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ALGEBRAIC PROPERTIES OF THE LIAPUNOV AND PERIOD CONSTANTS

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ABSTRACT. We give several algebraic properties of the Liapunov and period constants that simplify their effective computation. We apply them to get the first Liapunov and period constants and the second Liapunov constant for an arbitrary analytic system. Finally we apply them to some particular families of differential equations.

1. Introduction. Consider the differential equation:

$$\begin{aligned} \dot{x} &= -y + f(x,y), \\ \dot{y} &= x + g(x,y), \end{aligned} \tag{1}$$

where f and g are analytic functions in a neighborhood of $(0,0)$ and begin, at least, with second order terms. It is well known that the problem to determine if (1) has a center or a focus at the origin can be reduced to the study of the Poincaré return map, or equivalently to the computation of infinitely many real numbers, v_{2m+1}, $m \geq 1$, called the Liapunov constants. In fact, we have that if for some k, $v_3 = v_5 = \cdots = v_{2k-1} = 0$ and $v_{2k+1} \neq 0$ the origin is a focus, while if all v_{2m+1} are zero the origin is a center, see for instance [**1**].

A closely related problem is the following: Assume that (1) has a center, and consider the period of all its periodic orbits. When is this period independent of the orbit, or, in other words, when the center is isochronous? It turns out that the solution to this problem follows by computing infinitely many real numbers, T_{2m}, $m \geq 1$, called the *period constants* and by imposing that all of them vanish.

In [**5**] the authors give a survey of different ways to compute the Liapunov constants. From there it is clear that all the approaches involve a lot of computations. This implies that even with powerful

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computers the problem cannot be solved in the general case. The problem of determination of the period constants presents similar difficulties.

The aims of this paper are the following:

(1) First show that the Liapunov and period constants are polynomials in the coefficients with a rigid structure. In fact, we prove, by studying the effect of rotations and homoteties in (1), that such polynomials are quasi-homogeneous when we assign to each coefficient of f and g some special weights (see Theorem 1 of Section 3 for more details).

(2) Use the above properties and the study of special families of type (1) with centers to determine the coefficients of the polynomials (see Theorem 2 of Section 3).

Some quasi-homogeneity properties of the Liapunov constants were also established by other authors (see [**10, 13, 14**]).

By using the above approach, the results that we obtain, without using a computer, are:

Theorem A. *Let $\dot{z} = iz + F(z,\bar{z})$ be the complexified expression of* (1). *Denote by $F_i(z,\bar{z})$ the homogeneous part of $F(z,\bar{z})$ of degree i, and set $F_2(z,\bar{z}) = Az^2 + Bz\bar{z} + C\bar{z}^2$, $F_3(z,\bar{z}) = Dz^3 + Ez^2\bar{z} + Fz\bar{z}^2 + G\bar{z}^3$, $F_4(z,\bar{z}) = Hz^4 + Iz^3\bar{z} + Jz^2\bar{z}^2 + Kz\bar{z}^3 + L\bar{z}^4$, $F_5 = Mz^5 + Nz^4\bar{z} + Oz^3\bar{z}^2 + Pz^2\bar{z}^3 + Qz\bar{z}^4 + R\bar{z}^5$. Then*:

(i) $v_3 = 2\pi[\mathrm{Re}\,(E) - \mathrm{Im}\,(AB)]$.

(ii) $v_5 = (\pi/3)[6\mathrm{Re}\,(O) + \mathrm{Im}\,(3E^2 - 6DF + 6A\bar{I} - 12BI - 6B\bar{J} - 8CH - 2C\bar{K}) + \mathrm{Re}\,(-8C\bar{C}E + 4AC\bar{F} + 6A\bar{B}F + 6B\bar{C}F - 12B^2D - 4ACD - 6A\bar{B}\bar{D} + 10B\bar{C}\bar{D} + 4A\bar{C}G + 2BC\bar{G}) + \mathrm{Im}\,(6A\bar{B}^2C + 3A^2B^2 - 4A^2\bar{B}C + 4\bar{B}^3C)]$.

(iii) $T_2 = (2\pi/3)[-3\mathrm{Re}\,(AB) + 3B\bar{B} + 2C\bar{C} - 3\mathrm{Im}\,(E)]$.

The results (i) and (ii) have already been obtained by other authors using different methods; see, for instance, [**1, 5, 6**]. The result (iii) appears in [**8**] and in [**4**] (when $F(z,\bar{z}) = F_2(z,\bar{z})$.) We also want to comment that if we apply our method for the determination of v_7 it allows us to ensure that this Liapunov constant is a polynomial in the coefficients of the differential equation with less than 700 monomials. (This is not the sharpest bound.) On the other hand, if we only use the property that v_7 is a polynomial of degree 6, we get that it has more than $1.5 \cdot 10^8$ monomials.

When we apply our method to particular cases of equation (1) we can arrive further in the determination of the Liapunov and period constants. The results that we get are the following:

Theorem B. *Consider the Liénard equation*

$$\begin{cases} \dot{x} = -y + p(x) \\ \dot{y} = x \end{cases}$$

where $p(x) = a_2x^2 + a_3x^3 + \cdots$.

(i) *Assume that* $a_3 = a_5 = \cdots = a_{2m-1} = 0$ *and* $a_{2m+1} \neq 0$. *Then* $v_3 = v_5 = \cdots = v_{2m-1} = 0$ *and*

$$v_{2m+1} = \frac{(2m+1)!!}{(2m+2)!!} 2\pi a_{2m+1}.$$

(ii) *Assume that the origin is a center, that is,* $a_{2i+1} = 0$ *for* $i = 1, 2, 3, \ldots$. *Also assume that* $a_2 = a_4 = \cdots = a_{2m} = 0$ *and that* $a_{2m+2} \neq 0$. *Then* $T_2 = T_4 = \cdots = T_{4m} = 0$ *and*

$$T_{4m+2} = 2\pi \left(\frac{-2(4m+5)!!}{(2m+3)(4m+6)!!} + \frac{(4m+3)!!}{(4m+4)!!} \right) a_{2m+2}^2,$$

for $m = 1, 2, 3, \ldots$, *where* $n!! = n(n-2)\cdots 1$, (*respectively* $n!! = n(n-2)\cdots 2$) *when* n *is odd* (*respectively* n *even*).

Result (i) of the above theorem is already well known, see [**2, 16**]. As far as we know, the second one is a new result. At this point we want to stress that this result implies that all the period constants for the Liénard equation are nonnegative. This fact makes one wonder if the period of the periodic orbits of the Liénard equation increases with the distance to the origin. This is the situation when $p_2(x) = x^2$ as it is proved in [**3**]. Here we do not consider this problem.

Proposition C. *Consider the system*

$$\begin{cases} \dot{x} = -y + \alpha x^2 + \beta x^3 \\ \dot{y} = x + \gamma y^2 + \delta y^3 \end{cases}$$

where α, β, γ *and* δ *are real numbers. The following holds*:

(i) $v_3 = (\pi/4)(\beta + \delta)$, $v_5 = (\pi/24)(\alpha^2 - \gamma^2)(5\beta - 6\alpha\gamma)$, $v_7 = (3\pi/16)(\alpha^2 - \gamma^2)(\alpha^2 + \gamma^2)\alpha\gamma$ *and* $v_{2k+1} = 0$ *for all* $k \geq 4$. *Therefore the origin is a center if and only if* $\beta + \delta = 0$ *and either* $\alpha^2 - \gamma^2 = 0$ *or* $\beta = \alpha\gamma = 0$.

(ii) *Assume that the origin is a center. Then* $T_2 = (\pi/3)(\alpha^2 + \gamma^2)$, $T_4 = (3\pi/4)\beta^2$ *and* $T_{2k} = 0$ *for all* $k \geq 3$. *Therefore the origin is an isochronous center if and only if* $\alpha = \beta = \delta = \gamma = 0$.

The above family is studied in the last section. Obviously the computation of v_3, v_5 and T_2 follow from Theorem A. We will see in its proof that the computation of v_7 and T_4 can be reduced to the study of concrete systems. In fact, this is the main advantage of the method proposed in this paper, that is, to reduce the computation of the general Liapunov and period constants to the study of some concrete systems.

2. Definitions and preliminary results. Consider the equation (1) in complex coordinates, $z = x + iy = \mathrm{Re}\,(z) + i\mathrm{Im}\,(z)$, that is, $\dot{z} = iz + F(z, \bar{z})$.

By using the change of variables $r^2 = z\bar{z}$ and $\theta = \arctan(\mathrm{Im}\,(z)/\mathrm{Re}\,(z))$, equation (1) can be written as:

$$\dot{r} = \frac{1}{2r}[\dot{z}\bar{z} + z\dot{\bar{z}}], \qquad \dot{\theta} = \frac{1}{2ir^2}[\dot{z}\bar{z} - z\dot{\bar{z}}],$$

or, equivalently,

$$\text{(2)} \qquad \begin{aligned} \frac{dr}{d\theta} &= ir\frac{\dot{z}\bar{z} + z\dot{\bar{z}}}{\dot{z}\bar{z} - z\dot{\bar{z}}} = ir\frac{F\bar{z} + z\bar{F}}{2ir^2 + F\bar{z} - z\bar{F}} \\ &= \frac{(F\bar{z} + z\bar{F})/(2r)}{1 + (F\bar{z} - z\bar{F})/(2ir^2)}. \end{aligned}$$

Writing $F(z,\bar z) = \sum_{k\ge 2} F_k(z,\bar z) = \sum_{k\ge 2} r^k F_k(e^{i\theta}, e^{-i\theta})$, we obtain

$$\frac{F\bar z + z\bar F}{2r} = r^2 P_2(\theta) + r^3 P_3(\theta) + \cdots + r^n P_n(\theta) + \cdots,$$
$$\frac{F\bar z - z\bar F}{2ir^2} = rQ_2(\theta) + r^2 Q_3(\theta) + \cdots + r^{n-1} Q_n(\theta) + \cdots$$

where $P_k(\theta) = \operatorname{Re}(e^{-i\theta}F_k(e^{i\theta}, e^{-i\theta}))$ and $Q_k(\theta) = \operatorname{Im}(e^{-i\theta}F_k(e^{i\theta}, e^{-i\theta}))$. Notice that P_k and Q_k are homogeneous polynomials of degree $k+1$ in $e^{i\theta}$, $e^{-i\theta}$ and that P_k and Q_k have coefficients which are homogeneous polynomials of degree one in the coefficients of F and $\bar F$.

