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QUALITATIVE ANALYSIS OF A SINGULARLY-PERTURBED SYSTEM OF DIFFERENTIAL EQUATIONS RELATED TO THE VAN DER POL EQUATIONS

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ABSTRACT. A method to qualitatively analyze certain three-dimensional singularly-perturbed, nonautonomous, nonlinear systems is presented. The analysis involves the construction of a trapping region for solutions of the system. This method can be applied to the Oregonator model of the Belousov-Zhabotinskii reaction. One result is a new and clearer proof of Hastings and Murray's result that there is a nontrivial periodic solution of the model.

0. Introduction. This paper is concerned with the singularly-perturbed, nonautonomous, nonlinear ordinary differential equation:

$$\begin{aligned} \dot{x} &= \frac{1}{\varepsilon}(y - xz) + e_1(t) \\ \dot{y} &= -x + e_2(t) \\ \dot{z} &= \frac{1}{\varepsilon}(x^2/3 - 1 - z) + e_3(t) \end{aligned} \tag{1}$$

where $0 < \varepsilon \ll 1$, $e_i(t)$, $i = 1, 2, 3$ are bounded functions (for example, periodic functions with common period L), and with $e_2(t)$ small.

1. Motivation. There are certain muscle fibers in the heart known as the cardiac Purkinje fibers. The primary function of the Purkinje fibers is to transmit electrical pacemaker impulses in the heart. The Purkinje fiber also exhibits a secondary activity; if the fiber is not subject to any outside stimulus, then it spontaneously and regularly generates an electrical impulse. Noble derived a four-dimensional autonomous system of singularly-perturbed differential equations to model the Purkinje fiber by modifying the Hodgkin-Huxley equations [**9**].

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In the Nobel model, V represents the potential difference across the surface of the fiber, m and h measure the flow of sodium ions across the fiber surface, and n does the same for the flow of potassium ions. Using a transformation by Cronin [**5**], the Nobel model is:

$$\begin{aligned} \frac{dV}{dt} &= \frac{1}{\varepsilon}\frac{1}{\varepsilon'}(F(V,m,h,n)), & \frac{dm}{dt} &= \frac{1}{\varepsilon}\frac{1}{\varepsilon'}\frac{m_\infty(V)-m}{T_m(V)} \\ \frac{dh}{dt} &= \frac{1}{\varepsilon}\frac{h_\infty(V)-h}{T_h(V)}, & \frac{dn}{dt} &= \frac{n_\infty(V)-n}{T_n(V)} \end{aligned} \tag{2}$$

where $m_\infty, h_\infty, n_\infty, T_m, T_h$ and T_n are well-behaved functions of V, and F is a well-behaved function of all four variables. ε and ε' are small, fixed, positive numbers.

The Nobel model is a bit complicated. A simpler system which shares many characteristics of (2) is

$$\dot{x} = \frac{1}{\varepsilon}(y - xz), \qquad \dot{y} = -x, \qquad \dot{z} = \frac{1}{\varepsilon}(x^2/3 - 1 - z) \tag{3}$$

which is the case where the forcing terms $e_i(t)$ in (1) are all identically zero. Numerical experiments indicate the both (2) and (3) have a globally stable periodic solution. Both systems are singularly-perturbed, and both systems are of the form investigated by Mishchenko and Rosov [**8**]. Clearly (3) is an easier system to study than the Nobel model. We are also interested in determining how an outside force can affect the systems like (2) and (3). For example, how would an electrical impulse applied to the heart affect the behavior of a Purkinje fiber? This question can be considered by adding forcing terms to our model.

Although both the Noble equations and (1) can be solved numerically, a numerical solution does not explain why solutions behave the way that they do. To strengthen our understanding of the solutions, a qualitative analysis is in order. In this paper we develop qualitative methods that can be applied to systems such as (2).

At the conclusion of this paper, we will consider how our analysis applies to another three-dimensional, singularly-perturbed system, namely the Oregonator model of the Belousov-Zhabotinskii reaction [**10**] and prove a result about that model stronger than the result of Hastings and Murray [**7**]. We will also describe how to extend the analysis to the Noble model. For related work in this area, see Albrecht and Villari [**1**], Alexander, Doedel, and Othmer [**2**] and Chicone [**3**].

2. Overview. Mishchenko and Rosov's approach to systems such as (2) is to first consider the existence of a discontinuous solution, that is, where ε is zero, and then to construct some sort of tube around that discontinuous solution that will trap solutions of the system in question. (We do not need to prove that a discontinuous solution exists, which saves a lot of analytic work.) This approach can be seen in an analysis of the van der Pol system in the Liénard plane:

$$\dot{x} = \frac{1}{\varepsilon}(y - x^3/3 + x), \qquad \dot{y} = -x. \tag{4}$$

A trapping region for (4) was developed by Flanders and Stoker [**6**]. The region and the discontinuous solution can be seen in Figure 1.

For (3), we define a set $\tilde{S}$, which we call the *slow manifold*, as

$$\tilde{S} = \{(x, y, z) \mid y = xz \text{ and } z = x^2/3 - 1\}.$$

This is the set where $\dot{x} = \dot{z} = 0$ in (3). As indicated in Figure 2, $\tilde{S}$ is "S-shaped" (we can also consider it as the graph of $\tilde{y}(x) = x^3/3 - x$), and, in general, solutions of (1) are attracted by certain parts of the slow manifold and repelled by other parts. This situation is (purposely) reminiscent of (4).

In our analysis of (1), we shall show first that solutions of (1) are attracted (at least locally) to the "attracting" parts of $\tilde{S}$. Then we will prove the oscillatory behavior of solutions of (1)—they move between the "attracting" parts of $\tilde{S}$ and the "fast areas" indicated in Figure 2—by constructing a tube to trap solutions. In the case where the forcing terms $e_i(t)$ are all identically zero, the case of system (3), we can prove, using the Poincaré map, that we have a periodic solution.

