $U \cap GV = \phi$. Hence, X/G is Hausdorff. □

Proposition 7. *Let G act strongly properly discontinuously on X. Then, for any compact subset K of X the set $\{g \in G \mid gK \cap K \neq \phi\}$ is finite.*

Proof. Let K be any compact subset of X. Let x be any element of K. For any $y \in K$, we can find open sets U^y and V^y in X with $x \in U^y$, $y \in V^y$ and $E^y = \{g \in G \mid gU^y \cap V^y \neq \phi\}$ finite. The compactness of K yields a finite number of elements $y_1, \dots, y_r$ with $\cup_{i=1}^r V^{y_i} \supset K$. Let $W_x = \cap_{i=1}^r U^{y_i}$. Then W_x is an open set in X with $x \in W_x$. Moreover, $\Delta_x = \{g \in G \mid gW_x \cap K \neq \phi\} \subset \cup_{i=1}^r E^{y_i}$. Hence Δ_x is finite. Thus, for any $x \in K$, there exists an open set W_x of X with $x \in W_x$ and $\Delta_x = \{g \in G \mid W_x \cap K \neq \phi\}$ finite. The compactness of K yields a finite number of elements $x_1, \dots, x_k$ with $\cup_{i=1}^k W_{x_i} \supset K$. It follows that $\{g \in G \mid gK \cap K \neq \phi\}$ is a subset of $\cup_{i=1}^k \Delta_{x_i}$. Hence $\{g \in G \mid gK \cap K \neq \phi\}$ is finite. □

Remarks. 5. Let G act strongly properly discontinuously on X. Then for any $K \subset X$, $L \subset X$ with K and L compact, the set $\{g \in G \mid gK \cap L \neq \phi\}$ is finite. This is because $K \cup L$ is compact and $\{g \in G \mid gK \cap L \neq \phi\} \subset \{g \in G \mid g(K \cup L) \cap (K \cup L) \neq \phi\}$.

6. If X is locally compact, for any group G acting on X the orbit space X/G will be locally compact, even though X/G need not be Hausdorff. This is because for any $x \in X$ and any compact neighborhood N of x in X, $\eta(N)$ is a compact neighborhood of $\eta(x)$ in X/G.

7. In case X is locally compact, G acts on X strongly properly discontinuously on X if and only if $\{g \in G \mid gK \cap K \neq \phi\}$ is finite for any compact set K of X.

If G acts strongly properly discontinuously on X, then $\{g \in G \mid gK \cap K \neq \phi\}$ is finite by Proposition 7. For this part, we do not need the local compactness of X. Now assume X locally compact. For any x, y in X we find open sets U, V in X with $x \in U$, $y \in V$ and $\overline{U}, \overline{V}$ compact. Then $\{g \in G \mid gU \cap V \neq \phi\} \subset \{g \mid g(\overline{U} \cup \overline{V}) \cap (\overline{U} \cup \overline{V}) \neq \phi\}$

and this latter set is finite.

Definition 6. The action of G on X is said to be *free* if $G_x = \{e\}$ for all $x \in X$.

Definition 7. A continuous map $p : A \to B$ of topological spaces A, B which are *not necessarily Hausdorff* will be called a *covering map* if p is onto and for any $b \in B$ we can find an open set V in B with $b \in V$ satisfying the condition that $p^{-1}(V) = \cup_{\alpha \in J} V_\alpha$ a disjoint union of open sets V_α of A possessing the additional property that $p \mid V_\alpha \to V$ is a homeomorphism for each $\alpha \in J$.

Proposition 8. *Let G act freely and properly discontinuously on X. Then the canonical quotient map $\eta : X \to X/G$ is a covering map.*

Proof. Any element of X/G will be of the form $\eta(x)$ for some $x \in X$. From Proposition 3, there exists an open set V of X with $x \in V$ and $V \cap gV = \phi$ for any $g \neq e$ in G. Then $W = \eta(V)$ is an open set containing $\eta(x)$ in X/G. Moreover, $\eta^{-1}(W) = \cup_{g \in G} gV$ a disjoint union and $\eta \mid gV : gV \to W$ is a homeomorphism for each $g \in G$. □

Example 8.3 on pages 167, 168 and 169 of [**9**] describes a free, properly discontinuous action of the infinite cyclic group C on $\mathbf{R}^2$ (by means of C^∞ diffeomorphisms) with the quotient space $\mathbf{R}^2/C$ not Hausdorff. Also, Example 8.4 on pages 169, 170 of [**9**] describes a free discontinuous action of C on the infinite Möbius strip which is not properly discontinuous. This example is attributed to Joseph Auslander in [**9**]. These examples are somewhat complicated. In this section we will give easy examples to illustrate the same phenomena.

Example 4. Consider the action of C on $\mathbf{R}^2\backslash(0,0)$ given by $\tau^k(x,y) = (2^k x, y/2^k)$ for any $k \in \mathbf{Z}$, where τ is a generator of C. This action of C on $\mathbf{R}^2\backslash(0,0)$ is properly discontinuous but not strongly properly discontinuous.