In a neighborhood of $r = 0$, equation (2) can be expressed as

$$\frac{dr}{d\theta} = \frac{r^2 P_2(\theta) + r^3 P_3(\theta) + \cdots}{1 + rQ_2(\theta) + r^2 Q_3(\theta) + \cdots} = r^2 R_2(\theta) + r^3 R_3(\theta) + r^4 R_4(\theta) + \cdots. \tag{3}$$

Consider the solution of (3) such that it takes the value ρ when $\theta = 0$ and call it $r(\theta,\rho)$. Then

$$r(\theta,\rho) = u_1(\theta)\rho + u_2(\theta)\rho^2 + \cdots \quad \text{with} \quad u_1(0) = 1$$

and

$$u_k(0) = 0 \quad \text{for} \quad k \ge 2.$$

Let $h(\rho) = r(2\pi,\rho)$ be the return function. Then it is clear that system (1) has a center at $(0,0)$ if and only if $h(\rho) \equiv \rho$, which is equivalent to $u_1(2\pi) = 1$ and $u_k(2\pi) = 0$ for $k \ge 2$. It has a focus at $(0,0)$ if and only if there exists k such that $h^k(0) \ne 0$ or, equivalently, if there exists k such that $u_k(2\pi) \ne 0$. It is well known that the first k with $u_k(2\pi) \ne 0$ (if it exists) is an odd number, [**1**, p. 243].

Assume that system (1) satisfies $u_k(2\pi) = 0$ for $k = 2, 3, \ldots, 2m$ and $u_{2m+1}(2\pi) \ne 0$. We define

$$v_{2m+1} = u_{2m+1}(2\pi),$$

and we call it the mth-*Liapunov constant.*

Now suppose that system (1) has a center at the origin. Then near it

$$(4)\qquad \begin{aligned} T(\rho) &= \int_0^{2\pi} \frac{d\theta}{1 + (F\bar{z} - z\bar{F})/(2i(r(\theta,\rho))^2)} \\ &= \int_0^{2\pi} \frac{d\theta}{1 + \sum_{k\geq 2}(r(\theta,\rho))^{k-1}Q_k(\theta)} \end{aligned}$$

measures the time used to give a turn around the origin from the point $(\rho, 0)$. We call $T(\rho)$ the *period function.*

We say that a center is *isochronous* if $T(\rho)$ is constant. By writing

$$(5)\qquad \frac{1}{1 + \sum_{k\geq 2}(r(\theta,\rho))^{k-1}Q_k(\theta)} = 1 + \sum_{k\geq 1} H_k(\theta)(r(\theta,\rho))^k$$

we have the following expression for (4):

$$(6)\qquad \begin{aligned} T(\rho) &= \int_0^{2\pi} \left(1 + \sum_{k\geq 1} H_k(\theta)(r(\theta,\rho))^k\right) d\theta \\ &= 2\pi + \int_0^{2\pi} \sum_{k\geq 1} t_k'(\theta)\rho^k \, d\theta \\ &= 2\pi + \sum_{k\geq 1} t_k(2\pi)\rho^k. \end{aligned}$$

It is clear that the center is isochronous if and only if $t_k(2\pi) = 0$ for $k \geq 1$. We will prove (see Corollary 10) that the first k with $t_k(2\pi) \neq 0$ (if it exists) is an even number. This is a well known fact (see [**4**]), but here we present a different proof.

Assume that system (1) has a center at the origin, that $t_k(2\pi) = 0$ for $k = 1, 2, \ldots, 2m-1$ and $t_{2m}(2\pi) \neq 0$. We define

$$T_{2m} = t_{2m}(2\pi),$$

and we call it the mth-*period constant.*

We begin with three preliminary lemmas. The first two follow easily from [**7**, p. 14]. The third one can be proved by direct computations.

Lemma 1. *Let $R_k(\theta)$ for $k \geq 2$ be defined by equation* (3). *Then*:

$$R_k = (-1)^k \mathrm{Det} \begin{pmatrix} P_2 & 1 & 0 & \cdots & 0 \\ P_3 & Q_2 & 1 & \cdots & 0 \\ P_4 & Q_3 & Q_2 & \cdots & \\ \vdots & \vdots & \vdots & & \vdots \\ P_{k-1} & Q_{k-2} & Q_{k-3} & \cdots & 1 \\ P_k & Q_{k-1} & Q_{k-2} & \cdots & Q_2 \end{pmatrix}.$$

Particularly, for each $k \geq 2$, R_k is a polynomial of degree $3(k-1)$ in $e^{i\theta}$, $e^{-i\theta}$, and has coefficients which are polynomials of degree $k-1$ in the coefficients of F and $\bar{F}$.

Lemma 2. *Let $H_k(\theta)$ be defined by equation* (5). *Then*:

$$H_k = (-1)^k \mathrm{Det} \begin{pmatrix} Q_2 & 1 & 0 & \cdots & 0 \\ Q_3 & Q_2 & 1 & \cdots & 0 \\ Q_4 & Q_3 & Q_2 & \cdots & \\ \vdots & \vdots & \vdots & & \vdots \\ Q_k & Q_{k-1} & Q_{k-2} & \cdots & 1 \\ Q_{k+1} & Q_k & Q_{k-1} & \cdots & Q_2 \end{pmatrix}.$$

Particularly, for each $k \geq 1$, $H_k(\theta)$ is a polynomial of degree $3k$ in $e^{i\theta}$, $e^{-i\theta}$, and has coefficients which are polynomials of degree k in the coefficients of F and $\bar{F}$.

Lemma 3. *For $n, m \in \mathbf{N}$, let $I_{n,m}(\theta)$ be defined by*

$$I_{n,m}(\theta) = \int_0^\theta \sin^n \psi \cos^m \psi \, d\psi.$$

Then $I_{n,m}(\theta) = p(\theta) + k\theta$ where $p(\theta)$ denotes a trigonometric polynomial. Here

$$k = \frac{I_{n,m}(\alpha + 2\pi) - I_{n,m}(\alpha)}{2\pi} = \frac{1}{2\pi} \int_\alpha^{\alpha+2\pi} \sin^n \psi \cos^m \psi \, d\psi$$

for any $\alpha \in \mathbf{R}$. Moreover, if n or m are odd, $k = 0$.

We denote by $\lambda_k^n(\theta)$ the coefficient of ρ^k in the development of $(\sum_{i\geq 1} u_i(\theta)\rho^i)^n$, i.e.,

$$\left(\sum_{i\geq 1} u_i(\theta)\rho^i\right)^n = \lambda_n^n(\theta)\rho^n + \lambda_{n+1}^n(\theta)\rho^{n+1} + \cdots = \sum_{k\geq n} \lambda_k^n(\theta)\rho^k.$$

By using the development of the power series, we have that:

$$(7) \quad \lambda_k^n(\theta) = \sum_{\substack{a_1+a_2+\cdots+a_{k-1}=n \\ a_1+2a_2+3a_3+\cdots+(k-1)a_{k-1}=k}} \binom{n}{a_1 a_2 \cdots a_{k-1}} u_2^{a_2}(\theta)u_3^{a_3}(\theta)\cdots u_{k-1}^{a_{k-1}}(\theta)$$

for $k > n$ and $\lambda_n^n(\theta) = u_1^n(\theta) = 1$ for all θ.

Lemma 4. *Let* $r(\theta,\rho) = \sum_{n=1}^{\infty} u_n(\theta)\rho^n$ *be the solution of* (3) *such that it takes the value* ρ *when* $\theta = 0$. *Then*:

(i) $u_k(\theta) = \int_0^\theta P(R_2(\psi),\ldots,R_k(\psi),u_2(\psi),\ldots,u_{k-1}(\psi))\,d\psi$, *where* P *is a polynomial.*

(ii) *Given* θ, $u_k(\theta)$ *is a polynomial of degree* $k-1$ *with variables the coefficients of* $F_2, F_3, \ldots, F_k$ *and their conjugates.*

Proof. Since $r(\theta,\rho)$ is the solution of (3) with $r(0,\rho)=\rho$, we have:

$$\sum_{k\geq 2} R_k(\theta)[r(\theta,\rho)]^k = \sum_{k\geq 2} u_k'(\theta)\rho^k.$$

By comparing the coefficient of ρ^k for each k, we have that

$$u_k'(\theta) = \sum_{m=2}^{k} R_m(\theta)\lambda_k^m(\theta).$$

The assertion (i) now follows from (7).

We prove (ii) inductively: if the degree of $u_i(\theta)$ is $i-1$ for each $i = 1,2,\ldots,k-1$, then, from the above formula, $u_k'(\theta)$ has degree $(m-1)+a_2+2a_3+\cdots+(k-2)a_{k-1} = k-1$, and the same is true for

$u_k(\theta)$. On the other hand, if $u_i(\theta)$ is a polynomial with variables the coefficients of $F_2, F_3, \dots, F_i$ for each $i = 1, 2, \dots, k-1$, then, from the above formula again, $u'_k(\theta)$ is a polynomial with variables the coefficients of $F_2, F_3, \dots, F_{k-1}$ plus the coefficients which arise from $R_k(\theta)$. From Lemma 1 we see that $R_k(\theta)$ depends on the coefficients of $F_2, F_3, \dots, F_k$. So $u'_k(\theta)$ is a polynomial with variables the coefficients of $F_2, F_3, \dots, F_k$, and the same is true for $u_k(\theta)$. □

Remark 5. If $v_1 = v_3 = \cdots = v_{2k-1} = 0$, by using inductively Lemma 4 we obtain that $u_{2k}(\theta)$ and $u'_{2k+1}(\theta)$ are trigonometric polynomials. Hence, from Lemma 3, v_{2k+1} can be computed as $v_{2k+1} = u_{2k+1}(\alpha + 2\pi) - u_{2k+1}(\alpha)$ for any $\alpha \in \mathbf{R}$.