3. Analysis, Part I. Each part of the construction will be true for ε sufficiently small, with specific conditions on ε given in the proofs of the lemmas. In the statements of the lemmas, we omit the condition "for ε sufficiently small." The trapping region, $\Theta(\varepsilon)$, will be defined in six sections $\Theta_i(\varepsilon)$, $i = 1, 2, \ldots, 6$. To construct the region, we begin by defining

$$B_i = \max_{-\infty < t < \infty} |e_i(t)|, \quad i = 1, 2, 3,$$

and fixing three positive constants: a, γ_1 and γ_2 with $\gamma_1 + \gamma_2 < 1$, $2\gamma_1 < \gamma_2$, $a \ll 1$ (in particular, $a < 1/3$), and $B_2 < 1 - 2a$. Let

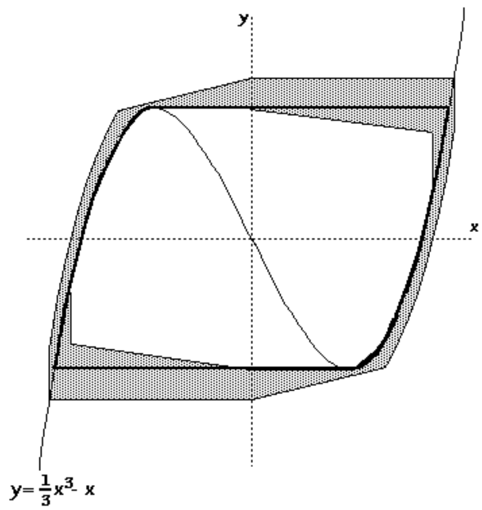

FIGURE 1. The trapping region of Flanders and Stoker.

$\hat{y} = (1/3)(a-1)^3 - (a-1)$. Then $0 < \hat{y} < 2/3$. Let $d = 1 - 2a - B_2$. Then $d > 0$. Later, we will define y^* such that $\hat{y} < y^* < 2/3$.

Cross sections of $\Theta_1(\varepsilon)$ will be ellipses in the planes $y = \bar{y}$, where $-3/2 < \bar{y} < y^*$. The center of each ellipse will be a point on $\tilde{S}$, which we will call $(\bar{x}_1(\bar{y}), \bar{y}, \bar{z}_1(\bar{y}))$, or simply $(\bar{x}_1, \bar{y}, \bar{z}_1)$. Clearly, for any $\bar{y} \in [-3/2, y^*]$, there is a unique value, $\bar{x}_1(\bar{y}) < -1$, such that $(\bar{x}_1, \bar{y}, \bar{x}_1^2 - 1) \in \tilde{S}$.

Define:

$$
\begin{aligned}
&Q = -\bar{z}_1, \qquad R = -\bar{x}_1, \qquad S = 2\bar{x}_1/3, \qquad T = -1 \\
&B = \frac{T-Q}{S} = \frac{\bar{x}_1^2 - 6}{2\bar{x}_1}, \\
&C = \frac{Q^2 + T^2 + 2(QT - RS)}{2S^2} = \frac{\bar{x}_1^2 + 12}{8}
\end{aligned}
$$

Q, R, S, T, B and C are continuous functions of y defined for $y < 2/3$.

The following is a technical lemma that we will refer to many times.

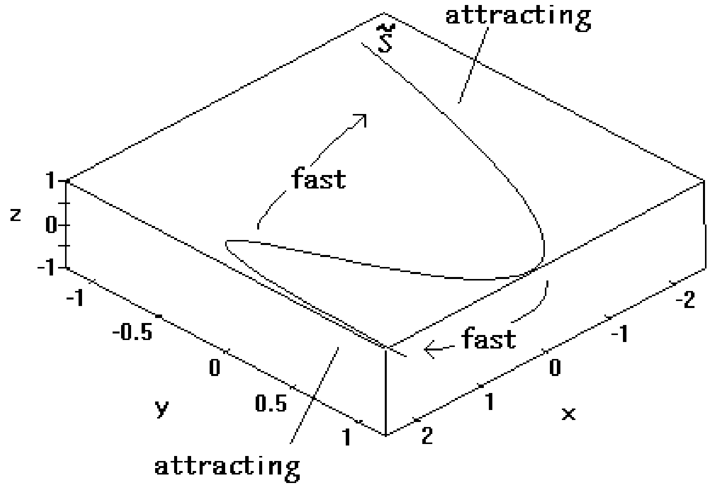

FIGURE 2. $\tilde{S}$ for (3), showing attracting and fast regions.

The proof is algebraically straightforward and is based on the fact that $QT - RS = \bar{x}_1^2 - 1 = \tilde{y}'(\bar{x}_1)$.

Lemma 1.1. *For all $y < 2/3$, we have that $Q+T < 0$, $QT-RS > 0$, $S \neq 0$, $2Q + BS < 0$, $BR + 2CT < 0$ and $(BQ + BT + 2R + 2CS)^2 - 4(2Q + BS)(BR + 2CT) < 0$.*

For (x, y, z) on or near the section of $\tilde{S}$ for which $x < -1$, define a function H_1 by:

$$H_1(x, y, z) = (x - \bar{x}_1)^2 + B(x - \bar{x}_1)(z - \bar{z}_1) + C(z - \bar{z}_1)^2$$

where $\bar{x}_1, \bar{z}_1, B$ and C are functions of y.