Let $(x,y) \in \mathbf{R}^2\backslash(0,0)$ with both x and y nonzero. As in Remark 3, we can get open intervals B, B' in $\mathbf{R}\backslash\{0\}$ with $x \in B$, $y \in B'$,

$2^k B \cap B = \phi$ and $2^k B' \cap B' = \phi$ for all $0 \neq k \in \mathbf{Z}$. Then $B \times B'$ is an open set containing (x, y) in $\mathbf{R}^2 \backslash (0,0)$ and satisfying $\tau^k(B \times B') \cap (B \times B') = (2^k B \cap B) \times (2^{-k} B' \cap B') = \phi$.

Let $(x, 0) \in \mathbf{R}^2 \backslash (0,0)$. Then $x \neq 0$ in $\mathbf{R}$. Let $r = |x| > 0$. The open ball $(B_r/4)(x, 0)$ of radius $r/4$ around $(x, 0)$ lies in $\mathbf{R}^2 \backslash (0,0)$. The translate $\tau^k((B_r/4)(x,0))$ is the region bounded by the ellipse with $(2^k x, 0)$ as its center, a semi-axis of length $2^k r/4$ parallel to the x-axis, the other semi-axis of length $r/2^{(k+2)}$ parallel to the y-axis (for any $0 \neq k \in \mathbf{Z}$). It is clear that $\tau^k((B_r/4)(x,0)) \cap (B_r/4)(x,0) = \phi$ for any $0 \neq k \in \mathbf{Z}$.

The proof for $(0, y)$ with $y \neq 0$ is similar. If the action of C on $\mathbf{R}^2 \backslash (0,0)$ is strongly properly discontinuous, then from Proposition 7, for any compact set K of $\mathbf{R}^2 \backslash (0,0)$ the set $\{\tau^k \in C \mid (\tau^k K) \cap K \neq \phi\}$ should be finite. For any closed curve γ in $\mathbf{R}^2 \backslash (0,0)$ surrounding the origin, it is easily seen that $\tau^k \gamma \cap \gamma \neq \phi$ for all $k \in \mathbf{Z}$. Thus the action of C on $\mathbf{R}^2 \backslash (0,0)$ is not strongly properly discontinuous. From Proposition 6, we conclude that $(\mathbf{R}^2 \backslash (0,0))/C$ is not Hausdorff. This example is actually mentioned on page 256 of R.S. Kulkarni's paper [**5**]. However, his terminology is different. What we refer to as strongly properly discontinuous, he calls properly discontinuous. Finally, it is clear that this action of C on $\mathbf{R}^2 \backslash (0,0)$ is free.

Example 5. Consider the subspace $P = \{(x, y) \in \mathbf{R}^2 \backslash (0,0) \mid x \geq 0, y \geq 0\}$. P is obtained from the quadrant $\{(x, y) \in \mathbf{R}^2 \mid x \geq 0, y \geq 0\}$ by deleting the origin. P is a C^∞ manifold with boundary; ∂P is the disjoint union $\{(a, 0) \in \mathbf{R}^2 \mid a > 0\} \cup \{(0, b) \in \mathbf{R}^2 \mid b > 0\}$. P is invariant under the action of C on $\mathbf{R}^2 \backslash (0,0)$ described in Example 4. Writing X and Y respectively for $\{(a, 0) \in \mathbf{R}^2 \mid a > 0\}$ and $\{(0, b) \in \mathbf{R}^2 \mid b > 0\}$ we see that X and Y are the connected components of ∂P, they are invariant under the action of C and the map $(a, 0) \to (0, 1/a)$ is a diffeomorphism of X onto Y respecting the action of C. Thus, if M^2 is obtained from P by identifying $(a, 0)$ in X with $(0, 1/a)$ in Y for each $a > 0$, then M^2 is a connected differentiable manifold without boundary, and the action of C on P induces an action of C on M^2. For this action of C on M^2 it is clear that each orbit is closed and discrete in M^2. Moreover, the action is free. In particular, it follows that C acts discontinuously on M^2. It is not difficult to see that M^2 is an infinite Möbius strip.

We claim that this action of C on M^2 is not properly discontinuous. Let $\pi : P \to M^2$ denote the canonical quotient map. For any $a > 0$ let us denote the element $\pi(a,0) = \pi(0,1/a)$ in M^2 by $\bar{a}$. For any open set U containing $\bar{a}$ in M^2, $\pi^{-1}(U)$ will necessarily contain the union $\{(x,y) \in P \mid (x-a)^2+y^2 < \varepsilon^2\} \cup \{(x,y) \in P \mid x^2+(y-1/a)^2 < \varepsilon^2\}$ for some $\varepsilon > 0$. Write B_1 for $\{(x,y) \in P \mid (x-a)^2+y^2 < \varepsilon^2\}$ and B_2 for $\{(x,y) \in P \mid x^2+(y-1/a)^2 < \varepsilon^2\}$. For any $k \geq 1$, the translate $\tau^k B_2$ is the intersection with P of the region bounded by the ellipse with center $(0, 1/(2^k a))$, semi-major axis of length $2^k\varepsilon$ parallel to the x-axis and semi-minor axis of length $\varepsilon/2^k$ parallel to the y-axis. It follows that $\tau^k B_2 \cap B_1 \neq \phi$ for infinitely many values of $k \geq 1$. In particular, $\tau^k U \cap U \supset \pi(\tau^k B_2 \cap B_1) \neq \phi$, for infinitely many values of k. This proves that the action of C on M^2 is not properly discontinuous.

