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# ON SOME QUASILINEAR SYSTEMS

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**0. Introduction.** In this paper we will be interested in systems of quasilinear equations whose action-functional is strongly indefinite. The general problem is the following

$$\text{(P)} \qquad \begin{cases} -\Delta_p u = F_u(u,v) & \text{in } \Omega \\ \Delta_q v = F_v(u,v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary, $N > 2$, $1 < p$, $q < N$ and $F$ is a $\mathcal{C}^1$-function. On the function $F$ we impose the following

**Coupling condition** (C). *The equations* $F_u(0,v) = 0$ *and* $F_v(u,0) = 0$ *have only finitely many solutions.*

We shall see later on that, due to this coupling condition, it is impossible for the Problem (P) to have solutions of the form $(u,0)$ or $(0,v)$.

Our goal is to give two methods for finding solutions of Problem (P). Section 1 will be devoted to studying Problem (P) by means of the reduction to a semilinear system following the ideas by [**8, 9**]. Here the method consists of a convenient splitting and using a duality argument (see [**7**] and [**5**]). The remaining sections will be dealing with a more general case using a Galerkin type argument in combination with the finite dimensional linking theorem (see [**1, 14**]). The Galerkin approach was previously used by other authors [**3, 11, 14**].

**1. The splitting method.** Due to technical obstructions, we study a particular case where we assume $2N/(N+2) < p < 2$, $q = 2$. Notice that the equations of Problem (P) are the Euler Lagrange equations of

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the functional

$$J_p(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(u,v)\,dx.$$

The functional $J_p$ is defined on the Banach space $W_0^{1,p}(\Omega)\times W_0^{1,2}(\Omega)$ and clearly $J_p$ is strongly indefinite, which makes it difficult to apply standard critical point theory for finding solutions of Problem (P). We shall, however, rewrite the equations of Problem (P) in such a way that the problem becomes semilinear and stated in a Hilbert space, which simplifies matters. The system, however, keeps it a strongly indefinite structure. Problem (P) can be equivalently formulated as follows. Define

$$w = |\nabla u|^{p-2}\nabla u;$$

consequently,

$$\nabla u = |w|^{(2-p)/(p-1)}w.$$

Then problem (P) becomes

$$\begin{cases} -\operatorname{div} w = F_u(u,v) \\ \Delta v = F_v(u,v) \\ \nabla u = |w|^{(2-p)/(p-1)}w \end{cases}. \tag{1.1}$$

In order to find weak solutions of Problem (P), we shall study (1.1), since these two formulations are equivalent. This splitting of the $p$-Laplacian allows us to consider the linear differential operator

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & -\operatorname{div} \\ 0 & \Delta & 0 \\ \nabla & 0 & 0 \end{pmatrix}. \tag{1.2}$$

It is very important to observe that $\operatorname{Ker}(\mathcal{A}) = N(\mathcal{A})$ is infinite dimensional.

*The functional analytic setting.* The operator $\mathcal{A}$ will be considered in the following functional analytic context, see Clément-van der Vorst [**8**] for notation,

(1) $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ where $H = L^2(\Omega)\times L^2(\Omega)\times (L^2(\Omega))^N$ and

$$D(\mathcal{A}) = H_0^1(\Omega)\times H^2\cap H_0^1(\Omega)\times H_{\operatorname{div}}(\Omega)$$

and the domain of the divergence operator is

$$\begin{aligned}H_{\mathrm{div}}(\Omega) = \{v \in (L^2(\Omega))^N \mid v = v_1 + v_2,\\ \operatorname{div} v_1 = 0, v_2 = \nabla u, u \in H^2 \cap H_0^1(\Omega)\}.\end{aligned}$$

The operator $\mathcal{A}$ is selfadjoint.

(2) With the scalar product in the Hilbert space $H$ and the operator $\mathcal{A}$ we have the following strongly indefinite quadratic form

$$\frac{1}{2}(\mathcal{A}\phi, \phi)_H = \int_\Omega w\nabla u\, dx - \frac{1}{2}\int_\Omega |\nabla v|^2\, dx \equiv Q(\phi). \tag{1.3}$$

Here $\phi = (u, v, w)^t$.