Lemma 1.2. *For any $\bar{y} < 2/3$, the set $\{(x, y, z) \mid H_1(x, \bar{y}, z) = \varepsilon^{\gamma_2}, y = \bar{y}\}$ is an ellipse with center $(\bar{x}_1(\bar{y}), \bar{y}, \bar{z}_1(\bar{y}))$.*

To prove this lemma, it is sufficient to show that $C > 0$ and

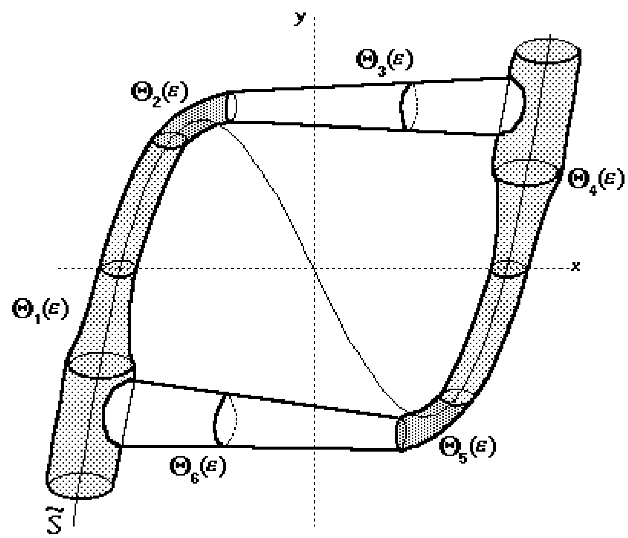

FIGURE 3. $\Theta(e)$, the trapping region for (1).

$B^2 - 4C < 0$. The proof is straightforward and so we omit it.

Define y^* to be the largest y less than $2/3$ such that all points in the set $\{(x, y, z) \mid H_1(x, y^*, z) = \varepsilon^{\gamma_2}\}$ satisfy $x \le -1 - \varepsilon^{\gamma_1}$. y^* depends on ε and y^* approaches $2/3$ as ε approaches 0. We require that ε is small enough so that $\hat{y} < y^*$.

Define the following sets (see Figure 4a)

$$\Theta_1'(\varepsilon) = \{(x, y, z) \mid H_1(x, y, z) \le \varepsilon^{\gamma_2}, -3/2 \le y \le y^*\}$$
$$C_0(\varepsilon) = \{(x, -3/2, z) \mid H_1(x, -3/2, z) \le \varepsilon^{\gamma_2}\}$$
$$C_1(\varepsilon) = \{(x, y^*, z) \mid H_1(x, y^*, z) \le \varepsilon^{\gamma_2}\}$$
$$D_1'(\varepsilon) = \{(x, y, z) \mid H_1(x, y, z) = \varepsilon^{\gamma_2}, -3/2 \le y \le y^*\}.$$

We will construct $\Theta_1(\varepsilon)$ later. It is not difficult to show that $dH_1 \ne 0$, hence $D_1'(\varepsilon)$ is a manifold. Now we demonstrate how solutions of (1) are attracted to the slow manifold.

Theorem 1. *For any $t_0 \in \mathbf{R}$ and for any solution of* (1) *with $(x(t_0), y(t_0), z(t_0)) \in \Theta_1'(\varepsilon)$, there exists $t_1 \ge t_0$ such that*

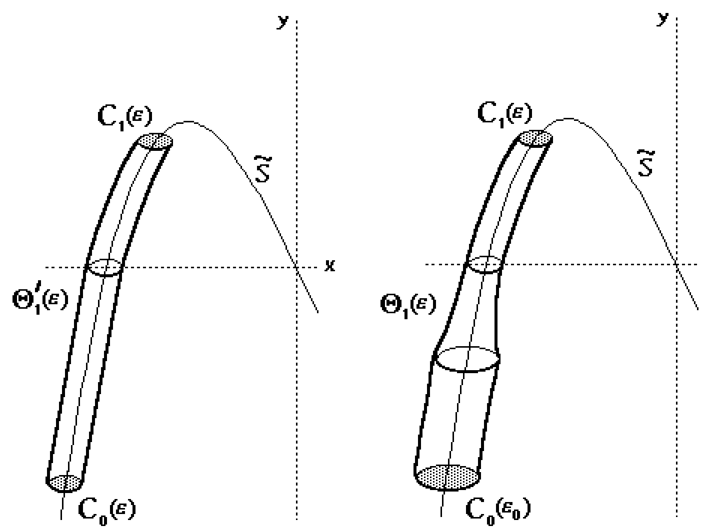

FIGURE 4. $\Theta_1'(\varepsilon)$ and $\Theta_1(\varepsilon)$.

$(x(t), y(t), z(t)) \in \Theta_1'(\varepsilon)$ *for* $t_0 \le t \le t_1$ *and* $(x(t_1), y(t_1), z(t_1)) \in C_1(\varepsilon)$.

Proof of Theorem 1. For all points of $\Theta_1'(\varepsilon)$, $\dot{y} = -x + e_2(t) > 1 - 2a - B_2 = d$. Clearly, in the case where $(x(t_0), y(t_0), z(t_0)) \in C_1(\varepsilon)$, the theorem is true with $t_1 = t_0$. In the case where $(x(t_0), y(t_0), z(t_0)) \in C_0(\varepsilon)$, then the solution will enter the interior of $\Theta_1'(\varepsilon)$.

In that case, or in the case where $(x(t_0), y(t_0), z(t_0)) \in D_1'(\varepsilon)$, we will show that the solution cannot "escape" $\Theta_1'(\varepsilon)$ through $D_1'(\varepsilon)$. Technically, we need to demonstrate that at all points of $D_1'(\varepsilon)$, $\vec{v} \cdot \vec{N} > 0$, where $\vec{v} = \langle \dot{x}, \dot{y}, \dot{z} \rangle$ is the tangent vector to the solution of (1) and $\vec{N}$ is an inner normal to $D_1'(\varepsilon)$. It follows from $\vec{v} \cdot \vec{N} > 0$ and from $\dot{y} > d$ that the solution must intersect $C_1(\varepsilon)$ in finite time.