Theorem 1. *Let X be a metric space and G a group acting on X by isometries. Then the following are equivalent*:

1. *G acts discontinuously on X.*
2. *G acts properly discontinuously on X.*
3. *G acts strongly properly discontinuously on X.*

Proof. Immediate consequence of Corollary 1, Corollary 2 and Proposition 6. □

Corollary 3. *In Example* 4, (*respectively Example* 5) $\mathbf{R}^2 \backslash (0,0)$ (*respectively* M^2) *cannot carry a metric invariant under that action of C.*

If G is a group acting by isometries on a metric space with each orbit closed, then we saw that $D(\eta(x), \eta(y)) = \inf_{g \in G} d(x, gy) = \inf_{h \in G, g \in G} d(hx, gy)$ is a metric on the orbit space X/G. We will refer to it as the metric on X/G derived from the metric d on X. In case G is free and properly discontinuous, the quotient map $\eta : X \to X/G$ will be a local isometry in the following sense. As in the proof of Proposition 8, we can find an open ball $B_r(x)$ around x for some $r > 0$

such that $B_r(x) \cap gB_r(x) = \phi$ for any $g \neq e$ in G. Thus, for any $y \in B_r(x)$ and $g \neq e$ in G we have $gy \notin B_r(x)$. In particular, for y, y' in $B_{r/4}(x)$ we have $d(y, y') < r/2$ and $d(y, gy') \geq 3r/4$ for any $g \neq e$ in G. Hence, $D(\eta(y), \eta(y')) = \inf_{g \in G} d(y, gy') = d(y, y')$. It follows that $\eta \mid B_{r/4}(x) \to \eta(B_{r/4}(x))$ is an isometry and $\eta(B_{r/4}(x))$ is open in X/G.

3. Proper and strongly proper actions. In this section G will be a Hausdorff topological group acting as a topological transformation group on a Hausdorff space X. Unlike in [**6, 7** and **12**], we do not make any assumptions about local compactness either of X or of G. For any two subsets A, B of X, let $E(A, B) = \{g \in G \mid A \cap gB \neq \phi\}$. For any h_1, h_2 in G and subsets A, B of X, it is easy to see that $E(h_1 A, h_2 B) = h_1 E(A, B) h_2^{-1}$ and $E(A, B) = E(B, A)^{-1}$.

Lemma 1. *Let A and B be subsets of X with one of them compact and the other closed in X. Then $E(A, B)$ is closed in G.*

Proof. Since $E(A, B) = E(B, A)^{-1}$, for proving this lemma we may assume that A is closed and that B is compact. Let g_α be a net in $E(A, B)$ converging to g in G. We need to show that $g \in E(A, B)$. Since $g_\alpha \in E(A, B)$ there exist elements $b_\alpha \in B$ with $g_\alpha b_\alpha \in A$. Since B is compact, passing to a subset if necessary we may assume that b_α converges to $b \in B$. Then $g_\alpha b_\alpha$ converges to gb, and since A is closed, we see that $gb \in A$. Thus $g \in E(A, B)$. □

Definition 8. We say that G *acts properly* on X if for each $x \in X$ there exists an open set U in X with $x \in U$ and $E(U, U)$ relatively compact in G.

When X is locally compact, the above definition is equivalent to saying that G acts locally properly on X as defined on page 265 of [**7**]. When G is locally compact and X is completely regular the above is equivalent to saying that X is a Cartan G-space in the sense of [**12**]. When G is discrete, then G acts properly on X if and only if G acts properly discontinuously on X according to Definition 4. This is precisely the motivation for our Definition 8.

Remark 8. Let G act properly on X. Let $x \in X$ and y be any point on the orbit Gx, say $y = hx$ with $h \in G$. If U is an open set in X with $x \in U$ and $E(U,U)$ relatively compact in G, then hU is open in X, $y \in hU$ and $E(U,hU) = E(U,U)h^{-1}$ is relatively compact in G.

Definition 9. We say that G *acts strongly properly* on X if for any x, y in X we can find open sets U and V in X with $x \in U$, $y \in V$ and $E(U,V)$ relatively compact in G.

When G is discrete, then G acts strongly properly on X if and only if G acts strongly properly discontinuously on X according to Definition 5. This is precisely the motivation for Definition 9.

The following lemma generalizes Proposition 7.