Lemma 6. *Let $t_k(\theta)$ and $H_k(\theta)$ be defined by equations* (5) *and* (6). *Then*:

(i) $t_k(\theta) = \int_0^\theta P(H_2(\xi), \dots, H_k(\xi), u_2(\xi), \dots, u_k(\xi))\, d\xi$, *where P is a polynomial.*

(ii) *Given θ, $t_k(\theta)$ is a polynomial of degree k with variables the coefficients of $F_2, F_3, \dots, F_{k+1}$ and their conjugates.*

The proof of Lemma 6 is similar to the proof of Lemma 4. We only want to put out that the formula used for the proof is:

$$t'_k(\theta) = H_1(\theta) u_k(\theta) + \sum_{m=2}^{k} H_m(\theta) \lambda_k^m(\theta). \tag{8}$$

From Lemmas 4 and 6 we see that the Liapunov constants v_{2m+1} and the period constants T_{2m} of system (1) are polynomials with variables the coefficients of F_i for $i = 2, 3, \dots, 2m+1$ and their conjugates. We put $F(z, \bar{z}) = \sum_{k+l \geq 2} A_{kl} z^k \bar{z}^l$, and we will use the following notation

$$v_{2m+1} = v_{2m+1}(F) = v_{2m+1}(F_2, F_3, \dots, F_{2m+1}) = v_{2m+1}(A_{kl}, \bar{A}_{kl})$$
$$T_{2m} = T_{2m}(F) = T_{2m}(F_2, F_3, \dots, F_{2m+1}) = T_{2m}(A_{kl}, \bar{A}_{kl}).$$

Lemma 7. *Let v_{2m+1} and T_{2m} be the Liapunov constant and the period constant associated with system* (1). *Then they can be written* *as*

$$\sum_{j\geq 1} \alpha_j M_j + \bar{\alpha}_j \bar{M}_j \tag{9}$$

where M_j *are monomials with variables the coefficients of* $F_2, F_3, \dots, F_{2m+1}$ *and* $\alpha_j \in \mathbf{C}$ *or as*

$$\sum_{j\geq 1} a_j \mathrm{Re}\,(M_j) + b_j \mathrm{Im}\,(M_j) \tag{10}$$

where $a_j, b_j \in \mathbf{R}$.

Proof. From Lemmas 4 and 6 we know that v_{2m+1} and T_{2m} are polynomials with variables the coefficients of $F_2, F_3, \dots, F_{2m+1}$ and their conjugates. So we can write them as $\sum_{j\geq 1} \beta_j M_j$ where M_j are monomials. Since v_{2m+1} and T_{2m} are real numbers, we have that

$$\begin{aligned}\sum_{j\geq 1} \beta_j M_j &= \sum_{j\geq 1} [\mathrm{Re}\,(\beta_j) + i\mathrm{Im}\,(\beta_j)][\mathrm{Re}\,(M_j) + i\mathrm{Im}\,(M_j)] \\ &= \sum_{j\geq 1} \mathrm{Re}\,(\beta_j)\mathrm{Re}\,(M_j) - \mathrm{Im}\,(\beta_j)\mathrm{Im}\,(M_j).\end{aligned}$$

Calling $\mathrm{Re}\,(\beta_j) = a_j$ and $-\mathrm{Im}\,(\beta_j) = b_j$, we obtain (10). Now substituting $\mathrm{Re}\,(M_j)$ and $\mathrm{Im}\,(M_j)$ by $(M_j + \bar{M}_j)/2$ and $(M_j - \bar{M}_j)/(2i)$, respectively, we have that

$$\sum_{j\geq 1} \beta_j M_j = \sum_{j\geq 1} \frac{1}{2}\beta_j M_j + \frac{1}{2}\bar{\beta}_j \bar{M}_j,$$

and calling $\alpha_j = (1/2)\beta_j$, the result follows. □

Lemma 8. *Assume that system* (1) *has a center at the origin. Let* $T(\rho)$ *be the period function defined in* (4), *and let* $r(\theta, \rho)$ *be the solution of* (3) *such that it takes the value* ρ *when* $\theta = 0$. *Then*:

(i) $T(\rho) - 2\pi = O(\rho^k)$ *with* $k \geq 2$.

(ii) $T(\rho) = T(-r(\pi, \rho))$.

Proof. From (6) and (8) we see that

$$T(\rho) = 2\pi + \left(\int_0^{2\pi} H_1(\theta)\, d\theta \right) \rho + O(\rho^2).$$

So we have to prove that

$$T_1 = \int_0^{2\pi} H_1(\theta)\, d\theta = 0.$$

From Lemma 1 we know that $H_1(\theta) = -Q_2(\theta)$ where $Q_2(\theta) = \operatorname{Im}[e^{-i\theta} F_2(e^{i\theta}, e^{-i\theta})]$ and $F_2(z, \bar{z})$ is homogeneous of degree two. So $Q_2(\theta)$ is a linear combination of $\sin\theta$, $\cos\theta$, $\sin 3\theta$ and $\cos 3\theta$. Since $T_1 = -\int_0^{2\pi} Q_2(\psi)\, d\psi$, result (i) follows.

In order to prove (ii), let $T_{\theta_0}(x)$ be the period function for the solution of (3) with initial conditions $\theta = \theta_0$, $r = x$. We claim that $T(-x) = T_\pi(x)$.

Let $s(\psi, x)$ be the solution of (3) with initial conditions $\theta = \pi$, $r = x$. Then

$$T_\pi(x) = \int_\pi^{3\pi} \frac{d\psi}{1 + \sum_{k\geq 1} s^k(\psi, x) Q_{k+1}(\psi)}.$$

Considering the change of variables $\psi = \xi + \pi$, we obtain

$$T_\pi(x) = \int_0^{2\pi} \frac{d\xi}{1 + \sum_{k\geq 1} s^k(\xi + \pi, x) Q_{k+1}(\xi + \pi)}.$$

Since Q_{k+1} is a trigonometric homogeneous polynomial of degree $k+2$, $Q_{k+1}(\xi + \pi) = (-1)^k Q_{k+1}(\xi)$. On the other hand, from [**1**, p. 241], $s(\xi + \pi, x) = -r(\xi, -x)$ and so $s^k(\xi + \pi, x) = (-1)^k r^k(\xi, -x)$. Hence, $s^k(\xi + \pi, x) Q_{k+1}(\xi + \pi) = r^k(\xi, -x) Q_{k+1}(\xi)$, and therefore,

$$T_\pi(x) = \int_0^{2\pi} \frac{d\xi}{1 + \sum_{k\geq 1} r^k(\xi, -x) Q_{k+1}(\xi)} = T(-x).$$

So the claim is proved.

Now let x be such that $r(\pi, \rho) = x$. Since $(0, \rho)$ and (π, x) are in the same periodic orbit, it is clear that $T(\rho) = T_\pi(x)$, and consequently,

$$T(\rho) = T_\pi(x) = T(-x) = T(-r(\pi, \rho)). \qquad \square$$

Corollary 9. *For each* $m \in \mathbf{N}$,

$$t_{2m+1}(2\pi) = \frac{1}{2} \sum_{j=2}^{2m} (-1)^j \lambda_{2m+1}^j(\pi) t_j(2\pi). \tag{11}$$

Proof. The equality $\sum_{k\geq 2} t_k(2\pi)(-r(\pi,\rho))^k = \sum_{k\geq 2} t_k(2\pi)\rho^k$ obtained in the above lemma, let us find some recurrent formulas for the period constants. Writing

$$\Big[\sum_{i\geq 1} u_i(\pi)\rho^i\Big]^n = \sum_{k\geq n} \lambda_k^n(\pi)\rho^k$$

and by comparing the coefficients of ρ^k for all $k \geq 2$, we obtain

$$t_k(2\pi) = \sum_{j=2}^{k} (-1)^j \lambda_k^j(\pi) t_j(2\pi).$$

Since $\lambda_n^n(\pi) = 1$ for all n, taking $k = 2m+1$, we have that

$$t_{2m+1}(2\pi) = \lambda_{2m+1}^2(\pi)t_2(2\pi) - \lambda_{2m+1}^3(\pi)t_3(2\pi) + \cdots + \lambda_{2m+1}^{2m}(\pi)t_{2m}(2\pi) - t_{2m+1}(2\pi)$$

and the result follows. □

Corollary 10. *Let* $T(\rho) = 2\pi + \sum_{k\geq 2} t_k(2\pi)\rho^k$ *be the period function associated to system* (1). *Assume that* $t_2(2\pi) = 0$, $t_3(2\pi) = 0, \dots, t_{k-1}(2\pi) = 0$ *and* $t_k(2\pi) \neq 0$. *Then* k *is an even number and the period constant is* $t_k(2\pi) = T_k$.

Proof. Suppose that $k = 2m+1$. Since $t_j(2\pi) = 0$ for all $j = 2, 3, \dots, 2m$, from the above result we obtain $t_{2m+1}(2\pi) = 0$, which is a contradiction. □

3. Algebraic Properties. The algebraic properties of the Liapunov and period constants are established in Theorems 1 and 2. Their proofs will appear at the end of this section.

Theorem 1. *Let v_{2m+1} and T_{2m} be the m-Liapunov and the m-period constants of system* (1). *Then they satisfy the following properties*:

(i) $v_{2m+1}(\lambda^{1-k+l}A_{kl}, \lambda^{-(1-k+l)}\bar{A}_{kl}) = v_{2m+1}(A_{kl}, \bar{A}_{kl})$ *and* $T_{2m}(\lambda^{1-k+l}A_{kl}, \lambda^{-(1-k+l)}\bar{A}_{kl}) = T_{2m}(A_{kl}, \bar{A}_{kl})$ *for all* $\lambda \in \mathbf{C}$, $|\lambda| = 1$.

(ii) $v_{2m+1}(\lambda^{k+l-1}A_{kl}, \lambda^{k+l-1}\bar{A}_{kl}) = \lambda^{2m}v_{2m+1}(A_{kl}, \bar{A}_{kl})$ *and* $T_{2m}(\lambda^{k+l-1}A_{kl}, \lambda^{k+l-1}\bar{A}_{kl}) = \lambda^{2m}T_{2m}(A_{kl}, \bar{A}_{kl})$ for all $\lambda \in \mathbf{R}$.

(iii) $v_{2m+1}(F) = v_{2m+1}(F + (iz + F)H)$ *for all analytic functions* $H(z, \bar{z})$ *such that* $H(0,0) = 0$.

Remark. Using (i) and (ii) of Theorem 1 there is a very easy way to list which are the monomials that appear in v_{2m+1} and in T_{2m}. For $K \in \mathbf{R}$, let $M = K(\prod_{i=1}^{r} A_{k_i l_i}^{m_i})(\prod_{i=r+1}^{r+s} \bar{A}_{k_i l_i}^{m_i})$ be a monomial of v_{2m+1}, respectively of T_{2m}. Then property (i) implies

$$(*) \qquad \sum_{i=1}^{r+s} m_i \varepsilon_i (1 - k_i + l_i) = 0,$$

and property (ii) implies

$$(**) \qquad \sum_{i=1}^{r+s} m_i (k_i + l_i - 1) = 2m,$$

where ε_i is 1, respectively -1, if $1 \le i \le r$, respectively $r+1 \le i \le r+s$. The monomials satisfying $(*)$ will be called monomials of weight zero.