In order to investigate $\vec{v} \cdot \vec{N}$, we transform (1) using $u = x - \bar{x}_1$,

$\nu = z - \bar{z}_1$ and get the following relationships:

$$\begin{aligned}\dot{x} &= \frac{1}{\varepsilon}(Qu + R\nu - u\nu) + e_1(t)\\ \dot{y} &= -(u + \bar{x}_1) + e_2(t)\\ \dot{z} &= \frac{1}{\varepsilon}(Su + T\nu + u^2/3) + e_3(t).\end{aligned}\tag{5}$$

Now $\vec{N} = \langle -\partial H_1/\partial x, -\partial H_1/\partial y, \partial H_1/\partial z\rangle$. Therefore,

$$\begin{aligned}\vec{v}\cdot\vec{N} &= -\frac{\partial H_1}{\partial x}\dot{x} - \frac{\partial H_1}{\partial y}\dot{y} - \frac{\partial H_1}{\partial z}\dot{z}\\ &= -(2u + B\nu)\dot{x} - (Bu + 2C\nu)\dot{z} - \frac{\partial H_1}{\partial y}\dot{y}\\ &= -\frac{1}{\varepsilon}[u^2(2Q + BS) + u\nu(BQ + BT + 2R + 2CS)\\ &\qquad + \nu^2(BR + 2CT) + \frac{1}{3}u^2(Bu + 2C\nu) - (2u + B\nu)u\nu]\\ &\quad - (2u + B\nu)e_1(t) - \frac{\partial H_1}{\partial y}\dot{y} - (Bu + 2C\nu)e_3(t).\end{aligned}$$

We wish to show that $\vec{v}\cdot\vec{N} > 0$ for ε sufficiently small. This can be attacked a few terms at a time. Since slices of $\Theta_1'(\varepsilon)$ are ellipses we have that $u^2 + \nu^2$ are $O(\varepsilon^{\gamma_2})$. So the term $(1/3)u^2(Bu + 2C\nu) - (2u + B\nu)u\nu$ is $O(\varepsilon^{3/2\gamma_2})$.

Using the Rayleigh quotient, we can show that there is a positive constant k_1, such that $u^2(2Q + BS) + uv(BQ + BT + 2R + 2CS) + \nu^2(BR + 2CT) < -k_1(QT - RS)\varepsilon^{\gamma_2}$, making use of the definition of $D_1'(\varepsilon)$. Since $QT - RS = \bar{x}_1^2 - 1$, and since $x < -1 - \varepsilon^{\gamma_1}$ for every point of $D_1'(\varepsilon)$, we have that $QT - RS > 2\varepsilon^{\gamma_1}$. Therefore, there is a positive constant k_2 such that $u^2(BQ + BS) + u\nu(BQ + BT + 2R + 2CS) + v^2(BR + 2CT) < -k_2\varepsilon^{\gamma_1+\gamma_2}$. A similar argument will show that $\partial H_1/\partial y$ is $O(\varepsilon^{\gamma_2 - 2\gamma_1})$, and clearly the other terms are bounded.

So, we have $\vec{v}\cdot\vec{N} > k_2\varepsilon^{\gamma_1+\gamma_2-1}$ plus other, bounded terms. Therefore, for ε sufficiently small, we have that $\vec{v}\cdot\vec{N} > 0$. $\square$

Let ε_0 be such that all the previous lemmas hold for $\varepsilon \in (0, \varepsilon_0)$.

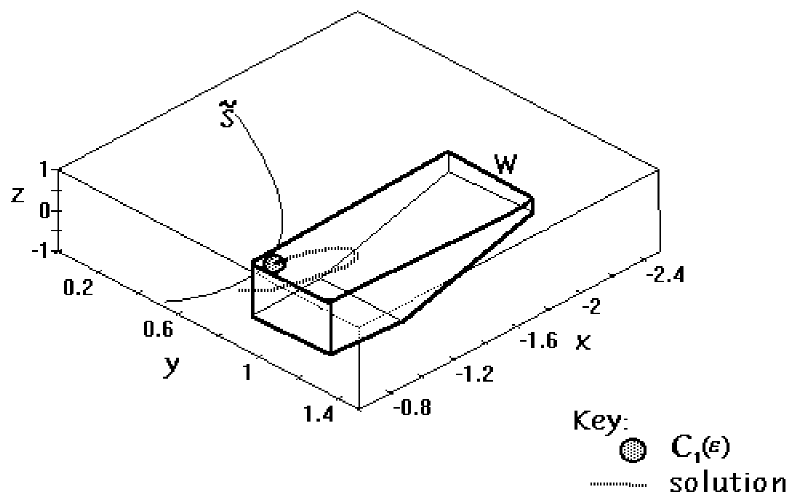

FIGURE 5. W and the behavior of a typical solution of (1).

Remark. For each $\bar{y} < 2/3$, $(\bar{x}_1(\bar{y}), \bar{z}_1(\bar{y}))$ is an equilibrium point for the system

$$\dot{x} = \bar{y} - xz, \qquad \dot{z} = x^2/3 - 1 - z. \tag{6}$$

The linear variational system corresponding to (6) is

$$\dot{u} = Qu + R\nu, \qquad \dot{v} = Su + T\nu. \tag{7}$$

As a result of Lemma 1.1, $(\bar{x}_1, \bar{z}_1)$ is an asymptotically stable equilibrium point of (6). Much of Theorem 1 is basically a proof that $H_1(x, \bar{y}, z)$ is a Lyapunov function of both (6) and (7) with respect to $(\bar{x}_1, \bar{z}_1)$.

4. Analysis, Part II. In Theorem 2, we determine that we can extend solutions past $C_1(\varepsilon)$ to the plane $x = -1 + 2a$. Define $z^* = \max\{-1, \hat{y}/(-1 + 2a)\}$. It is not hard to show that $z^* < -2/3$.