Lemma 2. *Suppose G acts strongly properly on X. For any compact subset K of X, the set $E(K,K)$ is compact in G.*

Proof. Choose any fixed $x \in K$. For any $y \in K$ we can find open sets U^y, V^y in X with $x \in U^y$, $y \in V^y$ and $E(U^y, V^y)$ relatively compact. The compactness of K yields a finite number of elements $y_1, \dots, y_r$ in K with $\cup_{i=1}^r V^{y_i} \supset K$. If $U_x = \cap_{i=1}^r U^{y_i}$, then U_x is open in X, $x \in U_x$ and $E(U_x, K)$ is relatively compact (because $E(U_x, K) \subset \cup_{i=1}^r E(U^{y_i}, V^{y_i})$). Using the compactness of K we can find a finite number of elements $x_1, \dots, x_s$ with $\cup_{j=1}^s U_{x_j} \supset K$ and $E(U_{x_j}, K)$ relatively compact. It follows that $E(K,K) \subset \cup_{j=1}^s E(U_{x_j}, K)$ is relatively compact in G. From Lemma 1, $E(K,K)$ is closed in G. Hence, $E(K,K)$ is compact in G. □

Corollary 4. *Let G act strongly properly on X. Then for any two compact subsets K, L of X, $E(K,L)$ is compact in G.*

The proof is similar to that of Lemma 2. We can also deduce this by noting that $E(K,L)$ is a closed subset of $E(K \cup L, K \cup L)$.

Proposition 9. *Let G act properly on X. Then X/G is Hausdorff if and only if the G-action is strongly proper.*

Proof. Assume that X/G is Hausdorff. Let x and y be any two elements of X. Because of Remark 5, to show that the G-action is strongly proper we may assume that $Gx \cap Gy \neq \phi$. Then the proof is exactly the same as the first part of Proposition 6. Conversely, assume that G acts strongly properly on X. Let x and y be elements of X satisfying $Gx \cap Gy = \phi$. Hausdorffness of X together with strong properness of the action of G yield open sets S and T in X with $x \in S$, $y \in T$, $S \cap T = \phi$ and $E(S,T)$ relatively compact. Let $K = \overline{E(S,T)}$ in G. Then $g \in G$, $S \cap gT \neq \phi \Rightarrow g \in E(S,T) \subset K$. From $Gx \cap Gy = \phi$ we see that $x \neq gy$ for any $g \in G$. In particular, $x \neq ky$ for any $k \in K$. We can find open sets A_k, B_k in X, N_k in G satisfying $x \in A_k \subset S$, $y \in B_k \subset T$ and $A_k \cap N_k B_k = \phi$. The compactness of K yields a finite number of elements $k_1, \ldots, k_r$ in K with $\cup_{i=1}^r N_{k_i} \supset K$. Then $U = \cap_{i=1}^r A_{k_i}$, $V = \cap_{i=1}^r B_{k_i}$ are open sets in X satisfying $x \in U$, $y \in V$ and $U \cap gV = \phi$ for all $g \in G$. Hence $GU \cap GV = \phi$. This shows that X/G is Hausdorff. □

When G is discrete, Proposition 9 yields Proposition 6.

4. Discrete subgroups of isometry groups. In this section X will denote a locally compact metric space and $I(X)$ the group of isometries of X. For any two subsets A and B of X, let $\Gamma(A,B) = \{g \in I(X) \mid g(A) \subset B\}$. The compact open topology on $I(X)$ is that topology for which sets of the form $\Gamma(K,U)$ with K compact and U open in X form a subbase. The pointwise convergence topology on $I(X)$ is the one for which sets of the form $\Gamma(\{x\},U)$ with $x \in X$ and U open in X form a subbase. Since $\{x\}$ is compact, one sees that the compact open topology is finer than the topology of pointwise convergence.

Proposition 10. *On $I(X)$ the compact open topology and the topology of pointwise convergence agree.*

Proof. We need only to show that for any compact set K and any open set U in X, the set $\Gamma(K,U)$ is open in the topology of pointwise convergence. Let $h \in \Gamma(K,U)$. Then $h(K)$ is compact and $h(K) \subset U$. Hence there exists an $\varepsilon > 0$ such that $V_\varepsilon(h(K)) \subset U$ where $V_\varepsilon(h(K)) = \{x \in X \mid d(x,h(y)) < \varepsilon \text{ for some } y \in K\}$. Since

K is compact, we can find a finite number of elements $y_1, \dots, y_r$ in K satisfying the condition that given any $y \in K$ there exists some j in $1 \leq j \leq r$ with $d(y, y_j) < \varepsilon/2$. Clearly, $N = \cap_{j=1}^{r} \Gamma(\{y_j\}, B_{\varepsilon/2}(h(y_j)))$ is an open set of $I(X)$ in the topology of pointwise convergence with $h \in N$. Let g be any element of N. Then

$$\begin{aligned} d(g(y), h(y_j)) &\leq d(g(y), g(y_j)) + d(g(y_j), h(y_j)) \\ &\leq d(y, y_j) + d(g(y_j), h(y_j)) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for any $y \in K$. Since $y_j \in K$ we see that $g(y) \in V_\varepsilon(h(K))$. Thus $g \in \Gamma(K, U)$. This shows that $h \in N \subset \Gamma(K, U)$. Hence, $\Gamma(K, U)$ is open in the topology of pointwise convergence. $\square$

It is known that the map $(g, h) \mapsto gh$ is a continuous map $I(X) \times I(X) \mapsto I(X)$ in the compact open topology (this follows from [**3**, Theorem 2.2]).