Performing integration by parts we have continuous extensions of $Q$ to:

(1) $E_0 = L^2(\Omega) \times H_0^1(\Omega) \times H_{\mathrm{div}}(\Omega)$,

(2) $E_1 = H_0^1(\Omega) \times H_0^1(\Omega) \times (L^2(\Omega))^N$.

Then by interpolation between $E_0$ and $E_1$, see [**8**], we obtain continuous extensions of $Q$ to the one-parameter family of spaces:

$$E_\alpha = [E_0, E_1]_\alpha = \Theta_1^\alpha(\Omega) \times H_0^1(\Omega) \times \Theta_3^{1-\alpha}(\Omega),$$

where

$$\begin{aligned}\Theta_1^\alpha(\Omega) &= D(\Delta^{a/2}) = \{u \in L^2(\Omega); (-\Delta)^{\alpha/2}u \in L^2(\Omega)\}\\ &= \begin{cases} H^\alpha(\Omega) & 0 \le \alpha < 1/2,\\ H_{00}^\alpha(\Omega) & \alpha = 1/2,\\ H^\alpha(\Omega) & 1/2 < \alpha \le 1,\end{cases}\end{aligned}$$

and

$$\begin{aligned}\Theta_3^{1-\alpha}(\Omega) = \{v \in (L^2(\Omega))^N;\ v = v_1 + v_2,\ \operatorname{div} v_1 = 0,\\ v_2 = \nabla u,\ u \in H^{2-\alpha} \cap H_0^1(\Omega)\}.\end{aligned}$$

On $\Theta_1^\alpha(\Omega)$ and $\Theta_3^{1-\alpha}(\Omega)$ we have the usual equivalent norms $\|\cdot\|_{\Theta_1^\alpha}$ and $\|\cdot\|_{\Theta_3^{1-\alpha}}$ and, therefore, $\|\cdot\|_{E_\alpha}^2 = \|\cdot\|_{\Theta_1^\alpha}^2 + \|\cdot\|_{H_0^1}^2 + \|\cdot\|_{\Theta_3^{1-\alpha}}^2$.

As in [**8**] we can associate with $\mathcal{A}$ a selfadjoint partial isometry $L$ such that

$$\frac{1}{2}\langle A\phi,\phi\rangle = \frac{1}{2}(L\phi,\phi)_{E_\alpha}.$$

We have the decomposition of $E_\alpha = E^+ \oplus E^- \oplus E^0$, where $E^+, E^-$ and $E^0$ are the eigenspaces of the operator $L$ associated with the eigenvalues $\lambda = 1$, $\lambda = -1$ and $\lambda = 0$.

Let us consider the operator

$$\partial_\nabla = \begin{pmatrix} 0 & -\mathrm{div} \\ \nabla & 0 \end{pmatrix} \tag{1.4}$$

on $\Theta_1^\alpha(\Omega) \times \Theta_3^{1-\alpha}(\Omega)$, with associated operator $L_\alpha$, see [**8**].

With the eigenvalues $\lambda = 1,\ -1,\ 0$ of $L_\alpha$, we have the eigenspaces $E_r^+$, $E_r^-$ and $E_r^0 = N(\mathrm{div}\,)$. Using the latter, we can characterize the spaces $E^+, E^-$ and $E^0$;

(1) $E^+ = \{\phi \in E_\alpha; (u,w) \in E_r^+, v = 0\}$

(2) $E^- = \{\phi \in E_\alpha; (u,w) \in E_r^-, v \in H_0^1(\Omega)\}$

(3) $E_0 = \{\phi \in E_\alpha; (u,w) \in E_r^0, v = 0\}$.

With this decomposition of $E_\alpha$ we have

(i) $\mathcal{Q} > 0$ in $E^+$,

(ii) $\mathcal{Q} < 0$ in $E^-$,

(iii) $\mathcal{Q} = 0$ in $E^0$.