Let $M = (\prod_{i=1}^{r} A_{k_i l_i}^{m_i})(\prod_{i=r+1}^{r+s} \bar{A}_{k_i l_i}^{m_i})$ be a monomial of variables the coefficients of F and their conjugates. We denote by $\operatorname{Unc}(M)$ the monomial

$$\operatorname{Unc}(M) = \left(\prod_{i=1}^{r} A_{k_i l_i}^{m_i}\right)\left(\prod_{i=r+1}^{r+s} A_{k_i l_i}^{m_i}\right),$$

and we call it the *unconjugate* of M.

Given two monomials M and N of the same degree, we say that M is *equivalent* to N if and only if $\operatorname{Unc}(M) = \operatorname{Unc}(N)$. It is obvious that the above relation is an equivalence relation. The equivalence classes will be called the unconjugacy classes.

Using special types of centers for system (1), reversible, holomorphic and Hamiltonian (see the next section for definitions), we obtain, respectively, properties (i), (ii) and (iii) of the next theorem.

Theorem 2. *For* $m \geq 1$, *let* v_{2m+1} *be the Liapunov constants of system* (1). *Then they satisfy the following properties*:

(i) *Set* $v_{2m+1} = R_1 + R_2 + \cdots + R_g$ *where each* R_j *is the sum of the monomials in the same unconjugacy class. Set* $R_j = \sum_{i=1}^{s_j} a_{ij}\mathrm{Re}\,(M_{ij}) + b_{ij}\mathrm{Im}\,(M_{ij})$ *and* $\varepsilon_{ij} = (-1)^{p_{ij}}$ *where* p_{ij} *is the number of variables of the monomial* M_{ij} *that are conjugate. Then when the degree of* R_j *is an even, respectively odd, number,* $\sum_{i=1}^{s_j} a_{ij}\varepsilon_{ij} = 0$, *respectively,* $\sum_{i=1}^{s_j} b_{ij}\varepsilon_{ij} = 0$.

(ii) *There are no monomials in* v_{2m+1} *that depend only on* $A_{k_1,0}$ *and* $\bar{A}_{k_2,0}$ *for* $0 \leq k_i \leq 2m+1$, $i = 1, 2$.

(iii) *If* F *is such that* $\mathrm{Re}\,(\partial F/\partial z) \equiv 0$, *then* $v_{2m+1}(F) = 0$.

Remark. Assume that M is a monomial of v_{2m+1} such that M and $\bar{M}$ are the only elements in its unconjugacy class. Then Theorem 2(i) says in particular that in the expression of v_{2m+1} only $\mathrm{Im}\,(M)$ appears, respectively $\mathrm{Re}\,(M)$, if the degree of M is even, respectively odd.

Remark. If, instead of system (1), we consider the more general system $\dot{x} = ax - by + f(x,y)$, $\dot{y} = bx + ay + g(x,y)$, with $b \neq 0$, that in complex coordinates is $\dot{z} = \alpha z + F(z,\bar{z})$, where $\alpha = a + bi$, its first Liapunov constant is $v_1 = e^{2\pi a/b} - 1$. Then, for $m \geq 1$, v_{2m+1} are defined if $a = 0$. Hence, it is clear that their Liapunov constants can be calculated using the Liapunov constants of system $\dot{z} = iz + F(z,\bar{z})/b$, that is, as $v_{2m+1}(F/b)$.

To prove Theorems 1 and 2, we need the following propositions.

Proposition 11. *The Liapunov constants* v_k *and the period constants* T_k *do not vary if we consider instead of system* (1) *the new system obtained by a rotation of* (1) *with* $w = e^{i\alpha}z$, $a \in \mathbf{R}$.

Proof. Let $r(\theta,\rho) = \rho + \sum_{i\geq 2} u_i(\theta)\rho^i$, respectively $\tilde{r}(\theta,\rho) = \rho + \sum_{i\geq 2} \tilde{u}_i(\theta)\rho^i$, be the solution of $dr/d\theta = \sum_{i\geq 2} R_k(\theta)r^k$ for system (1), respectively for the system obtained from (1) by a rotation through an angle α.

This rotation can be interpreted in the following way: consider the function, $\tilde{r}(\theta,\cdot)$, defined on the line $z = re^{i\alpha}$; then, for each $\theta \in [0,2\pi]$ we have $r(\alpha+\theta,\rho) = \tilde{r}(\theta, r(\alpha,\rho))$.

Taking $\theta = 2\pi$ we have the following equality:

$$\rho + \sum_{i\geq 2} u_i(\alpha+2\pi)\rho^i = r(\alpha,\rho) + \sum_{i\geq 2} \tilde{u}_i(2\pi)(r(\alpha,\rho))^i.$$

Writing $r(\alpha,\rho) = \rho + \sum_{i\geq 2} u_i(\alpha)\rho^i$, we obtain

$$\begin{aligned}\sum_{i\geq 2}&[u_i(\alpha+2\pi) - u_i(\alpha)]\rho^i \\ &= \sum_{i\geq 2} \tilde{u}_i(2\pi)[\rho + u_2(\alpha)\rho^2 + \cdots + u_n(\alpha)\rho^n + \cdots]^i.\end{aligned} \tag{12}$$

Now assume that $v_{2m+1} \neq 0$ is the first Liapunov constant for system (1) different from 0. Remark 5 implies that $u_i(\alpha+2\pi) - u_i(\alpha) = 0$ for $i = 2,\dots,2m$. Taking into account equality (12), we can deduce inductively that $\tilde{u}_i(2\pi) = 0$ for $i = 2,\dots,2m$ and that $v_{2m+1} = u_{2m+1}(\alpha+2\pi) - u_{2m+1}(\alpha) = \tilde{u}_{2m+1}(2\pi)$.

On the other hand, let $\tilde{T}(\rho)$ be the period function of system (1) after doing a rotation through an angle α. From Lemmas 2 and 3, it is easy to see that $\tilde{T}(r(\alpha,\rho)) = T(\rho)$ which implies that

$$\sum_{k\geq 2} \tilde{t}_k(2\pi)[r(\alpha,\rho)]^k = \sum_{k\geq 2} t_k(2\pi)\rho^k.$$

Now assume that $t_k(2\pi) = 0$ for $k = 2,3,\dots,m-1$ and $t_m(2\pi) \neq 0$. By comparing the coefficients of ρ^k we deduce that $\tilde{t}_k(2\pi) = 0$ for $k = 2,3,\dots,m-1$ and $\tilde{t}_m(2\pi) = t_m(2\pi)$. So the two period constants coincide. □

Proposition 12. *The Liapunov constants v_{2m+1} and the period constants T_{2m} are quasi-homogeneous polynomials of degree $2m$ with* *weights $k-1$ for F_k and $\bar{F}_k$, that is, for each $\lambda \neq 0$, $\lambda \in R$, we have*

$$v_{2m+1}(\lambda F_2, \lambda^2 F_3, \dots, \lambda^{2m} F_{2m+1}) = \lambda^{2m} v_{2m+1}(F_2, F_3, \dots, F_{2m+1}),$$
$$T_{2m}(\lambda F_2, \lambda^2 F_3, \dots, \lambda^{2m} F_{2m+1}) = \lambda^{2m} T_{2m}(F_2, F_3, \dots, F_{2m+1}).$$

Proof. Consider equation (1), $\dot{z} = iz + F(z, \bar{z})$ and do the change $z = \lambda\omega$. Then we obtain

$$\dot{\omega} = i\omega + G(\omega, \bar{\omega}) \quad \text{where} \quad \lambda G(\omega, \bar{\omega}) = F(z, \bar{z}). \tag{13}$$

Let P_n, Q_n and R_n, respectively $\tilde{P}_n, \tilde{Q}_n$ and $\tilde{R}_n$, be defined as before for system (1), respectively for system (5). A simple computation gives $\tilde{P}_n(\theta) = \lambda^{n-1} P_n(\theta)$, $\tilde{Q}_n(\theta) = \lambda^{n-1} Q_n(\theta)$ and $\tilde{R}_n(\theta) = \lambda^{n-1} R_n(\theta)$. So system (13) is obtained from system (1) by taking the coefficients $\lambda F_2, \lambda^2 F_3, \dots, \lambda^{2n} F_{2n+1}, \dots$.

Let $r(\theta, \rho)$, respectively $\tilde{r}(\alpha, \rho)$, be the solution of $dr/d\theta = \sum_{k \geq 2} r^k R_k(\theta)$, respectively $d\tilde{r}/d\alpha = \sum_{k\geq 2} \tilde{r}^k \tilde{R}_k(\alpha)$, where $z = re^{i\theta}$, respectively $\omega = \tilde{r}e^{i\alpha}$, such that $r(0, \rho) \equiv \rho$, respectively $\tilde{r}(0, \rho) \equiv \rho$. It is clear that the change $z = \lambda\omega$ gives $r = \lambda\tilde{r}$ and $\alpha = \theta$. We claim that $\lambda\tilde{r}(\theta, \rho/\lambda)$ is the solution of $dr/d\theta = \sum_{k\geq 2} r^k R_k(\theta)$ such that it takes the value ρ when $\theta = 0$:

$$\begin{aligned}\frac{d}{d\theta}\left[\lambda\tilde{r}\left(\theta, \frac{\rho}{\lambda}\right)\right] &= \lambda \sum_{k\geq 2} \tilde{r}^k \tilde{R}_k(\theta) \\ &= \lambda \sum_{k\geq 2} \tilde{r}^k \lambda^{k-1} R_k(\theta) \\ &= \sum_{k\geq 2} (\lambda\tilde{r})^k R_k(\theta).\end{aligned}$$

On the other hand, for $\theta = 0$, $\lambda\tilde{r}(0, \rho/\lambda) = \lambda(\rho/\lambda) = \rho$. So the claim is proved. From the uniqueness of solutions we obtain that $\lambda\tilde{r}(\theta, \rho/\lambda) = r(\theta, \rho)$.