Proposition 11. *The map $v : I(X) \to I(X)$ given by $v(g) = g^{-1}$ is continuous in the topology of pointwise convergence.*

Proof. Let $\Gamma(\{x\}, U)$ be any subbasic open set around $g_0^{-1}(x)$ where $g_0 \in I(X)$. This means U is open in X and $g_0^{-1}(x) \in U$. Choose an $\varepsilon > 0$ with $B_\varepsilon(g_0^{-1}(x)) \subset U$. We claim that $g_0 \in \Gamma(\{g_0^{-1}(x)\}, B_\varepsilon(x))$ and that $v(\Gamma(\{g_0^{-1}(x)\}, B_\varepsilon(x)) \subset \Gamma(\{x\}, U)$. Clearly, $g_0(g_0^{-1}(x)) = x \in B_\varepsilon(x)$. Also, if $h \in \Gamma(\{g_0^{-1}(x)\}, B_\varepsilon(x))$, then $d(h^{-1}x, g_0^{-1}(x)) = d(x, hg_0^{-1}(x)) = d(hg_0^{-1}(x), x) < \varepsilon$. Hence $h^{-1}x \in B_\varepsilon(g_0^{-1}x) \subset U$. This means $v(h) = h^{-1} \in \Gamma(\{x\}, U)$. This proves the continuity of v in the pointwise convergence topology. $\square$

It is straightforward to see that $I(X)$ is Hausdorff in the pointwise convergence topology. As an immediate consequence of Propositions 10 and 11 above and Theorem 5.3 in [**10**] or Theorem 2.4.2 of [**3**], we get the following

Theorem 2. *Let X be a locally compact metric space. Then $I(X)$ is a Hausdorff topological group in the compact open topology. The map $(g, x) \mapsto gx$ is a continuous map $I(X) \times X \to X$.*

The following results are proved in [**4**, Section 2, Chapter 4] for connected Riemannian manifolds. Their proofs are valid more generally. We include their proofs in their greater generality. Observe that a sequence $\{g_n\}_{n\geq 1}$ in $I(X)$ will converge to g in $I(X)$ if and only if $\lim_{n\to\infty} g_n(x) = g(x)$ in X for each $x \in X$.

Proposition 12. *Let X be a locally compact metric space. Let $\{g_n\}_{n\geq 1}$ be a sequence of elements in $I(X)$ converging pointwise at every point of $A \subset X$. Then $\{g_n\}_{n\geq 1}$ converges pointwise at all points of $\overline{A}$.*

Proof. Let $p \in \overline{A}$. Choose $r > 0$ such that the open ball $B_r(p) = \{x \in X \mid d(x,p) < r\}$ is relatively compact. Let $0 < \varepsilon < r$ and choose an $a \in A$ with $d(a,p) < \varepsilon/3$. Since $\{g_n(a)\}_{n\geq 1}$ converges in X, there exists an integer N such that $d(g_m(a), g_n(a)) < \varepsilon/3$ for all $m \geq N$, $n \geq N$. Hence

$$\begin{aligned}\delta(g_m(p), g_n(p)) &\leq d(g_m(p), g_m(a)) + d(g_m(a), g_n(a)) + d(g_n(a), g_n(p))\\ &\leq d(p,q) + d(g_m(a), g_n(a)) + d(a,p)\\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \quad \text{whenever } m \geq N, n \geq N.\end{aligned}$$

In particular, $g_n(p) \in B_\varepsilon(g_N(p))$ for $n \geq N$. Since $B_\varepsilon(g_N(p)) = g_N(B_\varepsilon(p)) \subset g_N(B_r(p))$, we see that $B_\varepsilon(g_N(p))$ is relatively compact. This means that we can choose a subsequence $\{g_{n_k}\}_{k\geq 1}$ of $\{g_n\}$ with $\lim_{k\to\infty} g_{n_k}(p)$ existing in X. Let $b = \lim_{k\to\infty} g_{n_k}(p)$. From (1) we have $d(g_m(p), g_{n_k}(p)) < \varepsilon$ whenever $m \geq N$ and $n_k \geq N$. Allowing k to tend to infinity, we get $d(g_m(p), b) \leq \varepsilon$ for all $m \geq N$. This implies that $\lim_{m\to\infty} g_m(p) = b$.