*Sobolev embeddings.* In the last section we saw that the quadratic form $Q$ is well-defined on the one-parameter family of interpolation spaces $E_\alpha$, $\alpha \in [0,1]$. If we include the nonlinear part of Problem (P), we need embeddings of $E_\alpha$ in proper $L^p$-spaces, the so-called Sobolev embeddings. Using the characterization of $E_\alpha$ in terms of Sobolev spaces, we have from the Sobolev embeddings:

$$\begin{cases} E^+ \oplus E^- \hookrightarrow L^{r_1+1} \times L^{r_2+1} \times L^{r_3+1} \equiv X \\ r_1 + 1 < \dfrac{2N}{N-2\alpha}, \qquad r_2 + 1 < \dfrac{2N}{N-2}, \\ r_3 + 1 < \dfrac{2N}{N-2+2\alpha}. \end{cases}$$

For $r_1$ and $r_3$, this yields the relation

$$\frac{1}{r_1+1}+\frac{1}{r_3+1}>\frac{N-1}{N}.$$

Furthermore, $E_0 \subset (L^2)^N$ and, see, e.g., [**8**],

(i) $E_0 \cap X$ is closed in $X$,

(ii) $E^+ \oplus E^- \oplus (E_0 \cap X)$ is dense in $X$,

(iii) $(E_0 \cap X)$ is dense in $E_0$.

*Formulation of the Hamiltonian.* System (1.1) has, of course, a variational structure, and the equations of System (1.1) are the Euler-Lagrange equations of the function

$$\mathcal{L}(\phi)=\frac{1}{2}\langle \mathcal{A}\phi,\phi\rangle-\mathcal{H}(\phi)=Q(\phi)-\mathcal{H}(\phi),$$

where $Q$ is defined above and $\mathcal{H}$ is given by

$$\mathcal{H}(\phi)=\frac{p-1}{p}\int_\Omega |w|^{p/(p-1)}\,dx+\int_\Omega F(u,v)\,dx.$$

We assume that $F$ is a smooth function which is strictly convex and meets the following growth conditions

(1) $F_u, F_v$ are strictly monotone with respect to the preorder in $\mathbf{R}^2$.

(2) $c_1|u|^{r_1} \le |F_u(u,v)| \le c_2|u|^{r_1}+\sum |u|^{a-1}|v|^b$,

(3) $c_3|v|^{r_3} \le |F_v(u,v)| \le c_4|v|^{r_3}+\sum |u|^{a}|v|^{b-1}$,

(4) $uF_u+vF_v-\gamma F \ge A(|u|^{r_1}+|v|^{r_3})-B$ for $\gamma>2$ with

$$\frac{a}{r_1+1}+\frac{b}{r_3+1}<1$$

and $r_1, r_2, r_3$ subcritical. On the other hand, we need that for fixed $p$ the $w$-component in $\mathcal{H}$ must satisfy

$$\frac{p}{p-1}<\frac{2N}{N-2+2\alpha}\quad \text{for some } 0<\alpha<1, \tag{1.5}$$

which can indeed be met when $2N/(N+2)<p<2$.

In this way we obtain that $\mathcal{H}$ is well-defined and strictly convex on $E_\alpha$ for some appropriate $\alpha \in (0,1)$, depending on the growth of the function $F(u,v)$. We have that

$$\mathcal{L}(\phi) = \mathcal{Q}(\phi) - \mathcal{H}(\phi)$$

is a proper $C^1$-functional on $E_\alpha$. We are now in a position to give an existence result using an abstract critical point theorem due to Benci and Fortunato [**5**], which is based on a dual method, see [**7**].

**Theorem 1.1.** *Suppose p verifies* $2N/(N+2) < p < 2$ *and* $F$ *satisfies the growth conditions mentioned above. Then Problem* (P) *has at least one nontrivial solution. Moreover, if* $F(u,v)$ *is even, Problem* (P) *has infinitely many nontrivial solutions.*

*Proof of Theorem* 1.1. The Sobolev embeddings and the properties of the Hamiltonian allow us to consider the idea of combining duality with critical point theory due to Clarke and Ekeland [**7**]. See also Benci and Fortunato [**5**], where this method is developed. For the sake of completeness, we include the idea of the proof. More precisely, consider $\mathcal{H}^*$ the Legendre transform of $\mathcal{H}$ and $\mathcal{K}$ compact extension of the inverse of $\mathcal{A}$, i.e.,

$$\mathcal{K} : E_\alpha^* \overset{i^*}{\hookrightarrow} R(\mathcal{A})^* \overset{\text{Riesz}}{\equiv} R(\alpha) \overset{(\mathcal{A})^{-1}}{\longrightarrow} R(\mathcal{A}) \overset{i}{\hookrightarrow} E^\alpha.$$