Theorem 2. *For any* $t_1 \in \mathbf{R}$ *and for any solution of* (1) *with* $(x(t_1), y(t_1), z(t_1)) \in C_1(\varepsilon)$*, there exists* $t_2 > t_1$ *such that* $x(t_2) = -1+2a$*,* $y^* < y(t_2) < 1$*, and* $z^* \leq z(t_2) \leq 1$*.*

Proof of Theorem 2. Define W and W_1 (see Figure 5) by:

$$\begin{aligned}
W = &\left\{(x,y,z) \mid -\frac{9}{4} \leq x \leq -\frac{5}{4}z^* - \frac{9}{4}, y^* \leq y, -\frac{4}{5}x - \frac{9}{5} \leq z \leq 1\right\} \\
&\cup \left\{(x,y,z) \mid -\frac{5}{4}z^* - \frac{9}{4} \leq x \leq -1+2a, y^* \leq y, z^* \leq z \leq 1\right\} \\
W_1 = &\left\{(x,y,z) \mid -\frac{9}{4} \leq x \leq -1+2a, y^* \leq y, z = 1\right\} \\
&\cup \left\{\left(-\frac{9}{4}, y, z\right) \mid y^* \leq y, 0 \leq z \leq 1\right\} \\
&\cup \left\{(x,y,z) \mid -\frac{9}{4} \leq x \leq -\frac{5}{4}z^* - \frac{9}{4}, y^* \leq y, z = -\frac{4}{5}x - \frac{9}{5}\right\} \\
&\cup \left\{(x,y,z) \mid -\frac{5}{4}z^* - \frac{9}{4} \leq x \leq -1+2a, y^* \leq y, z = z^*\right\}.
\end{aligned}$$

W will serve as a gutter to funnel solutions from $C_1(\varepsilon)$ to the plane $x = -1+2a$. As in Theorem 1, we need to demonstrate that for $\vec{v} = \langle \dot{x}, \dot{y}, \dot{z} \rangle$ and $\vec{N}$ a vector normal to W_1, which is pointing into W, we have that $\vec{v} \cdot \vec{N} > 0$ along W_1. To do this, we consider each section of W_1 separately.

For example, along $\{(x,y,z) \mid -9/4 \leq x \leq -1+2a, y^* \leq y, z = 1\}$ we have that $\vec{N} = \langle 0, 0, -1 \rangle$. Therefore,

$$\begin{aligned}
\vec{v} \cdot \vec{N} = -\dot{z} &= -\frac{1}{\varepsilon}(x^2/3 - 1 - z) - e_3(t) \\
&= -\frac{1}{\varepsilon}(x^2/3 - 2) - e_3(t) \\
&\geq -\frac{1}{\varepsilon}((1/3)(-9/4)^2 - 2) - B_3 \\
&\geq \frac{1}{\varepsilon}(5/16) - B_3,
\end{aligned}$$

which is positive for ε sufficiently small. The other sections are dealt with similarly. In the case of the points of W where an inner normal is not defined, we can make arguments similar to the ones above.

Since $C_1(\varepsilon) \subset W$, we have that $(x(t_1), y(t_1), z(t_1)) \in W$. It is straightforward to show that solutions of (1), while in W, behave as demonstrated in Figure 5, that is, although solutions may at first pull away from the $x = -1 + 2a$ plane, they are eventually attracted to the plane once z is large enough. Again, this behavior holds when ε is sufficiently small. So there exists $t_2 > t_1$ such that $x(t_2) = -1 + 2a$ and $z^* \le z(t_2) \le 1$. As far as $y(t_2)$ goes, either $y(t_2) < 3/4$, or else there exists $t^* > t_1$, such that $y(t^*) = 3/4$. In this case, to show that $y(t_2) < 1$, we note that $\dot{y}(t) = -x(t) + e_2(t) < 2 + B_2$. It can then be shown without too much difficulty, using a series of estimates based on the shape and size of W, that an upper bound on $t_2 - t^*$ is $1/[4(2+B_2)]$, so we get that $y(t_2) - 3/4 = y(t_2) - y(t^*) < (2 + B_2)(t_2 - t^*) < 1/4$, so $y(t_2) < 1$. $\square$

5. Analysis, Part III. Let $Q_1(y, \varepsilon)$ be a smooth function of y and ε, defined on $[-3/2, 2/3] \times [0, \varepsilon_0]$, that satisfies $Q_1(y, \varepsilon) = \varepsilon_0^{\gamma_2}$ if $-3/2 \le y \le -\hat{y}/2$, that $dQ/dy < 0$ if $-\hat{y}/2 \le y \le 0$, and that $Q_1(y, \varepsilon) = \varepsilon^{\gamma_2}$ if $0 \le y \le 2/3$.

Define the sets (see Figures 3 and 4b):

$$\Theta_1(\varepsilon) = \{(x, y, z) \mid H_1(x, y, z) \le Q_1(y, \varepsilon), -3/2 \le y \le y^*\}$$
$$D_1(\varepsilon) = \{(x, y, z) \mid H_1(x, y, z) = Q_1(y, \varepsilon), -3/2 \le y \le y^*\}.$$

Lemma 3.1. *For any $t_0 \in \mathbf{R}$ and any solution of* (1) *with $(x(t_0), y(t_0), z(t_0)) \in \Theta_1(\varepsilon)$, there exists $t_1 \ge t_0$ such that $(x(t), y(t), z(t)) \in \Theta_1(\varepsilon)$ for $t_0 \le t \le t_1$, and $(x(t_1), y(t_1), z(t_1)) \in C_1(\varepsilon)$.*

The proof of this lemma is very similar to the proof of Theorem 1, except that now $\vec{N}$ is an inner normal to $D(\varepsilon)$, so $\vec{N} = \langle -\partial H_1/\partial x, \partial H_1/\partial y + \partial Q_1/\partial y, \partial H_1/\partial z\rangle$. The complications this change causes are minor.