Proposition 13. *Let X be a connected locally compact separable metric space and $\{f_n\}_{n\geq 1}$ be a sequence of elements in $I(X)$. Suppose that there exists an element $x_0 \in X$ with $\{f_n(x_0)\}_{n\geq 1}$ convergent in X. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges pointwise at all $x \in X$.*

Proof. Let $S = \{q \in X \mid f_n(q) \mid n \geq 1\}$ is relatively compact. We will first show that S is closed in X. Let $\{p_i\}_{i\geq 1}$ be any sequence in S converging to $p \in X$. Let $E = \{f_n(p) \mid n \geq 1\}$. Let $\{e_\mu\}_{\mu\geq 1}$

be any sequence of elements in E. We will show that there exists a subsequence $\{e_{\mu_k}\}$ of $\{e_\mu\}$ with $\{e_{\mu_k}\}$ convergent in X. This will prove that E is relatively compact, hence $p \in S$. Since $p_i \in S$ the set $E_i = \{f_n(p_i) \mid n \geq 1\}$ is relatively compact in X. Since $e_\mu \in E$ we can write e_μ as $f_{n_\mu}(p)$ (though we cannot guarantee that n_μ will increase with μ). Let $e_\mu(i) = f_{n_\mu}(p_i)$. Then $\{e_\mu(i)\}_{\mu\geq 1}$ is a sequence in E_i. By induction on i, we can find sequences $\mu_1^{(i)} < \mu_2^{(i)} < \mu_3^{(i)} < \cdots$ with the following properties:

(a) $\{\mu_n^{(i+1)}\}_{n\geq 1}$ is a subsequence of $\{\mu_n^{(i)}\}_{n\geq 1}$ for each $i \geq 1$ and

(b) $\lim_{n\to\infty} e_{\mu_n^{(i)}}(i)$ exists in X.

Let $v_k = \mu_k^{(i)}$. Then $\lim_{k\to\infty} e_{v_k}(i)$ exists in X for each $i \geq 1$. Equivalently, $\lim_{k\to\infty} f_{n_{v_k}}(p_i)$ exists in X. Since p exists in the closure of $\{p_i \mid i \geq 1\}$, from Proposition 12 we see that $\lim_{k\to\infty} f_{n_{v_k}}(p)$ exists in X. This means that $\lim_{k\to\infty} e_{v_k}$ exists in X. Hence, $p \in S$, showing that S is closed. Next we will show that S is open in X. Let $q \in S$. Choose $r > 0$ such that $B_r(q) = \{x \in X \mid d(x,q) < r\}$ is relatively compact. Consider $U = B_{r/4}(q) = \{x \in X \mid d(x,q) < r/4\}$. We will show that $U \subset S$. Let $u \in U$; i.e., $d(u,q) < r/4$. Let $F = \{f_n(u) \mid n \geq 1\}$. We need to show that F is relatively compact. Let $\{\gamma_\mu\}_{\mu\geq 1}$ be any sequence in F. Then $\gamma_\mu = f_{n_\mu}(u)$ (we cannot assert that n_μ will increase with μ). If $\beta_\mu = f_{n_\mu}(q)$, then $\{\beta_\mu\}_{\mu\geq 1}$ is a sequence in the set $\{f_n(q) \mid n \geq 1\}$. Since $q \in S$, the set $\{f_n(q) \mid n \geq 1\}$ is relatively compact. Hence $\{\beta_\mu\}$ admits a subsequence $\{\beta_{\mu_k}\}_{k\geq 1}$ with $\lim_{k\to\infty} \beta_{\mu_k}$ existing in X. Let $b = \lim_{k\to\infty} \beta_{\mu_k} = \lim_{k\to\infty} f_{n_{\mu_k}}(q)$. We may, without loss of generality, assume $\beta_{\mu_k} \in B_{r/4}(b)$ for all $k \geq 1$; i.e., $d(f_{n_{\mu_k}}(q), b) < r/4$. Since $f_n \in I(X)$, from $d(u,q) < r/4$, we get $d(f_{n_{\mu_k}}(u), f_{n_{\mu_k}}(q)) < r/4$. It follows that $d(f_{n_\mu}(u), b) < r/2$ or $f_{n_{\mu_k}}(u) \in B_{r/2}(b)$ or $\gamma_{\mu_k} \in B_{r/2}(b)$. Since $B_r(q)$ is relatively compact, we see that $f_{n_{\mu_k}}(B_r(q))$ is relatively compact. We claim that $B_{r/2}(b) \subset f_{n_{\mu_k}}(B_r(q))$. Let $c \in B_{r/2}(b)$. Then $d(c, f_{n_{\mu_k}}(q)) = d(c,b) + d(b, f_{n_{\mu_k}}(q)) < r/2 + r/4$. Hence, $d(f_{n_{\mu_k}}^{-1}(c), q) < r$ or $f_{n_{\mu_k}}^{-1}(c) \in B_r(q)$. This yields $c \in f_{n_{\mu_k}}(B_r(q))$, proving the claim.