Then the new functional is defined for $\psi \in W = \overline{(E^+)^*} \oplus \overline{(E^-)^*}$ and given by

$$l(\phi) = \mathcal{H}^*(\psi) - \frac{1}{2}\langle w, \mathcal{K}(w)\rangle.$$

We can apply the mountain pass theorem in [**2**] and we get a critical point $w \in W$. This critical point verifies

$$\mathcal{K}(w) = \mathcal{P} d\mathcal{H}^*(w)$$

where the last equality is in $R(\mathcal{A})$ and $\mathcal{P}$ is the projection. This implies the existence of a $w_0 \in \overline{(E_0 \cap \mathcal{X})}$ such that $w_0 = \mathcal{K}(w) = d\mathcal{H}^*(w)$. Call $\phi = w_0 + \mathcal{K}(w)$, then we get $\mathcal{A}(\phi) = d\mathcal{H}(\phi)$, which is equivalent to system (1.1). For the rest of the details, see [**4, 5**] and [**8**].

*Remark* 1.2. The case $q \neq 2$ and $2 > p \geq 2N/(N+2)$ can be handled in a similar way.

The case $q \neq 2$ and $1 < p \leq 2N/(N-2)$ or $p \geq 2$ can also be considered. For example, by making an additional splitting for the $v$-equation (one obtains a semilinear system of four equations, $\partial_T$ system). The proper functional analytic setting in a Banach space can be found in [**9**]. A slightly different system was considered there but the main ideas remain the same. The method for finding critical points seems more delicate in this case, and the duality method does not seem straightforward since the Hamiltonian will no longer be convex. It remains an open question how to find a critical point for this semilinear system.

**2. The Galerkin approach.** In this section we consider the following particular system, that is, we take $F$ to be

$$F(u,v) = G(u,v) - \frac{1}{r+1}|u|^{r+1} - \frac{1}{s+1}|v|^{s+1}.$$

Without using the splitting as described in Section 1, we search for solutions of Problem (P) by means of a Galerkin type method. The problem is

$$(2.1) \qquad \begin{cases} -\Delta_p u = |u|^{r-1}u + G_u(u,v) & \text{in } \Omega, \\ \Delta_q v = |v|^{s-1}v + G_v(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

The numbers $r$ and $s$ satisfy $p \leq r+1 < pN/(N-p)$ and $q \leq s+1 < qN/(N-q)$. On $G$ we impose the following conditions:

(g1) $G \in \mathcal{C}^1$, $G_u(0,0) = 0 = G_v(0,0)$ and $G \geq 0$,

(g2) $|\nabla G(u,v)| \leq c_1 + c_2|u|^d + c_3|v|^d$, where $\max(p,q) < d+1 \leq \min(r+1, s+1)$,

(g3) $|G_u(u,0)| = o(|u|^{p-1})$, as $|u| \to 0$,

(g4) $0 < G(u,v) < (1/p)uG_u + (1/q)vG_v$ for $|u|, |v| \geq \eta > 0$.

(g5) *Coupling condition.* The equations $G_u(0,v) = 0$ and $G_v(u,0) = 0$ possess only finitely many solutions.

We have

**Theorem 2.1.** *Let $\Omega$, $p$, $q$, $r$ and $s$ as before, and suppose that $G$ satisfies the conditions* (g1)–(g5). *Then Problem* (2.1) *has at least one weak solution* $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ *with nontrivial components.*

The Lagrangian associated with Problem (2.1) is given by

$$J(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \frac{1}{q}\int_\Omega |\nabla u|^q \, dx - \int_\Omega G(u,v)\, dx - \frac{1}{r+1}\int_\Omega |u|^{r+1} - \frac{1}{s+1}\int_\Omega |v|^{s+1}. \tag{2.2}$$

By conditions (g1) and (g2), $J$ is a $\mathcal{C}^1$-functional on the Banach space

$$\mathbf{E} = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega).$$

Weak solutions must be understood as critical points of this functional.