We now define the set $\Theta_2(\varepsilon)$, an extension of $\Theta_1(\varepsilon)$. We wish for any solution of (1) which intersects $C_1(\varepsilon)$ to enter $\Theta_2(\varepsilon)$ and remain in $\Theta_2(\varepsilon)$ until the solution intersects the plane $x = -1 + 2a$. One way to do this is to construct $\Theta_2(\varepsilon)$ as a union of all of the solutions of (1) which intersect $C_1(\varepsilon)$.

For each initial point $(x_0, y_0, z_0) \in C_1(\varepsilon)$, and for $t_1 \in \mathbf{R}$, let $X(t; x_0, y_0, z_0, t_1)$ be the solution of (1) with $X(t_1; x_0, y_0, z_0, t_1) = (x_0, y_0, z_0)$. Define

$$\Theta_2(\varepsilon) = \bigcup_{\substack{(x_0,y_0,z_0)\in C_1(\varepsilon)\\ t_1\in(-\infty,\infty)}} \{X(t; x_0, y_0, z_0, t_1) \mid t_1 \le t \le t_2\}$$

$$C_2(\varepsilon) = \{X(t_2; x_0, y_0, z_0, t_1) \mid (x_0, y_0, z_0) \in C_1(\varepsilon), t_1 \in (-\infty, \infty)\}$$

where t_2 is defined above for each initial condition. (See Figure 3.)

In the same way, we can define the function $H_4(x, y, z)$ about the section of $\tilde{S}$ where $x > 1$, and we can construct sets $\Theta_4(\varepsilon)$, $C_4(\varepsilon)$, $\Theta_5(\varepsilon)$ and $C_5(\varepsilon)$. So we see, for instance, that the section of $\Theta_4(\varepsilon)$ for which $\hat{y}/2 \le y \le 3/2$ is defined using ε_0 instead of ε.

We have considered solutions with initial points in $\Theta_1(\varepsilon)$ and have shown that they eventually cross the plane $x = -1 + 2a$. We now wish to extend solutions past that plane. In an informal sense, the solutions will jump from the plane $x = -1 + 2a$ to the set $\Theta_4(\varepsilon)$. In order to describe this jump formally, it is necessary that a section of $\Theta_4(\varepsilon)$, which is the "target" of the jump, be defined independently of ε, as we have done. In the same way, $\Theta_1(\varepsilon)$ will be a "target" for solutions at the plane $x = 1 - 2a$.

To analyze the jump, we define the set

$$C = \{(x, y, z) \mid x = -1 + 2a, \hat{y} \le y \le 1, z^* \le z \le 1\}$$

and we transform (1) using the change of variables $t = \varepsilon\tau$.

$$\begin{aligned} \frac{dx}{d\tau} &= y - xz + \varepsilon e_1(\varepsilon\tau) \\ \frac{dy}{d\tau} &= -\varepsilon x + \varepsilon e_2(\varepsilon\tau) \\ \frac{dz}{d\tau} &= x^2/3 - 1 - z + \varepsilon e_3(\varepsilon\tau). \end{aligned} \tag{7}$$

For $\varepsilon = 0$, this is

$$\frac{dx}{d\tau} = y_0 - xz, \qquad \frac{dz}{d\tau} = x^2/3 - 1 - z \tag{8}$$

where y_0 is some fixed number.

We see that (8) resembles (6), and two facts about (8) are first, that if $y_0 > 2/3$, then there exists a unique point, $(\bar{x}_4, \bar{z}_4)$ which is a stable node of (8), with $\bar{x}_4 > 1$; and second, that there exists $T > 0$ such that for any $(x_0, y_0, z_0) \in C$ the solution of (8) with $x(\tau_2) = x_0$ and $z(\tau_2) = z_0$ satisfies $H_4(x(\tau_3), y_0, z(\tau_3)) < (1/2)\varepsilon_0^{\gamma_2}$, where $\tau_3 = \tau_2 + T$. This second fact follows from the first and because C is a compact set.

Lemma 3.2. *For any $t_2 \in \mathbf{R}$ and any solution of* (1) *with $(x(t_2), y(t_2), z(t_2)) \in C_2(\varepsilon)$, there exists $t_3 > t_2$ such that $(x(t_3), y(t_3), z(t_3)) \in \Theta_4(\varepsilon)$.*

Proof of Lemma 3.2. To prove this lemma, we use the dependence of solutions of ordinary differential equations on parameters. Let ε_1 be a number such that $0 < \varepsilon_1 \leq \varepsilon_0$ and such that if $0 < \varepsilon < \varepsilon_1$ then all of the previous lemmas hold.

We can think of (7) as defining a map:

$$\Phi : C \times \mathbf{R} \times [0, \varepsilon] \longrightarrow \mathbf{R}^3$$

as follows: For each $(x_0, y_0, z_0) \in C$, $\tau_2 \in \mathbf{R}$ and $\varepsilon = [0, \varepsilon_1]$, let

$$(x(\tau; x_0, y_0, z_0, \tau_2, \varepsilon), y(\tau; x_0, y_0, z_0, \tau_2, \varepsilon), z(\tau; x_0, y_0, z_0, \tau_2, \varepsilon))$$

be the solution of (7) with $(x(\tau_2), y(\tau_2), z(\tau_2)) = (x_0, y_0, z_0)$ and define

$$\Phi(x_0, y_0, z_0, \tau_2, \varepsilon) = (x(\tau_2 + T), y(\tau_2 + T), z(\tau_2 + T)).$$

Φ is a continuous function in all of its variables, and we have that

$$\Phi(x_0, y_0, z_0, \tau_2, 0) \in \{(x, y_0, z) \mid H_4(x, y_0, z) < (1/2)\varepsilon_0^{\gamma_2}\}.$$