Since $f_{n_{\mu_k}}(B_r(q))$ is relatively compact, we see that $B_{r/2}(b)$ is relatively compact. From $\gamma_{\mu_k} \in B_{r/2}(b)$, we see that there exists a subsequence of $\{\gamma_\mu\}_{\mu\geq 1}$ which converges in X. This proves that F is relatively compact; hence, $U \subset S$. S is nonempty since $x_0 \in S$. The connectedness of X implies that $S = X$. This means for each $x \in X$, the set $\{f_n(x) \mid n \geq 1\}$ is relatively compact in X. In particular, there exists a sequence $n_1 < n_2 < n_3, \ldots$ *depending on* x with $\lim_{k\to\infty} f_{n_k}(x)$ existing in X.

Next we show that there exists a sequence $n_1 < n_2 < n_3 < \cdots$ independent of x with $\lim_{k\to\infty} f_{n_k}(x)$ existing in X for all $x \in X$. By assumption, there exists a countable dense set $A = \{a_j \mid j \geq 1\}$. Our earlier argument shows that $\{f_n(a_j) \mid n \geq 1\}$ is relatively compact for each $j \geq 1$. Using a diagonal process, we can get a sequence $n_1 < n_2 < n_3 < \cdots$ with the property that $\lim_{k\to\infty} f_{n_k}(a_j)$ exists for all $j \geq 1$. Now Proposition 12 shows that $\lim_{k\to\infty} f_{n_k}(x)$ exists for each $x \in X$. □

Proposition 14. *Let X be a locally compact, connected, separable metric space. Let $\{f_n\}_{n\geq 1}$ be a sequence in $I(X)$, converging pointwise to $f : X \to X$. Then $f \in I(X)$. Moreover, $\lim_{n\to\infty} f_n = f$ in the compact open topology.*

Proof. For any x, y in X, we have $d(f(x), f(y)) = \lim_{n\to\infty} d(f_n(x), f_n(y)) = d(x, y)$. We claim that $f(X) = X$. Let $a \in X$ and $b = f(a)$. Then $\lim_{n\to\infty} d(b, f_n(a)) = d(b, f(a)) = d(b, b) = 0$. Hence, $\lim_{n\to\infty} d(f_n^{-1}b, a) = 0$ or $\lim_{n\to\infty} f_n^{-1}(b) = a$. From Proposition 13, there exists a subsequence $\{f_{n_k}^{-1}\}$ of $\{f_n^{-1}\}$ such that $\lim_{k\to\infty} f_{n_k}^{-1}(x)$ exists for each $x \in X$.

Let $p \in X$ and $q = \lim_{k\to\infty} f_{n_k}^{-1}(p)$. Then

$$\begin{aligned} d(f(q), p) &= \lim_{n\to\infty} d(f_n(q), p) \\ &= \lim_{k\to\infty} d(f_{n_k}(q), p) \\ &= \lim_{k\to\infty} d(q, f_{n_k}^{-1}(p)) \\ &= d(q, q) = 0. \end{aligned}$$

Hence, $p = f(q)$. This shows that $f(X) = X$, hence $f \in I(X)$.

Since f_n converges to f pointwise, from Proposition 10 we see that f_n converges to f in the compact open topology. □

Proposition 15. *Let G be a Hausdorff topological group and H a discrete subgroup of G. Then H is closed in G.*

Proof. For any h in H, the set $\{h\}$ is closed in G and $H = \cup_{h\in H}\{h\}$. We will show that $\{h\}_{h\in H}$ is locally finite. Since H is discrete, there exists an open set U in G with $U \cap H = \{e\}$. Since G is a topological group there exists an open set V in G with $e \in V$ and $V^{-1}V \subset U$. For any $g \in G$, the set gV is an open set in G with $g \in gV$. We will show that $gV \cap H$ can have at most one element. Let $h_j \in gV \cap H$ with $j = 1, 2$. Then $h_j = gv_j$ with $v_j \in V$. Hence, $h_1^{-1}h_2 = v_1^{-1}g^{-1}gv_2 = v_1^{-1}v_2 \in V^{-1}V \subset U$. Thus, $h_1^{-1}h_2 \in H \cap U = \{e\}$ implying $h_1 = h_2$. This shows that the cardinality $|gV \cap H|$ of $gV \cap H$ is at most 1. Hence $\{h\}_{h\in H}$ is a locally finite family of closed sets. Hence $H = \cup_{h\in H}\{h\}$ is closed in G. □

Theorem 3. *Let X be a connected, locally compact, separable metric space. Let G be a discrete subgroup of $I(X)$ in the compact open topology. Then the action of G on X is strongly properly discontinuous.*

Proof. Because of Theorem 1, we have only to show that the action of G on X is discontinuous. Let $\{f_n\}_{n\geq 1}$ be an infinite sequence of distinct elements in G. Suppose, for some $x_0 \in X$, the sequence $\{f_n(x_0)\}_{n\geq 1}$ converges in X. Then Propositions 12 and 13 imply that there exists a subsequence $\{f_{n_k}\}_{k\geq 1}$ of $\{f_n\}_{n\geq 1}$ with f_{n_k} converging to some $f \in I(X)$ in the compact open topology. Since G is discrete in $I(X)$, from Proposition 15, we see that G is closed in $I(X)$. Hence, $f \in G$. For any neighborhood V of f in $I(X)$, $V \cap G$ will contain f_{n_k} for $k \geq a$ suitable k_0. Since the f_n's are distinct, this contradicts the assumption that G is discrete in $I(X)$. □