With the hypothesis (g1)–(g5), we do not pursue full generality since one can allow for $G$ to satisfy less restrictive growth conditions. For instance, (g2) can be generalized. A good example for $G$ can be a function $G(u,v) \equiv h(u+v)$ with $h(t)$ satisfying similar conditions. The strategy of the proof is as follows:

1) Consider a family of projected problems on a finite dimensional family of subspaces, and we look for positive critical values by means of the linking theorem [**1, 14**].

2) We prove a compactness property which is stronger than the usual Palais-Smale condition: the (PS)*-condition.

3) Using the latter condition we pass to the limit and we find a positive critical value of $J$.

## 3. The proof of Theorem 2.1.

The space $\mathbf{E}$ is a separable Banach space and therefore there exists a filtering of finite dimensional subspaces such that

1) $E_n = E_n^u \times E_n^v$

2) $E_n \subset E_{n+1}$

3) $\overline{\cup E_n} = \mathbf{E}$. The spaces $E_n^u$ and $E_n^v$ are considered to be the same.

The restriction of $J$ to $E_n$ is denoted by $J_n$.

**Definition 3.1.** We say that the functional $J$ satisfies the $(PS)^*$-condition if, for any sequence $\{z_n\} \subset \mathbf{E}$, with $z_n \in E_n$, such that $|J(z_n)| \le C$ and $J_n'(z_n) \to 0$, implies that there exists a convergent subsequence and the limit is a *critical point* for $J$ (see also [**3**]).

**Lemma 3.2.** *Let $\Omega$, $p$, $q$, $r$ and $s$ be as before, and suppose that $G$ satisfies* (g1), (g2) *and* (g4). *Then the functional defined by* (2.1) *satisfies condition $(PS)^*$.*

*Proof of Lemma* 3.2. First we observe the following.

Let $\tilde{z}_n$ be a vector in $\mathbf{E}$ such that $P_n\tilde{z}_n = z_n$ is the vector $z_n$ seen as an element of $\mathbf{E}$. Then

$$J_n(z_n) = J(P_n\tilde{z}_n).$$

Now

$$\langle J_n'(z_n), \phi_n\rangle = \langle J'(P_n\tilde{z}_n), P_n\tilde{\phi}_n\rangle, \quad \forall\, \phi_n \in E_n,$$

and thus if we write $P_n\tilde{z}_n = z_n$ and $P_n\tilde{\phi}_n = \phi_n$ as a vector in $\mathbf{E}$, we have

$$\langle J_n'(z_n), \phi_n\rangle = \langle J'(z_n), \phi_n\rangle.$$

Suppose $\{z_n\} \subset \mathbf{E}$ is a sequence as indicated in Definition 3.1. Then

(3.0)

$$\begin{aligned} C + \varepsilon\|z_n\|_{\mathbf{E}} &\le J(z_n) - \left\langle J'(z_n), \left(\frac{1}{p}u_n, \frac{1}{q}v_n\right)\right\rangle \\ &= \left(\frac{1}{p} - \frac{1}{r+1}\right)\int_\Omega |u_n|^{r+1} \\ &\quad + \left(\frac{1}{q} - \frac{1}{s+1}\right)\int_\Omega |v_n|^{s+1} \\ &\quad + \frac{1}{p}\int_\Omega u_n G_u(u_n, v_n) + \frac{1}{q}\int_\Omega v_n G_v(u_n, v_n) \\ &\quad - \int_\Omega G(u_n, v_n) \end{aligned}$$

$$\geq \left(\frac{1}{p} - \frac{1}{r+1}\right) \int_\Omega |u_n|^{r+1} + \left(\frac{1}{q} - \frac{1}{s+1}\right) \int_\Omega |v_n|^{s+1} - C.$$

Furthermore,

$$\left| \int_\Omega |\nabla u_n|^p - \int_\Omega |u_n|^{r+1} - \int_\Omega u_n G_u(u_n, v_n) \right| \leq \varepsilon \|u_n\|_{W_0^{1,p}(\Omega)} \tag{3.1}$$

$$\left| \int_\Omega |\nabla v_n|^q - \int_\Omega |v_n|^{s+1} - \int_\Omega v_n G_v(u_n, v_n) \right| \leq \varepsilon \|v_n\|_{W_0^{1,q}(\Omega)}. \tag{3.2}$$