Using the fact that C is compact, we have for ε sufficiently small,

$$\begin{aligned}\Phi(x_0, y_0, z_0, \tau_2, \varepsilon) \in \{(x, y, z) \mid H_4(x, y_0, z) < \varepsilon_0^{\gamma_2}, (1/2)\hat{y} < y < 3/2\} \\ \subset \Theta_4(\varepsilon).\end{aligned}$$

We once again make the change of variables $t = \varepsilon\tau$, and then we have the statement of Lemma 3.2. $\square$

Now we can define the rest of the annulus $\Theta(\varepsilon)$. For each initial point $(x_0, y_0, z_0) \in C_2(\varepsilon)$, let $X(t; x_0, y_0, z_0, t_2)$ be the solution of (1) with

$$X(t_2; x_0, y_0, z_0, t_2) = (x_0, y_0, z_0).$$

Define

$$\Theta_3(\varepsilon) = \bigcup_{\substack{(x_0,y_0,z_0)\in C_2(\varepsilon)\\ t_2\in(-\infty,\infty)}} \{X(t; x_0, y_0, z_0, t_2) \mid t_2 \le t \le t_3\}$$

where t_3 is defined above. Define $\Theta_6(\varepsilon)$, which connects $\Theta_5(\varepsilon)$ with $\Theta_1(\varepsilon)$ in a similar fashion.

We can now define the annulus $\Theta(\varepsilon)$ to be the union of the six parts. (See Figure 3.) It should be clear from the construction that solutions of (1) with initial points in the annulus, near the slow manifold, will flow through the annulus and that oscillations will occur. In the case of system (3), where the forcing terms $e_i(t)$ are all identically zero, we have an autonomous system, and using a Poincaré map, slicing through $\Theta_1(\varepsilon)$, we find that there is a nontrivial periodic solution of (3) contained in $\Theta(\varepsilon)$.

6. Application to a model of a chemical reaction and to the Nobel model. A quite similar analysis can be made for the Oregonator model of the Belousov-Zhabotinskii reaction. This reaction involves the mixing of sulfuric acid, malonic acid, cerium ammonium and sodium bromate. Rather than directly approaching a stable equilibrium, the reaction oscillates between a yellow state (which corresponds to a relatively large concentration of cerium ions) and a colorless state (when the concentration of cerium ions is small) [**10**].

The Oregonator model, developed by Field and Noyes, is a mathematical model of this reaction with three variables. In the model ξ is the concentration of the hydrogen bromate, η is the concentration of bromium ions, and ρ is the concentration of cerium ions. There are also parameters ω, h, q and p, which have the following approximate values:

$$\omega \approx 0.0002, \qquad h \approx 0.5, \qquad q \approx 0.000008, \qquad p \approx 300.$$

The model is

$$\omega \frac{d\xi}{d\tau} = \xi + \eta - q\xi^2 - \xi\eta$$

$$\frac{d\eta}{d\tau} = -\eta + 2h\rho - \xi\eta \tag{11}$$

$$p\frac{d\rho}{d\tau} = \xi - \rho$$

where ξ, η and ρ are all positive.

Hastings and Murray [**7**] showed that there exists a cube $\mathbf{B} \subset \mathbf{R}^3$ such that solutions of (11) which start in $\mathbf{B}$ or are on the boundary of $\mathbf{B}$ at $\tau = 0$ must lie entirely in $\mathbf{B}$ for $\tau > 0$. Further, they subdivided $\mathbf{B}$ into eight smaller cubes, $\mathbf{B}_i$, $i = 1, 2, \dots, 8$ and proved that solutions in $\mathbf{B}$ actually travel through six of the eight cubes in the sequential order. Hastings and Murray also demonstrated that (11) has nontrivial periodic solution.

In an analysis similar to that of (1), we can construct a trapping region for (11) and use it with a Poincaré map to prove that (11) has a nontrivial periodic solution. This improves on the result of Hastings and Murray in two ways. First, we can extend the analysis to a model with periodic forcing terms. This would be analogous to, for example, adding a certain amount of cerium ammonium to the reaction every 25 seconds. Using the analysis of (1) as a model, it can be seen that, with the forcing terms, the reaction still oscillates between a yellow and colorless state, although we cannot determine whether or not the oscillation occurs at a periodic rate. Second, in the case where the forcing terms do not exist, we have the existence of a nontrivial periodic solution, as before, but we can locate the periodic solution more precisely and in such a way that the underlying structure of the situation is better represented.

To see how we can apply our approach to the Oregonator model, we begin by making the following change of variables: $\varepsilon = 1/p$, $t = \tau/p$,

$\zeta = \omega\xi$, $c = 1/\omega$ and $k = q/\omega^2$. We get a new system:

$$\begin{aligned} \frac{d\zeta}{dt} &= \frac{1}{\varepsilon}(c\zeta + \eta - k\zeta^2 - c\zeta\eta) \\ \frac{d\eta}{dt} &= \frac{1}{\varepsilon}(-\eta + 2h\rho - c\zeta\eta) \\ \frac{d\rho}{dt} &= c\zeta - \rho \end{aligned} \tag{12}$$

where $\varepsilon \approx 1/300$, $h \approx 0.5$, $c \approx 5000$ and $k \approx 200$.

This system has three equilibrium points, one of which is

$$\begin{aligned} \zeta_0 &= \frac{1}{2ck}(\sqrt{[(2h-1)c^2 + k]^2 + 4c^2k(2h+1)} - k - (2h-1)c^2) \\ \eta_0 &= \frac{2ch\zeta_0}{1 + c\zeta_0} \\ \rho_0 &= c\zeta_0 \end{aligned}$$

ζ_0, η_0 and ρ_0 are all positive.