Writing $r(\theta, \rho) = \sum_{k=1}^{\infty} u_k(\theta)\rho^k$, respectively $\tilde{r}(\theta, \rho) = \sum_{k=1}^{\infty} \tilde{u}_k(\theta)\rho^k$, we have that $\lambda \sum_{k=1}^{\infty} \tilde{u}_k(\theta)(\rho/\lambda)^k = \sum_{k=1}^{\infty} u_k(\theta)\rho^k$ and consequently

$\tilde{u}_k(\theta) = \lambda^{k-1} u_k(\theta)$. Applying the definition of the Liapunov constants, we obtain:

$$\begin{aligned} v_{2m+1}(\lambda F_2, \lambda^2 F_3, \dots, \lambda^{2m} F_{2m+1}) &= \tilde{u}_{2m+1}(2\pi) = \lambda^{2m} u_{2m+1}(2\pi) \\ &= \lambda^{2m} v_{2m+1}(F_2, F_3, \dots, F_{2m+1}). \end{aligned}$$

Set $D_m^k = \{\mathbf{a} = (a_1, a_2, \dots, a_{k-1}) \in \mathbf{N}^{k-1} : a_1 + a_2 + \dots + a_{k-1} = m,$ $a_1 + 2a_2 + \dots + (k-1)a_{k-1} = k\}$. Then, by using (7–8), we have that:

$$\begin{aligned} \tilde{t}_k(2\pi) &= \int_0^{2\pi} \tilde{H}_1(\theta)\tilde{u}_k(\theta) + \sum_{m=2}^{k} \tilde{H}_m(\theta) \\ &\quad \left[\sum_{\mathbf{a} \in D_m^k} \binom{m}{a_1 a_2 \cdots a_{k-1}} \tilde{u}_2^{a_2}(\theta)\tilde{u}_3^{a_3}(\theta) \cdots \tilde{u}_{k-1}^{a_{k-1}}(\theta) \right] d\theta \\ &= \int_0^{2\pi} \lambda^k H_1(\theta) u_k(\theta) \\ &\quad + \sum_{m=2}^{k} \lambda^m H_m(\theta) \left[\sum_{\mathbf{a} \in D_m^k} \binom{m}{a_1 a_2 \cdots a_{k-1}} \right. \\ &\qquad \left. \lambda^{a_2 + 2a_3 + \dots + (k-2)a_{k-1}} u_2^{a_2}(\theta) u_3^{a_3}(\theta) \cdots u_{k-1}^{a_{k-1}}(\theta) \right] d\theta \\ &= \lambda^k t_k(2\pi). \end{aligned}$$

So $\tilde{T}_k = \lambda^k T_k$, and the result follows. □

Proof of Theorem 1. (i) Consider equation (1) and do a rotation of angle α, $w = e^{i\alpha} z$. This new equation is $\dot{w} = iw + G(w, \bar{w})$ where $G(w, \bar{w}) = e^{i\alpha} F(e^{-i\alpha} w, e^{i\alpha} \bar{w})$, i.e.,

$$\begin{aligned} G(w, \bar{w}) &= e^{i\alpha} \sum A_{kl} (e^{-i\alpha} w)^k (e^{i\alpha} \bar{w})^l \\ &= \sum A_{kl} e^{i(1-k+l)\alpha} w^k \bar{w}^l. \end{aligned}$$

From Proposition 11 the Liapunov and the period constants are invariant under the change. So (i) of Theorem 1 is proved.

(ii) Let $M = K(\prod_{i=1}^{r} A_{k_i l_i}^{m_i})(\prod_{i=r+1}^{r+s} \bar{A}_{k_i l_i}^{m_i})$ be a monomial of v_{2m+1} or of T_{2m}, and recall that A_{kl} is the coefficient of $z^k \bar{z}^l$ in $F(z, \bar{z})$. From

Proposition 12, we know that:

$$\left(\prod_{i=1}^{r}(\lambda^{k_i+l_i-1}A_{k_il_i})^{m_i}\right)\left(\prod_{i=r+1}^{r+s}(\lambda^{k_i+l_i-1}\bar{A}_{k_il_i})^{m_i}\right) = \lambda^{2m}\left(\prod_{i=1}^{r}A_{k_il_i}^{m_i}\right)\left(\prod_{i=r+1}^{r+s}\bar{A}_{k_il_i}^{m_i}\right).$$

Hence, (ii) of Theorem 1 is proved.

(iii) Consider an analytic function such that $H(z,\bar{z}) \in \mathbf{R}$ for each $z \in \mathbf{C}$ and $H(0,0) = 0$. Near $(0,0)$ the orbits of (1) and $\dot{z} = [iz + F(z,\bar{z})][1 + H(z,\bar{z})]$ are the same. Furthermore, both equations have associated the same equation (2) in polar coordinates. Hence, (iii) of Theorem 1 follows. □

As it has been mentioned in Section 1, the knowledge of special cases of systems of type (1) with a center can be used to obtain properties of the Liapunov constants.

Definitions. a) Following the notation of [**14**], if, for some real α, system (1) is invariant under the change of variables $w = e^{i\alpha}\bar{z}$, $t' = -t$, we will say that it is *reversible*.

b) If the components of system (1) satisfy the Cauchy-Riemann equations, we will say that it is *holomorphic*.

c) If the divergence of the vector field associated with system (1) is zero, we will say that it is *Hamiltonian*.

Proposition 13. a) *System* (1) *is reversible if and only if, for some* α, $A_{kl} = -\bar{A}_{kl}e^{i(1-k+l)\alpha}$ *for all* k *and* l.

b) *System* (1) *is holomorphic if and only if* $F(z,\bar{z}) \equiv F(z)$.

c) *System* (1) *is Hamiltonian if and only if* $\mathrm{Re}\,(\partial F/\partial z) \equiv 0$.

Furthermore, in the three cases the origin of system (1) *is a center.*

Proof. a)

$$\begin{aligned}\frac{dw}{dt'} &= -e^{i\alpha}\frac{d\bar{z}}{dt} \\ &= -e^{i\alpha}(-i\bar{z} + \bar{F}(z,\bar{z})) \\ &= iw - \sum \bar{A}_{kl} e^{i(1-k+l)\alpha} w^k \bar{w}^l.\end{aligned}$$

So system (1) is invariant if and only if $A_{kl} = -\bar{A}_{kl} e^{i(1-k+l)\alpha}$. Let $x(t)$ and $y(t)$ be a solution of (1) such that, for $t = 0$, the point $(x(0), y(0))$ lies on the line $z = re^{(\alpha/2)i}$. Then, for r small enough, there exists t_0 such that $(x(t_0), y(t_0))$ also lies on $z = re^{(\alpha/2)i}$. It is easy to prove that the symmetric curve of $(x(t), y(t))$ with respect to the line $z = re^{(\alpha/2)i}$, when $t \in [0, t_0]$ is also a solution of (1). Hence a) follows.

b) Cauchy-Riemann equations in complex coordinates are $\partial F(z,\bar{z})/\partial \bar{z} = 0$, so $F(z,\bar{z}) \equiv F(z)$. In order to prove that the origin is a center, write, for $z \neq 0$,

$$\frac{i}{iz + F(z)} = \frac{1}{z} + \frac{-F(z)}{z(iz + F(z))}$$

and observe that $-F(z)/[z(iz + F(z))]$ is a holomorphic function in a neighborhood of $z = 0$. Hence, we can consider the function $H(z) = \ln|z| + \mathrm{Re}\,(S(z))$ with $S'(z) = -F(z)/[z(iz + F(z))]$. Let $z(t)$ be a solution of $\dot{z} = iz + F(z)$. Then

$$\frac{dH(z(t))}{dt} = \mathrm{Re}\left[\left(\frac{1}{z} + \frac{-F(z)}{z(iz + F(z))}\right)(iz + F(z))\right] = \mathrm{Re}\,(i) = 0.$$

So the solutions of $\dot{z} = iz + F(z)$ are contained in the curves $\ln|z(t)| + \mathrm{Re}\,(S(z)) = k$. Since the system is analytic, the origin is either a center or a focus. Assume that the origin is a focus. Then there exists a path $z(t)$ that tends to $z = 0$. Since, on the path, $\ln|z(t)| = k - \mathrm{Re}\,(S(z(t)))$, and $S(z)$ is a continuous function near the origin, we have a contradiction. So the origin must be a center, and b) follows.

c) Direct computations give that the divergence of the vector fields associated to (1) is $2\mathrm{Re}\,(\partial F/\partial z)$. Furthermore, it is well known that, for planar Hamiltonian systems, the only critical points with positive index are centers. Hence, c) is proved. □

Lemma 14. *Let M be a monomial of weight zero in variables the coefficients of system* (1). *Let d be the degree of M. If d is an even,* *respectively odd, number, then the imaginary, respectively real, part of M, restricted to reversible systems, is zero.*

Proof. Set $M = (\prod_{i=1}^{r} A_{k_i l_i}^{m_i})(\prod_{i=r+1}^{r+s} \bar{A}_{k_i l_i}^{m_i})$. From Proposition 13 we know that a system is reversible if and only if $A_{kl} = -\bar{A}_{kl} e^{i(1-k+l)\alpha}$. Hence,

$$M|_{A_{kl}=-\bar{A}_{kl}e^{i(1-k+l)\alpha}} = (-1)^r e^{i\sum_{i=1}^{r} m_i(1-k_i+l_i)\alpha} \left(\prod_{i=1}^{r+s} \bar{A}_{k_i l_i}^{m_i} \right),$$

and

$$\bar{M}|_{A_{kl}=-\bar{A}_{kl}e^{i(1-k+l)\alpha}} = (-1)^s e^{i\sum_{i=1}^{s} m_i(1-k_i+l_i)\alpha} \left(\prod_{i=1}^{r} \bar{A}_{k_i l_i}^{m_i} \right).$$

Since M has weight zero, we have that $\sum_{i=1}^{r} m_i(1-k_i+l_i) = \sum_{i=1}^{s} m_i(1-k_i+l_i)$. So, according to $r+s$ is even or odd, the lemma follows. □

Proof of Theorem 2. (i) Set $v_{2m+1} = R_1 + R_2 + \cdots + R_g$ where each R_j is the sum of the monomials in the same unconjugacy class. Since $v_{2m+1} = \sum_{j\geq 1} \alpha_j M_j + \bar{\alpha}_j \bar{M}_j$ and M_j and $\bar{M}_j$ are in the same unconjugacy class, it is clear that each R_j is real and so it can be written as $R_j = \sum_{i=1}^{t_j} a_{ij} \mathrm{Re}\,(M_{ij}) + b_{ij} \mathrm{Im}\,(M_{ij})$ with $a_{ij}, b_{ij} \in \mathbf{R}$ for all $i = 1, 2, \ldots, s_j$ and for all $j = 1, 2, \ldots, g$, and also as $R_j = \sum_{i=1}^{t_j} (\alpha_{ij} M_{ij} + \bar{\alpha}_{ij} \bar{M}_{ij})$ with $\alpha_{ij} \in \mathbf{C}$. Since all the monomials of R_j are in the same unconjugacy class, we define $\mathrm{Unc}\,(R_j) = \mathrm{Unc}\,(M_{ij})$ for all $i = 1, 2, \ldots, t_j$. Notice that $\mathrm{Unc}\,(R_j)$ for $j = 1, 2, \ldots, g$ are linearly independent.