We continue with the inequality (3.1),

$$\begin{aligned} \int_\Omega |\nabla u_n|^p - \varepsilon \|u_n\|_{W_0^{1,p}(\Omega)} &\leq \left| \int_\Omega |u_n|^{r+1} + \int_\Omega u_n G_u(u_n, v_n) \right| \\ &\leq \int_\Omega |u_n|^{r+1} + \int_\Omega |u_n| |G_u(u_n, v_n)| \\ &\leq \int_\Omega |u_n|^{r+1} + c \int_\Omega |u_n| \\ &\quad + C \int_\Omega |u_n| (|u_n|^{d+1} + |v_n|^{d+1}) \end{aligned} \tag{3.3}$$

and, similarly for (3.2),

$$\begin{aligned} \int_\Omega |\nabla v_n|^q - \varepsilon \|v_n\|_{W_0^{1,q}(\Omega)} &\leq \int_\Omega |v_n|^{s+1} + c \int_\Omega |v_n| \\ &\quad + C \int_\Omega |v_n| (|u_n|^{d+1} + |v_n|^{d+1}). \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) and using (g2) yields

$$\begin{aligned} \int_\Omega |\nabla u_n|^p + \int_\Omega |\nabla v_n|^q - \varepsilon \|z_n\|_{\mathbf{E}} &\leq \int_\Omega |u_n|^{r+1} + \int_\Omega |v_n|^{s+1} + c \int_\Omega (|u_n| + |v_n|) \\ &\quad + C \int_\Omega (|u_n|^{d+1} + |v_n|^{d+1}) \\ &\leq C + C \int_\Omega |u_n|^{d+1} + \int_\Omega |v_n|^{d+1}. \end{aligned}$$

Eventually, this gives

$$\|z_n\|_{\mathbf{E}}^{\alpha} - \varepsilon\|z_n\|_{\mathbf{E}} \le C + C\int_\Omega |u_n|^{d+1} + \int_\Omega |v_n|^{d+1} \tag{3.5}$$

for some $\alpha > 1$. Next we substitute (3.0) in (3.5), which gives

$$\|z_n\|_{\mathbf{E}}^{\alpha} - \varepsilon\|z_n\|_{\mathbf{E}} \le C + \varepsilon\|z_n\|_{\mathbf{E}},$$

proving that $\|z_n\|_{\mathbf{E}} \le C$. Because $\mathbf{E}$ is compactly embedded into $L^{r+1}(\Omega) \times L^{s+1}(\Omega)$, this provides the following statements

(i) $z_n \rightharpoonup z$ weakly in $\mathbf{E}$

(ii) $z_n \to z$ strongly in $L^{r+1}(\Omega) \times L^{s+1}(\Omega)$

(iii) $z_n \to z$ almost everywhere.

Because $\mathbf{E}$ is uniformly convex and $\|z_n\|_{\mathbf{E}} \to \|z\|_{\mathbf{E}}$, we obtain a strong convergence of $\{z_n\}$ in $\mathbf{E}$. Thus,

$$J(z_n) \to J(z) = c.$$

Finally we need to verify whether or not $z$ is a critical point of $J$. We fix a number $M$. Then

$$|\langle J'(z_n), \phi_m\rangle| \le \varepsilon_n\|\phi_m\|$$

for $m \le M \le n$, provided $M$ is large, and $\varepsilon_n \to 0$ as $n \to \infty$. Along a subsequence, this yields

$$\langle J'(z), \phi_m\rangle = 0, \quad \forall\, \phi_m \in P_m(\mathbf{E}).$$

The latter holds for all $M \in \mathbf{N}$. Since $\cup E_n$ is dense in $\mathbf{E}$, we get that

$$\langle J'(z), \phi\rangle = 0, \quad \forall\, \phi \in \mathbf{E},$$

which proves the lemma. □

*Remark* 3.3. If we take a (PS)-sequence in a fixed $E_n$, it follows from the proof of Lemma 3.2 that the functionals $J_n$ satisfy the (PS)-condition in any finite dimensional subspace $E_n$. This observation will be useful later on.