By setting $x = \zeta - \zeta_0$, $y = \rho_0 - \rho$ and $z = \eta - \eta_0$, we can then reformulate (12) as

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{\varepsilon}(\alpha x + \beta z - kx^2 - cxz) \\ \frac{dy}{dt} &= -cx - y \\ \frac{dz}{dt} &= \frac{1}{\varepsilon}(\gamma x + \delta z - 2hy - cxz) \end{aligned} \tag{13}$$

where

$$\begin{aligned} \alpha &= c - c\eta_0 - 2k\zeta_0, \qquad & \beta &= 1 - c\zeta_0 \\ \gamma &= -c\eta_0, & \delta &= \beta - 2 \end{aligned}$$

and where $x > -\zeta_0$, $y < \rho_0$ and $z > -\eta_0$. It is not difficult to show that α, β, γ and δ are all negative.

Finally we add forcing terms to (13) to get:

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{\varepsilon}(\alpha x + \beta z - kx^2 - cxz) + e_1(t) \\ \frac{dy}{dt} &= -cx - y + e_2(t) \\ \frac{dz}{dt} &= \frac{1}{\varepsilon}(\gamma x + \delta z - 2hy - cxz) + e_3(t). \end{aligned} \tag{14}$$

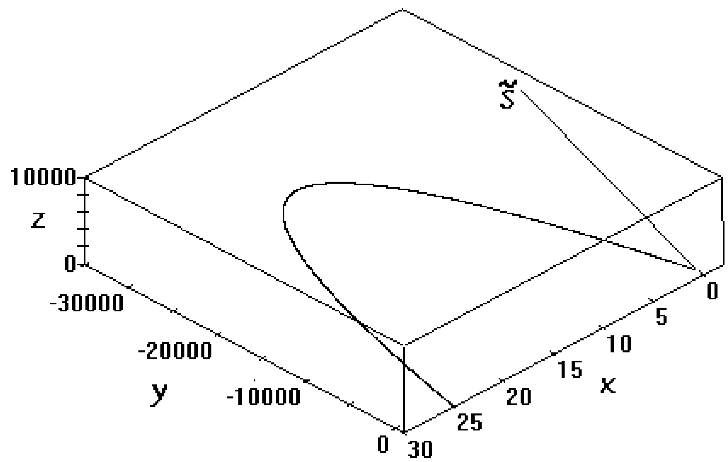

FIGURE 6. $\tilde{S}$ for (13), the Oregonator model of the Belousov-Zhabotinskii reaction.

The system (14) is quite similar to the system (1) in two key ways. First of all, if we define the slow manifold, $\tilde{S}$, by

$$\tilde{S} = \{(x, y, z) \mid \alpha x + \beta z - kx^2 - cxz = 0 \text{ and } \gamma x + \delta z - 2hy - cxz = 0\}$$

it is not difficult to show that $\tilde{S}$ is an "S-shaped" curve in $\mathbf{R}^3$. (See Figure 6.) Second, like before, we can define functions Q, R, S, T, B and C, this time by:

$$\begin{aligned} Q &= \alpha - 2k\bar{x}_1 - c\bar{z}_1 & R &= \beta - c\bar{x}_1 \\ S &= \gamma - c\bar{z}_1 & T &= \delta - c\bar{x}_1 \\ B &= \frac{T - Q}{S} & C &= \frac{Q^2 + T^2 + 2(QT - RS)}{S^2}. \end{aligned}$$

It turns out that Lemma 1.1 can be proved for these functions, too. Lemma 1.1, a technical lemma, is the key to the construction, because satisfying the lemma means that points on the slow manifold are asymptotically stable equilibrium points of what is commonly referred to as the "fast system." This is a vital criterion for constructing the trapping region $\Theta(\varepsilon)$ for (14) and should be considered when analyzing other systems similar to (1) and (14). In the case of (14), we get a trapping region similar to Figure 3. Solutions oscillate, with solutions with $y \gg 0$ corresponding to the yellow state of the reaction and solutions with $y \ll 0$ corresponding to the clear state of the reaction.

System (2), the Nobel model, is related to this work in that it, too, has an "S-shaped" slow manifold. In this case, we define the slow manifold as

$$\tilde{S} = \{(V, m, h, n) \mid F(V, m_\infty(V), h_\infty(V), n) = 0, \\ m = m_\infty(V), h = h_\infty(V)\}$$

which is a one-dimensional curve in Euclidean 4-space. The equation $F(V, m_\infty(V), h_\infty(V), n) = 0$ implicitly defines n as an "S-shaped" function of V for $-90 \leq V \leq 50$ (the physiologically significant values of V), and $m_\infty(V)$ and $h_\infty(V)$ are well-behaved, monotonic functions of V. The attracting parts of this slow manifold are the sections where $V < -77$ or $V > -20$, and the behavior of solutions of (2) near these parts of the slow manifold is similar to the behavior described earlier for $\Theta_1(\varepsilon)$ and $\Theta_4(\varepsilon)$.

The "fast areas" of the system are where $-77 < V < -20$, and in this area n is nearly constant with respect to t (approximately either 0.415 or 0.6997, corresponding in (3) to y equaling $\pm 2/3$). The other variables are determined by

$$\begin{aligned} \varepsilon \frac{dV}{dt} &= \frac{1}{\varepsilon'}(F(V, m, h, n)) \\ \varepsilon \frac{dm}{dt} &= \frac{1}{\varepsilon'} \frac{m_\infty(V) - m}{T_m(V)} \\ \varepsilon \frac{dh}{dt} &= \frac{h_\infty(V) - h}{T_h(V)} \end{aligned} \tag{15}$$

which is also a singular-perturbation problem, this time with perturbation parameter ε'. It is not difficult to show that, for small ε', solutions of (15) are attracted to a stable equilibrium point in a way similar to what was described in Section 5 of this paper so that, like solutions of (1), solutions of (2), either with or without forcing terms, oscillate between the "attracting" parts of $\tilde{S}$ and the "fast areas." For more details of this analysis of the Nobel mode, see Cronin [**4**].

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