An immediate consequence of Theorem 3 is the following:

Corollary 5. *In Example* 3 *there is no metric on $\mathbf{R}^n$ which is invariant under the action of C.*

Remark 9. In Example 1, under the usual (Riemannian) metric on each copy of $\mathbf{R}^n$, $C \times C$ is a discrete subgroup of the group $I(M^n)$

of isometries of M^n. Here M^n is a locally compact separable metric space. Still the action of $C \times C$ on M^n is nowhere discontinuous. This example shows that the assumption that X be connected cannot be dispensed with in Theorem 3.

5. Some miscellaneous results. When X is a locally compact separable metric space, the isometry group $I(X)$ will be locally compact and second countable. If, further, X is connected, the isotropy group $I(X)_x$ at any $x \in X$ will be compact. Even though these results are not directly related to the topic of discrete groups and discontinuous actions, they follow from the techniques used earlier. We will include proofs of these results.

Proposition 16. *Let X be a locally compact, separable metric space. Then $I(X)$ is a Hausdorff topological group satisfying the second axiom of countability. If X is further connected, then $I(X)$ is locally compact.*

Proof. A separable metric space satisfies the second axiom of countability. Since X is also locally compact, there exists a countable base $\{U_n\}_{n\geq 1}$ for the topology of X with each $\overline{U}_n$ compact. Let $\mathcal{F}$ be the family of sets $\{\Gamma(\overline{U}_k, U_l)\}$ where k and l vary over integers ≥ 1, and let $\mathcal{B}$ be the family constituted by finite intersections of set from $\mathcal{F}$. We will show that $\mathcal{B}$ is a base for the compact open topology on $I(X)$. Let $f \in I(X)$ and $\Gamma(K, V)$ with K compact, V open in X be any subbasic open set containing f. Then $f(K) \subset V$. For any $x \in K$, we can find an integer $l(x) \geq 1$ with $f(x) \in U_{l(x)} \subset V$. Then $x \in f^{-1}(U_{l(x)})$ and $f^{-1}(U_{l(x)})$ is open. We can find an integer $k(x) \geq 1$ with $x \in U \subset \overline{U}_{k(x)} \subset f^{-1}(U_{l(x)})$. The compactness of K yields a finite number of elements $x_1, \dots, x_r$ with $\cup_{i=1}^r U_{k(x_i)} \supset K$. Writing k_i for $k(x_i)$ and l_i for $l(x_i)$, we see that $\cup_{i=1}^r U_{k_i} \supset K$, $f(\overline{U}_{k_i}) \subset \underline{U_{l_i}} \subset V$. It follows that $f \in \cap_{i=1}^r \Gamma(\overline{U}_{k_i}, U_{l_i}) \subset \Gamma(K, V)$. The set $\cap_{i=1}^r \Gamma(\overline{U}_{k_i}, U_{l_i})$ is in $\mathcal{B}$. This shows that $\mathcal{B}$ is a base for the compact open topology on $I(X)$.

Now assume that X is further connected.

Given any $f \in I(X)$, the set $\Gamma(\{x_0\}, V)$ is an open set containing f where x_0 is any chosen point in X and V any open set in X containing $f(x_0)$. We may choose V to satisfy the condition that $\overline{V}$

is compact in X. We claim that $\Gamma(\{x_0\}, V)$ is relatively compact in $I(X)$. Since $I(X)$ is second countable, to prove that $\Gamma(\{x_0\}, V)$ is relatively compact, we have only to show that any sequence $\{f_n\}_{n\geq 1}$ of elements in $\Gamma(\{x_0\}, V)$ will admit a subsequence converging in $I(X)$. Since $\overline{V}$ is compact, the sequence $\{f_n(x_0)\}_{n\geq 1}$ of elements in V admits a subsequence $\{f_{n_k}(x_0)\}_{k\geq 1}$ converging to some element in $\overline{V} \subset X$. Now Propositions 13 and 14 yield a subsequence of the sequence $\{f_{n_k}\}_{k\geq 1}$ converging to some $f \in I(X)$. □

Proposition 17. *Let X be a connected, locally compact, separable metric space. Then, for each $x \in X$, the isotropy group $I(X)_x$ is a compact subgroup of $I(X)$.*

Proof. Let V be any relatively compact open neighborhood of x in X. Then, from the proof of Proposition 16, we see that $\Gamma(\{x\}, V)$ is a relatively compact subset of $I(X)$. It is clear that $I(X)_x$ is a subset of $\Gamma(\{x\}, V)$ and is closed in $I(X)$. Hence, $I(X)_x$ is a closed subset of the compact set $\overline{\Gamma(\{x\}, V)}$ and as such is itself compact. □

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