Next we investigate some geometric properties of $J$ in order to find critical points by means of a mini-max characterization. Define

$$\mathcal{S} = \{(u,0); \|u\|_{W_0^{1,p}(\Omega)} = \rho\}.$$

Let $e^+ \in E_1^+$ with $\|e^+\| = 1$ and fixed $\rho, r_1, r_2 > 0, r_1 > \rho$. Consider

$$\begin{aligned}\mathcal{Q} =&\{(te^+, v); e^+ \in W_0^{1,p}(\Omega),\ 0 \le t \le r_1,\\ &v \in W_0^{1,q}(\Omega), \|v\|_{W_0^{1,q}(\Omega)} \le r_2\}.\end{aligned}$$

Then $\partial\mathcal{Q}$ is given by

$$\begin{aligned}\{(0,w)|; |v\|_{W_0^{1,q}(\Omega)} \le r_2\} \cup \{(r_1e^+, w); \|v\|_{W_0^{1,q}(\Omega)} < r_2\}\\ \cup \{(te^+, w); 0 \le t \le r_1, \|v\|_{W_0^{1,q}(\Omega)} = r_2\}.\end{aligned}$$

We can prove the following lemma.

**Lemma 3.4.** *Let $\Omega, p, q, r$ and $s$ be as before, and suppose that $G$ satisfies* (g1)–(g3). *Then there are numbers $0 < \rho < r_1$ and $r_1 \le r_2$ such that*

$$J|_{\mathcal{S}} \ge \alpha > 0, \qquad J|_{\partial\mathcal{Q}} \le 0, \qquad J|_{\mathcal{Q}} < \infty.$$

*Proof of Lemma* 3.4. We start with $\mathcal{S}$:

$$J(u,0) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega G(u,0)\,dx - \frac{1}{r+1}\int_\Omega |u|^{r+1}\,dx,$$

then

$$\begin{aligned}J(u,0) \ge \frac{1}{p}\int_\Omega |\nabla u|^p\,dx&\\ -\varepsilon\int_\Omega |u|^p\,dx - C(\varepsilon)\int_\Omega |u|^{d+1}\,dx&\\ -C\bigg(\int_\Omega |\nabla u|^p\,dx\bigg)^{(r+1)/p}& \ge \alpha > 0,\end{aligned}$$

provided $\rho > 0$ is small enough.

As for $\partial\mathcal{Q}$ we proceed as follows.

a)

$$J(0,v) = -\frac{1}{q}\int_\Omega |\nabla u|^q\,dx - \int_\Omega G(0,v)\,dx - \frac{1}{s+1}\int_\Omega |v|^{s+1} \le 0.$$

b)

$$\begin{aligned}
J(r_1e^+,v) &= \frac{1}{p}r_1^p - \frac{1}{q}\int_\Omega |\nabla v|^q\,dx \\
&\quad - \int_\Omega G(r_1e^+,v)\,dx \\
&\quad - \frac{1}{r+1}\int_\Omega |r_1e^+|^{r+1} - \frac{1}{s+1}\int_\Omega |v|^{s+1} \\
&\le \frac{1}{p}r_1^p - c\frac{1}{r+1}r_1^{r+1} - \frac{1}{q}\int_\Omega |\nabla v|^q\,dx \\
&\quad - \int_\Omega G(r_1e^+,v)\,dx - \frac{1}{s+1}\int_\Omega |v|^{s+1} \\
&\le \frac{1}{p}r_1^p - c\frac{1}{r+1}r_1^{r+1} \le 0
\end{aligned}$$

if $r_1 > 0$ is sufficiently large.

c)

$$\begin{aligned}
J(te^+,v) &= \frac{1}{p}t^p - \frac{1}{q}r_2^q - \int_\Omega G(te^+,v)\,dx \\
&\quad - \frac{1}{r+1}\int_\Omega |te^+|^{r+1} - \frac{1}{s+1}\int_\Omega |v|^{s+1} \\
&\le \frac{1}{p}t^p - \frac{1}{q}r_2^q - \int_\Omega G(te^+,v)\,dx \le 0
\end{aligned}$$

if $r_2 \ge r_1$.