Now consider a reversible center with $\alpha = 0$. Then, if M is a monomial which appears in v_{2m+1}, then from Proposition 13, $\bar{A}_{kl} = -A_{kl}$ and so $M = (-1)^s \mathrm{Unc}\,(M)$. Set $M_{ij} = (\prod_{i=1}^{r_j} A_{k_i l_i}^{m_i})(\prod_{i=r_j+1}^{r_j+s_j} \bar{A}_{k_i l_i}^{m_i})$. Then

$$R_j = \sum_{i=1}^{t_j} \alpha_{ij} M_{ij} + \bar{\alpha}_{ij} \bar{M}_{ij}$$

$$
\begin{aligned}
&= \sum_{i=1}^{t_j} \alpha_{ij}(-1)^{s_j}\,\mathrm{Unc}\,(M_{ij}) + \bar{\alpha}_{ij}(-1)^{r_j}\,\mathrm{Unc}\,(\bar{M}_{ij}) \\
&= \sum_{i=1}^{t_j} [(-1)^{s_j}\alpha_{ij} + (-1)^{r_j}\bar{\alpha}_{ij}]\mathrm{Unc}\,(R_j) \\
&= \gamma_j\,\mathrm{Unc}\,(R_j),
\end{aligned}
$$

and consequently, $v_{2m+1} = \sum_{j=1}^{g} \gamma_j\,\mathrm{Unc}\,(R_j)$. Since all the Liapunov constants are zero on the centers and $\mathrm{Unc}\,(R_j)$ for $j = 1, 2, \dots, g$ are linearly independent, we get $\gamma_j = 0$ for all $j = 1, 2, \dots, g$.

Assume that the degree of R_j is an even, respectively odd, number. From Lemma 14 we know that $\mathrm{Im}\,(M_{ij}) = 0$, respectively $\mathrm{Re}\,(M_{ij}) = 0$, for all $i = 1, 2, \dots, t_j$, and so

$$
\gamma_j = \sum_{i=1}^{t_j} a_{ij}\varepsilon_{ij}, \quad \text{respectively } \gamma_j = \sum_{i=1}^{t_j} b_{ij}\varepsilon_{ij}
$$

where $\varepsilon_{ij} = (-1)^{s_j}$. Hence, (i) is proved.

(ii) Consider the system $\dot{z} = iz + A_{20}z^2 + A_{30}z^3 + \cdots + A_{n0}z^n + \cdots$. From Proposition 13 b), it has a center at the origin. Therefore, we have that $v_{2m+1} \equiv 0$ restricted to the above systems. Then (ii) follows.

(iii) It is clear from Proposition 13. □

4. Proofs of the main results. Theorems A and B and Proposition C are applications of the above results to the effective computation of Liapunov constants. We also use the following well-known result about quadratic systems.

Theorem 15 (Dulac, Kaptein, Zoladek, see [**14**]). *The point* $z = 0$ *is a center for* $\dot{z} = iz + Az^2 + Bz\bar{z} + C\bar{z}^2$ *if and only if one of the following conditions is satisfied*:

(i) $B = 0$.

(ii) $2A + \bar{B} = 0$.

(iii) $\mathrm{Im}\,(AB) = \mathrm{Im}\,(A^3C) = \mathrm{Im}\,(\bar{B}^3C) = 0$.

(iv) $A - 2\bar{B} = |C| - |B| = 0$.

Proof of Theorem A. We know that v_3 has degree two on the coefficients of F_2, F_3 and their conjugates. The linear part of v_3 is composed of monomials arising from F_d, with $d = k + l$ satisfying $k + l - 1 = 2$ (see $(**)$ after Theorem 1) i.e., $d = 3$. Since the only monomials with weight zero in F_3 are E and $\bar{E}$, from Theorem 2(i) we deduce that the linear part of v_3 is $\alpha \mathrm{Re}\,(E)$. The quadratic part of v_3 is composed of the product of two monomials arising from F_{d_1} and F_{d_2} with $d_1 + d_2 = 4$. Since $d_i = 2$ or $d_i = 3$ for $i = 1, 2$ we have that $d_1 = d_2 = 2$. The monomials of weight zero in F_2 are AB, $\bar{A}\bar{B}$ and $A\bar{A}$, $B\bar{B}$, $C\bar{C}$. So we have four unconjugacy classes for the monomials of degree two. Now, from the remark after Theorem 2 we see that the quadratic part of v_3 reduces to $\beta \mathrm{Im}\,(AB)$ and so

$$v_3 = \alpha \mathrm{Re}\,(E) + \beta \mathrm{Im}\,(AB).$$

Now take, for instance, Hamiltonian quadratic system $\dot{z} = iz + Az^2 + Bz\bar{z} + C\bar{z}^2$ with $2A + \bar{B} = 0$, and consider the system $\dot{z} = (iz + Az^2 + Bz\bar{z} + C\bar{z}^2)(1 + z + \bar{z}) = iz + (A + i)z^2 + (B + i)z\bar{z} + C\bar{z}^2 + Az^3 + (A + B)z^2\bar{z} + (B + C)z\bar{z}^2 + C\bar{z}^3$. From (iii) of Theorem 1, we know that v_3 must be identically zero for each A, B, C with $B = -2\bar{A}$. So we have:

$$\begin{aligned} v_3 &= \alpha \mathrm{Re}\,(A + B) + \beta \mathrm{Im}\,((A + i)(B + i)) \\ &= (-\alpha - \beta)\mathrm{Re}\,(A) \equiv 0 \end{aligned}$$

which implies that $\alpha = -\beta$ and, consequently, $v_3 = \alpha(\mathrm{Re}\,(E) - \mathrm{Im}\,(AB))$.

Now we are going to calculate v_5, taking into account that $v_3 \equiv 0$, that is, $\mathrm{Re}\,(E) = \mathrm{Im}\,(AB)$.

The only monomials with weight zero satisfying $(**)$ after Theorem 1, with $m = 2$, are the following:

$$\begin{aligned} &O \\ &A\bar{I}, AJ, BI, B\bar{J}, CH, C\bar{K}, D\bar{D}, DF, E^2, E\bar{E}, F\bar{F}, G\bar{G} \\ &ABE, AB\bar{E}, A\bar{A}E, B\bar{B}E, C\bar{C}E, A^2F, \bar{B}^2F, AC\bar{F}, A\bar{B}F, \\ &\quad B\bar{C}F, A^2\bar{D}, B^2D, ACD, A\bar{B}\bar{D}, B\bar{C}\bar{D}, A\bar{C}G, BC\bar{G} \\ &A^3C, A^2B^2, A^2\bar{B}C, \bar{B}^3C, A\bar{B}^2C, AB^2\bar{B}, ABC\bar{C}, A^2\bar{A}B, \\ &\quad A^2\bar{A}^2, B^2\bar{B}^2, C^2\bar{C}^2, A\bar{A}B\bar{B}, A\bar{A}C\bar{C}, B\bar{B}C\bar{C}. \end{aligned}$$

Taking into account (i) of Theorem 2, we can say that $\mathrm{Re}\,(O)$ is the only monomial appearing in the linear part of v_5. In the same way, the quadratic part of v_5 is a linear combination of the imaginary part of the monomials $A\bar{I}, AJ, BI, B\bar{J}, CH, C\bar{K}, DF, E^2$, and the real part of $E\bar{E}+E^2$. The part of v_5 of degree three is a linear combination of the real part of ABE, $AB\bar{E}$, $A\bar{A}E$, $B\bar{B}E$, $C\bar{C}E$, A^2F, $\bar{B}^2F$, $AC\bar{F}$, $A\bar{B}F$, $B\bar{C}F$, $A^2\bar{D}$, B^2D, ACD, $A\bar{B}\bar{D}$, $B\bar{C}\bar{D}$, $A\bar{C}G$, $BC\bar{G}$, and the imaginary part of $ABE+AB\bar{E}$. The part of degree four of v_5 is a linear combination of the imaginary part of A^3C, A^2B^2, $A^2\bar{B}C$, B^3C, $A\bar{B}^2C$, $AB^2\bar{B}$, $ABC\bar{C}$, $A^2\bar{A}B$, and the real part of $A\bar{A}B\bar{B}-ABAB$. Furthermore, by using (ii) of Theorem 2, we have that the monomial $A^2\bar{D}$ does not appear.

Using that $v_3\equiv 0$, i.e., $\mathrm{Im}\,(AB)=\mathrm{Re}\,(E)$ we obtain the following relations:

$$\begin{aligned}
2\mathrm{Re}\,(ABE) &= \mathrm{Im}\,(A^2B^2)-\mathrm{Im}\,(E^2)\\
2\mathrm{Re}\,(AB\bar{E}) &= \mathrm{Im}\,(A^2B^2)+\mathrm{Im}\,(E^2)\\
\mathrm{Im}\,(AB^2\bar{B}) &= B\bar{B}\mathrm{Im}\,(AB)=\mathrm{Re}\,(B\bar{B}E)\\
B\bar{B}\mathrm{Im}\,(AB) &= \mathrm{Re}\,(B\bar{B}E)\\
\mathrm{Im}\,(ABC\bar{C}) &= \mathrm{Re}\,(C\bar{C}E)\\
\mathrm{Im}\,(ABA\bar{A}) &= \mathrm{Re}\,(A\bar{A}E)\\
\mathrm{Re}\,(A\bar{A}B\bar{B}-ABAB) &= \mathrm{Re}\,(E\bar{E}+E^2)\\
&= \mathrm{Im}\,(ABE+AB\bar{E})=2\mathrm{Im}^2(AB).
\end{aligned}$$

So, until now we know that $v_5=\alpha_1\mathrm{Re}\,(O)+\mathrm{Im}\,(\alpha_2E^2+\alpha_3DF+\alpha_4AJ+\alpha_5A\bar{I}+\alpha_6BI+\alpha_7B\bar{J}+\alpha_8CH+\alpha_9C\bar{K})+\mathrm{Re}\,(\alpha_{10}A\bar{A}E+\alpha_{11}B\bar{B}E+\alpha_{12}C\bar{C}E+\alpha_{13}A^2F+\alpha_{14}B^2\bar{F}+\alpha_{15}AC\bar{F}+\alpha_{16}A\bar{B}F+\alpha_{17}B\bar{C}F+\alpha_{18}B^2D+\alpha_{19}ACD+\alpha_{20}A\bar{B}\bar{D}+\alpha_{21}B\bar{C}\bar{D}+\alpha_{22}A\bar{C}G+\alpha_{23}BC\bar{G})+\mathrm{Im}\,(\alpha_{24}A\bar{B}^2C+\alpha_{25}A^2B^2+\alpha_{26}A^2\bar{B}C+\alpha_{27}A^3C+\alpha_{28}B^3\bar{C})+\alpha_{29}\mathrm{Im}^2(AB)$.