Finally,

$$J(te^+,v) \le \frac{1}{p}r_1^p < \infty$$

and

$$J(z) > -\infty, \quad z \in \mathcal{Q}.$$

This completes the proof. □

We observe now that, by the choice of the subspaces $E_n$ the geometric properties of $J$ are preserved after the restriction to $E_n$. To be more precise:

(i) $S_n = \mathcal{S} \cap E_n$,

(ii) $Q_n = \mathcal{Q} \cap E_n$ and

(iii) $J_n|_{S_n} \geq \alpha > 0$, $J_n|_{\partial Q_n} \leq 0$, $J_n|_{Q_n} < \infty$.

*Proof of Theorem* 2.1. From Remark 3.3 it follows that $J_n$ satisfies (PS) for every $n \geq 2$ and for Lemma 3.4 it follows that $J_n$ satisfies the conditions of the linking theorem, see [**1, 4** and **14**] and this gives that, for every $n$, $J_n$ has a *nontrivial critical point*, $z_n \in E_n$ with critical value

$$0 < \alpha \leq J_n(z_n) = c_n \leq C.$$

We thus have a sequence $\{z_n\} \subset E_n$ for which $\|J_n'(z_n)\| = 0$ and $|J_n(z_n)| \leq C$. Then since $J$ satisfies the (PS)$^*$-condition, it follows that there exists a subsequence $\{z_{n_k}\} \subset \mathbf{E}$ converging to $z \in \mathbf{E}$, which is a nontrivial critical point of $J$, $u \not\equiv 0$ or $v \not\equiv 0$.

As for the proof that both components are nontrivial, we argue as follows. Suppose that $v \equiv 0$. Then $u$ satisfies the equation

$$-\Delta_p u = |u|^r u + G_u(u, 0).$$

From regularity theory for this equation, see [**10, 15**], it follows that $u \in C^0(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$. On the other hand, we know from (g5) that the equation

$$G_v(u, 0) = 0,$$

has only finitely many solutions and thus, due to the continuity of $u$, this gives that $u = C$, a constant, in $\overline{\Omega}$. The boundary condition on $u$ then determines the constant to be zero, which would imply that $(u, v) \equiv (0, 0)$, a contradiction. The same argument holds for $v$. This proves that the solution has nontrivial components. □

## REFERENCES

**1.** A. Ambrosetti, *Critical points and nonlinear variational problems*, Mémoire No. 49, Société Mathématique de France **120** (1992).

**2.** A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

**3.** A. Bahri and H. Berestycki, *Existence of forced oscillations for some nonlinear differential equations*, Comm. Pure Appl. Math. **37** (1984), 403–442.

**4.** P. Bartolo, V. Benci and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity*, J. Nonlinear Anal. **9** (1983), 981–1012.

**5.** V. Benci and D. Fortunato, *The dual method in critical point theory. Multiplicity results for indefinite functional*, Ann. Math. Pura Appl. **34** (1982), 215–242.

**6.** H. Brezis, J.M. Coron and L. Nirenberg, *Free vibrations for a nonlinear wave equations and a theorem of P. Rabinowitz*, Comm. Pure Appl. Math. **33** (1980), 667–684.

**7.** F.H. Clarke and I. Ekeland, *Hamiltonian trajectories having prescribed minimal period*, Comm. Pure Appl. Math. **33** (1980), 103–116.

**8.** P. Clement and R. van der Vorst, *Interpolation spaces for $\partial_T$-systems and applications to critical points theory*, Panamer. Math. J. **4** (1994), 1–45.

**9.** ———, *On a semilinear elliptic system*, Differential Integral Equations **8** (1995), 1317–1329.

**10.** E. Di Benedetto, $\mathcal{C}^{1,a}$ *local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), 827–850.

**11.** G. Fournier, D. Lupo, M. Ramos and M. Willem, *Limit relative category and critical point theory*, preprint.

**12.** J. Hulshof and R. van der Vorst, *Differential systems with strongly indefinite functional*, J. Funct. Anal. **114** (1993), 32–58.

**13.** J. Ne**C**cas, *Les methodes directes en Theorie des Equations Elliptiques*, Acad. Tchescoslovaque des Sciences Praga, 1967.

**14.** P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, American Math. Society, 1984.

**15.** P. Tolksdorff, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126–150.

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