In order to determine the above constants, we take quadratic and cubic systems with a center and we multiply the vector fields associated to them by different real factors. Using (iii) of Theorem 1 we know that the new system also has $v_5\equiv 0$, and we obtain a lot of relations between the α_i's.

The cases that we consider are the following:

a) Quadratic system of type i) of Theorem 15 multiplied by $1+kz+\bar{k}\bar{z}+\lambda z\bar{z}$, where $\lambda \in \mathbf{R}$.

b) Quadratic system of type ii) and iv) of Theorem 15 multiplied by $1+kz+\bar{k}\bar{z}$.

c) The holomorphic cubic system $iz + Az^2 + Dz^3$ multiplied by $1+(z+\bar{z})$.

d) The holomorphic cubic system $iz+Dz^3$ multiplied by $1+(z+\bar{z})^2$.

e) A Hamiltonian cubic system.

Just as an example of the method, we do the computations in the first case with $k=1$ and $\lambda=0$. That is, we take the quadratic system $iz + Az^2 + C\bar{z}^2$ that has a center at the origin. Multiplying it by $1+z^2+\bar{z}^2$, we obtain

$$\dot{z} = iz + Az^2 + C\bar{z}^2 + iz^3 + iz^2\bar{z} + Az^4 + (A+C)z^2\bar{z} + C\bar{z}^4$$

which satisfies $v_3 = 0$ and

$$\begin{aligned} v_5 &= \alpha_4 \mathrm{Im}\,(A(A+C)) + \alpha_8 \mathrm{Im}\,(AC) + \alpha_{13}\mathrm{Re}\,(A^2 i) \\ &\quad + \alpha_{15}\mathrm{Re}\,(-ACi) + \alpha_{19}\mathrm{Re}\,(ACi) + \alpha_{27}\mathrm{Im}\,(A^3 C) \\ &= (\alpha_4 - \alpha_{13})\mathrm{Im}\,(A^2) + (\alpha_4 + \alpha_8 + \alpha_{15} - \alpha_{19})\mathrm{Im}\,(AC) \\ &\quad + \alpha_{27}\mathrm{Im}\,(A^3 C) + \alpha_{29}\mathrm{Im}^{\,2}(AB). \end{aligned}$$

Then, from (iii) of Theorem 1, $v_5 \equiv 0$. Hence $\alpha_4 = \alpha_{13}$, $\alpha_4+\alpha_8+\alpha_{15} = \alpha_{19}$, $\alpha_{27}=0$ and $\alpha_{29}=0$.

At the end, we obtain an overdetermined system with one degree of freedom. By solving it we have that $\alpha_i = ak_i$ for $i=1,2,\dots,29$. In order to determine the constant a (and the constant α for v_3) we compute the Liapunov constants for the family of systems $\dot{z} = iz + z^{m+1}\bar{z}^m$. These systems satisfy $dr/d\theta = r^{2m+1}$. The solution of $dr/d\theta = r^{2m+1}$ with $r(0,\rho)=\rho$ is

$$\begin{aligned} r(\theta,\rho) &= \rho[1-2m\rho^{2m}\theta]^{-1/(2m)} \\ &= \rho\left[\binom{-1/(2m)}{0} + \binom{-1/(2m)}{1}(-2m\rho^{2m}\theta) + \cdots\right] \\ &= \rho + \theta\rho^{2m+1} + \cdots . \end{aligned}$$

Hence $v_{2m+1} = u_{2m+1}(2\pi) = 2\pi$. For $m = 1$, $\dot{z} = iz + z^2\bar{z}$ satisfies $v_3 = \alpha = 2\pi$ and (i) follows. For $m = 2$, $\dot{z} = iz + z^3\bar{z}^2$ satisfies $v_5 = a[6\mathrm{Re}\,(O)] = 2\pi$. Therefore, $a = \pi/3$ and (ii) is proved.

Now, from Theorem 1, we know that T_2 is formed by a sum of monomials of weight zero in the coefficients of F_2 and F_3, satisfying the relation $(**)$ after Theorem 1. From this we see that the only monomials which can appear are: $E, \bar{E}, AB, \bar{A}\bar{B}, A\bar{A}, B\bar{B}$, and $C\bar{C}$. Since we assume that $v_3 = \alpha(\mathrm{Re}\,(E) - \mathrm{Im}\,(AB)) = 0$, we have that

$$T_2 = a\mathrm{Re}\,(AB) + b\mathrm{Re}\,(E) + c\mathrm{Im}\,(E) + dA\bar{A} + eB\bar{B} + fC\bar{C}$$

with $a, b, c, d, e, f \in \mathbf{R}$.

In order to determine the coefficients a, b, c, d, e, f, g, we consider the following systems:

(1) $\dot{z} = iz + (i/2)z\bar{z} + (i/2)\bar{z}^2$

(2) $\dot{z} = iz + (1/4)z^2 - (5/4)\bar{z}^2$

(3) $\dot{z} = iz - (i/2)z^2 + (i/2)z\bar{z}$

(4) $\dot{z} = iz + (1/4)z^2 + (1/2)z\bar{z} + (1/4)\bar{z}^2$

(5) $\dot{z} = iz + iz^2\bar{z}$.

(6) $\dot{z} = iz + (1+i)z^2\bar{z}^2 + iz\bar{z} + z^3 + z^2\bar{z} + z\bar{z}^2 + \bar{z}^3$.

By computing the period constant T_2 for the above systems, we obtain, substituting the results obtained in the expression of T_2, that $a = -2\pi$, $b = 0$, $c = -2\pi$, $d = 0$, $e = 2\pi$ and $f = 4\pi/3$.

We give an example of the procedure used in order to obtain the above equations. Consider system (1) and write it in polar coordinates:

$$\begin{cases} \dot{r} = r^2(2\cos^2\xi\sin\xi) \\ \dot{\xi} = 1 + r(\cos^3\xi - \cos\xi\sin^2\xi) \end{cases} .$$

Then we have that

$$H_1(\xi) = -(\cos^3\xi - \cos\xi\sin^2\xi)$$
$$H_2(\xi) = (\cos^3\xi - \cos\xi\sin\xi)^2$$

and

$$\begin{aligned} u_2(\xi) &= \int_0^{\xi} R_2(\theta)\,d\theta \\ &= \int_0^{\xi} 2\cos^2\theta\sin\theta\,d\theta \\ &= -\frac{2}{3}\cos^3\xi + \frac{2}{3}. \end{aligned}$$

Hence,

$$T_2 = \int_0^{2\pi} H_1(\xi)u_2(\xi) + H_2(\xi)\,d\xi = \frac{5\pi}{6}.$$

Therefore, $e/4 + f/4 = 5\pi/6$. □

Remark 16. In the computation of v_3, first we have obtained that its only monomials are $\mathrm{Re}\,(E)$ and $\mathrm{Im}\,(AB)$, that is, $v_3 = \alpha\mathrm{Re}\,(E) + \beta\mathrm{Im}\,(AB)$. In order to relate α and β, we would like to comment on a different method than the one used in the proof of Theorem A. Note that we already know v_3 for the quadratic case $\dot{z} = iz + F_2$, because $E = 0$. Consider the real function $H(z,\bar{z}) = kz + \bar{k}\bar{z}$ and apply Theorem 1 (iii). Then

$$v_3(F_2) = v_3(F_2 + (iz + F_2)(kz + \bar{k}\bar{z})),$$

or equivalently,

$$\beta\mathrm{Im}\,(AB) = \alpha\mathrm{Re}\,(kB + \bar{k}A) + \beta\mathrm{Im}\,[(A + ki)(B + \bar{k}i)].$$

Hence $(\alpha + \beta)\mathrm{Re}\,(kB + \bar{k}A) \equiv 0$ and then $\alpha = -\beta$. Of course, we have obtained the same result as in Theorem A, but here we have used the Liapunov constant v_3 for a simple case (the quadratic one) to compute the general v_3 Liapunov constant, that is, we use a kind of recursive procedure.

Proof of Theorem B. We begin by proving that $v_{2m+1} = \alpha_{2m+1}a_{2m+1}$. The Liénard equation can be written in complex notation:

$$\begin{aligned} \dot{z} = iz &+ \frac{a_2}{4}z^2 + \frac{a_2}{2}z\bar{z} + \frac{a_2}{4}\bar{z}^2 + \frac{a_3}{8}z^3 \\ &+ \frac{3a_3}{8}z^2\bar{z} + \frac{3a_3}{4}z\bar{z}^2 + \frac{a_3}{8}\bar{z}^3 + \cdots . \end{aligned}$$

By using Theorem A (i) we have that $v_3 = (3/4)\pi a_3$. Now we proceed by induction: assume that $a_3 = a_5 = \cdots = a_{2m-1} = 0$ and $a_{2m+1} \neq 0$. From (**) after Theorem 1 we know that the monomials of v_{2m+1} arise from F_{d_i} with $\sum_{i=1}^{2r+1}(d_i - 1) = 2m$. By the induction hypothesis we know that $F_d \equiv 0$ for $d = 3, 5, \ldots, 2m-1$, and consequently v_{2m+1} only depends on F_d for $d = 2, 4, \ldots, 2m$ and $d = 2m+1$.

By using Theorem 1 (ii) we can deduce that $v_{2m+1} = \alpha_{2m+1}a_{2m+1} + p(a_2, a_4, \ldots, a_{2m})$ for some polynomial p. Observe that if $a_{2i+1} = 0$ for all $i \geq 1$, the Liénard equation is reversible and by Proposition 13 it has a center at the origin. Therefore $p(a_2, a_4, \ldots, a_{2m}) \equiv 0$ and $v_{2m+1} = \alpha_{2m+1}a_{2m+1}$.

In order to determine the constants α_i we consider the family of systems $\dot{x} = -y + x^m$, $\dot{y} = x$. These systems satisfy

$$\begin{aligned}\frac{dr}{d\theta} &= r^m \cos^{m+1}\theta[1 - r^{m-1}\sin\theta\cos^m\theta]^{-1} \\ &= r^m \cos^{m+1}\theta[1 + r^{m-1}\sin\theta\cos^m\theta + \cdots] \\ &= r^m \cos^{m+1}\theta + r^{2m-1}\sin\theta\cos^{2m+1}\theta + \cdots .\end{aligned}$$

Let $r(\theta, \rho) = u_1(\theta)\rho + u_2(\theta)\rho^2 + \cdots$ be the solution of (6) with $r(0, \rho) \equiv \rho$. Then

$$\begin{aligned}\frac{dr}{d\theta} &= u_1'(\theta)\rho + u_2'(\theta)\rho^2 + \cdots \\ &= [u_1(\theta)\rho + u_2(\theta)\rho^2 + \cdots]^m \cos^{m+1}\theta \\ &\quad + [u_1(\theta)\rho + u_2(\theta)\rho^2 + \cdots]^{2m-1}\sin\theta\cos^{2m+1}\theta + \cdots .\end{aligned}$$

Hence $u_i'(\theta) = 0$ for each $i = 1, 2, \ldots, m-1$ and so $u_i(\theta) = k_i$ for $i = 1, 2, \ldots, m-1$. From $u_i(0, \rho) \equiv \rho$ we deduce that $u_1(\theta) = 1$ and $u_i(\theta) = 0$ for $i = 2, 3, \ldots, m-1$. For $i = m$ we have that $u_m'(\theta) = \cos^{m+1}\theta$ and so $u_m(2\pi) = \int_0^{2\pi}\cos^{m+1}\theta\, d\theta$. From Lemma 3, for m even, this integral is zero and for m odd direct computations give that

$$v_m = u_m(2\pi) = \frac{m!!}{(m+1)!!}2\pi.$$

Hence the part (i) of the theorem is proved.

In order to see (ii), first we prove that $T_{4k+2} = \alpha_{2k+2}a_{2k+2}^2$. We proceed by induction. Theorem A iii) implies that $T_2 = (\pi/3)a_2^2$.

Assume that $a_2 = a_4 = \cdots = a_{2k} = 0$ and $a_{2k+2} \neq 0$. By induction hypothesis, we know that $T_{4k-2} = 0$ and consequently $T_{4k-1} = 0$. We claim that $T_{4k} = 0$.

Set $M = \mu(\prod_{i=2k+2}^{4k} a_i^{m_i})$ be a monomial of T_{4k}, with $\mu \neq 0$. From (**) after Theorem 1, we have that $m_{2k+2}(2k+1) + m_{2k+4}(2k+3) + \cdots + m_{4k}(4k-1) = 4k$. From this equality it is clear that $m_j \leq 1$ for all $j = 2k+2, \dots, 4k$. On the other hand, if there exist s and t with $s < t$ and $m_s = m_t = 1$, then the above equality says that $s-1+t-1 \leq 4k$ which is a contradiction with the fact $s-1 \geq 2k$ and $t-1 > 2k$. Since it is also not possible to have a unique j with $m_j = 1$, we deduce that $m_j = 0$ for all $j = 2k+2, \dots, 4k$. Hence $M = \mu$ is a constant and therefore $M = 0$.

Now we are going to see which monomials are in T_{4k+2}. Set $M = \mu(\prod_{i=2k+2}^{4k+2} a_i^{m_i})$ with $\mu \neq 0$. Again, from (**) after Theorem 1, we have that $m_{2k+2}(2k+1) + m_{2k+4}(2k+3) + \cdots + m_{4k+2}(4k+1) = 4k+2$. We claim that $m_{2k+2} = 2$. If $m_{2k+2} \neq 2$, then there exists at least s and t such that $2k+2 \leq s < t \leq 4k+2$ and $m_s = 1$, $m_t = 1$. But it implies that $s-1+t-1 \leq 4k+2$, which is a contradiction with $s-1 \geq 2k+1$ and $t-1 > 2k+1$. So M does not contain linear terms because $m_{2k+2} = 2$ and $m_j = 0$ for all $j > 2k+2$; that is, $M = \mu a_{2k+2}^2$ with $\mu \in \mathbf{R}$.

In order to determine the constant μ we consider the equation

$$\begin{cases} \dot{x} = -y + x^{2k+2} \\ \dot{y} = x. \end{cases}$$

After some computations we obtain that

$$T_{4k+2} = \mu = 2\pi\left(\frac{-2(4k+5)!!}{(2k+3)(4k+6)!!} + \frac{(4k+3)!!}{(4k+4)!!}\right). \qquad \square$$

Proof of Proposition C. In complex coordinates, the system under consideration can be written as:

$$\dot{z} = iz + Az^2 + 2\bar{A}z\bar{z} + A\bar{z}^2 + dz^3 + 3ez^2\bar{z} + 3dz\bar{z}^2 + e\bar{z}^3,$$

where

$$A = (\alpha - i\gamma)/4, \qquad d = (\beta - \delta)/8 \in \mathbf{R} \quad \text{and} \quad e = (\beta + \delta)/8 \in \mathbf{R}.$$

First of all, observe that from Proposition 13 the above system has a reversible center at the origin if and only if either $d = e = \mathrm{Im}\,(A^2) = 0$ or $e = \mathrm{Re}\,(A^2) = 0$.

On the other hand, from Theorem A we know that $v_3 = 6\pi e$ and assuming $e = 0$, $v_5 = (8\pi/3)\mathrm{Re}\,(A^2)[5d + 12\mathrm{Im}\,(A^2)]$. So $v_5 = 0$ if and only if either $\mathrm{Re}\,(A^2) = 0$ or $d = (-12/5)\mathrm{Im}\,(A^2)$. Hence, from the above observation the case $\mathrm{Re}\,(A^2) = 0$ is finished.

Consider now condition $d = (-12/5)\mathrm{Im}\,(A^2)$. By using Theorems 1 and 2, the only monomials that can appear in v_7 are: $A\bar{A}d^2$, $\mathrm{Re}\,(A^3\bar{A}d)$, $\mathrm{Im}\,(A^3\bar{A}d)$, $A^3\bar{A}^3$, $\mathrm{Re}\,(A^5\bar{A})$, and $\mathrm{Im}\,(A^5\bar{A})$. Observe that $(A\bar{A})^3 = A\bar{A}(\mathrm{Re}^2(A^2) + \mathrm{Im}^2(A^2))$. Hence, when $d = (-12/5)\mathrm{Im}\,(A^2)$, we have that

$$v_7 = \alpha_1 A\bar{A}\mathrm{Re}^2(A^2) + \alpha_2 A\bar{A}\mathrm{Im}^2(A^2) + \alpha_3 A\bar{A}\mathrm{Re}\,(A^2)\mathrm{Im}\,(A^2).$$

Note that when $e = d = 0$, $A = a \in \mathbf{R}$, the origin of our system is a reversible center. Therefore $v_7|_{e=d=0, A=a\in\mathbf{R}} = \alpha_1 a^6 = 0$ and, as a consequence, $\alpha_1 = 0$. Arguing in the same way but using the family of reversible centers $d = e = 0$ and $A = a + ai$, $a \in \mathbf{R}$ (so $\mathrm{Re}\,(A^2) = 0$) we get that $\alpha_2 = 0$. Therefore, $v_7 = \alpha_3 A\bar{A}\mathrm{Re}\,(A^2)\mathrm{Im}\,(A^2)$. Finally, to obtain α_3 we consider the particular case $A = (1+5i)/4$, $d = -3/2$ and $e = 0$, and we make all the tedious but straightforward computations explained in Lemmas 1, 2 and 4. We get $v_7 = 585\pi$ and so $\alpha_3 = -384\pi$. Hence we have proved that when $v_3 = v_5 = v_7 = 0$ our system has a reversible center at the origin. Therefore the proof ends by expressing all the Liapunov constants in terms of the original parameters.

(ii) By using the notation of the proof of (i) and Theorem A, we get that when $e = 0$, $T_2 = (16/3)\pi A\bar{A}$. From Theorem 1, $T_4 = \alpha d^2$. By computing T_4 for the system when $A = 0$, $e = 0$ and $d = 1$ (in a similar way to the one used in the proof of (i) for computing v_7) we obtain that $\alpha = 12\pi$ and so the proposition follows. □

Note that the final expression of v_7 depends only on its computation for a particular system. We have chosen $A = (1 + 5i)/4$, $d = -3/2$ and $e = 0$, and we have obtained $v_7 = 585\pi$. This result has also been obtained by our colleague A. Guillamon by using the software explained in [**11**]. On the other hand, observe that the system studied in Proposition C when $\alpha = \gamma = \beta + \delta = 0$ is a center with homogeneous nonlinearities of degree 3. By applying the results of [**12**] we can prove that the condition to be isochronous is $\delta = 0$.

REFERENCES

1. A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier, *Theory of bifurcations of dynamic systems on a plane*, John Wiley and Sons, New York, 1967.

2. T.R. Blows and N.G. Lloyd, *The number of small-amplitude limit cycles of Liénard equations*, Math. Proc. Cambridge Philos. Soc. **95** (1984), 751–758.

3. C. Chicone, *The monotonicity of the period function for planar Hamiltonian vector fields*, J. Differential Equations **69** (1987), 310–321.

4. C. Chicone and M. Jacobs, *Bifurcation of critical periods for planar vector fields*, Trans. Amer. Math. Soc. **312** (1989), 433–486.

5. W.W. Farr, Chengzhi Li, I.S. Labouriau and W.F. Langford, *Degenerate Hopf bifurcation formulas and Hilbert's 16th problem*, SIAM J. Math. Anal. **20** (1989), 13–29.

6. F. Göbber and K.D. Williamowski, *Liapunov approach to multiple Hopf bifurcation*, J. Math. Anal. Appl. **71** (1979), 333–350.

7. I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series and products*, Academic Press, Inc., New York, 1980.

8. B. Hassard and Y.H. Wan, *Bifurcation formulae derived from center manifold theory*, J. Math. Anal. Appl. **63** (1978), 297–312.

9. A. Lins, W. de Melo and C.C. Pugh, *On Liénard's equation*, Lecture Notes in Math. **597** (1976), 335–357.

10. Liu Yi-Rong and Li Ji-Bin, *Theory of values of singular point in complex autonomous differential systems*, Sci. China Ser. A. **33** (1990), 10–23.

11. N.G. Lloyd and J.M. Pearson, *REDUCE and the bifurcation of limit cycles*, Symbol. Comput. Artificial Intelligence **9** (1990), 215–224.

12. I.I. Pleshkan, *A new method of investigating the isochronicity of a system of two differential equations*, Differential Equations **5** (1969), 796–802.

13. K.S. Sibirskii, *Algebraic invariants of differential equations and matrices*, Kshinev, Shtiint-sa (1976), in Russian.

14. H. Zoladek, *Quadratic systems with center and their perturbations*, J. Differential Equations **109** (1994), 223–273.

15. ———, *On a certain generalization of the Bautin's theorem*, Nonlinearity **7** (1994), 273–280.

16. Carlos Zuppa, *Order of cyclicity of the singular point of Liénard's polynomial vector fields*, Bol. Soc. Bras. Mat. **12** (1981), 105–111.

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