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ON QUASI-POSITIVE DEFINITE FUNCTIONS AND REPRESENTATIONS OF HYPERGROUPS IN QP_n SPACES

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ABSTRACT. The purpose of this paper is to transfer the results about the relation between quasi-positive definite functions on groups and semigroups and their representations in spaces with an indefinite metric to the case of hypergroups. This work may be considered to continue the studies concerning the generalizations of Godement's theory about cyclic unitary representations of locally compact groups in Hilbert spaces and positive definite functions [**3**].

0. Introduction. The relation between cyclic unitary representations of a topological group G in Pontryagin spaces and quasi-positive definite functions on G has been thoroughly investigated by K. Sakai [**7**]. In [**1**], C. Berg and Z. Sasvári have studied indefinite functions on semigroups and their relation to representations in spaces with an indefinite metric. The present paper starts from these two articles, which suggested that the same problem can be transferred to the case of hypergroups. The main difficulty of this program consists in defining quasi-positive functions on hypergroups such that this definition becomes compatible with the ones for groups and semigroups. In addition, one should be able to relate them to representations of hypergroups in indefinite spaces. In Section 1 we define quasi-positive functions on hypergroups and we construct the indefinite space (and finally the Pontryagin space) associated with a quasi-positive function. In Section 2 we relate such functions to representations in indefinite spaces. In the last section we give examples of quasi-positive functions by starting with the study of functions of finite rank.

Hypergroups are locally compact Hausdorff spaces whose regular complex-valued Borel measures form an algebra, which has similar properties as the convolution algebra $(\mathcal{M}(G), *)$ of a topological group

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G. For basic references, one can consult [**5**] or [**8**]. Hypergroups naturally arise as double coset spaces of locally compact groups by compact subgroups. Our notation concerning hypergroups will in general agree with Jewett's in [**5**]. In particular, K always stands for a hypergroup (same as convo in [**5**]). However, we shall denote by $\mathcal{C}(K)$ the bounded continuous complex-valued functions on K and by $\mathcal{M}(K)$ the bounded regular complex-valued Borel measures on K. Throughout this paper $\mathcal{M}^+(K)$ (the nonnegative elements of $\mathcal{M}(K)$) is endowed with the cone topology, see [**5**, 2.2]. It is known [**5**, Lemma 2.2A and 2.2B] that the mapping $x \mapsto p_x$ (the unit point mass at x) is a homeomorphism of K onto a closed subset of $\mathcal{M}^+(K)$ and that the set of measures in $\mathcal{M}^+(K)$ with finite support is dense in $\mathcal{M}^+(K)$. If μ is a measure and f is a μ-integrable function we denote by $\int_K f\,d\mu$ or $\mu(f)$ the integral of f with respect to μ. If f is a Borel function on K and $x, y \in K$, we define

$$_xf(y) = f(x * y) = \int_K f\,d(p_x * p_y)$$

if this integral exists. By [**5**, Lemma 6.1F], if μ and ν are in $\mathcal{M}(K)$ and f is a bounded Borel function

$$\int_K f\,d(\mu * \nu) = \int_K \int_K f(x * y)\,d\mu(x)\,d\nu(y).$$

From Theorem I.3.4 of [**8**], we conclude that, if f is in $\mathcal{C}(K)$ and μ in $\mathcal{M}(K)$, then the function on K, $x \mapsto (\mu * f)(x)$, defined by

$$(\mu * f)(x) = \int_K f(y^- * x)\,d\mu(y)$$

is also in $\mathcal{C}(K)$. Moreover, for any $\mu, \nu \in \mathcal{M}(K)$

$$(\mu * \nu) * f = \mu * (\nu * f).$$

The involution on K is denoted by $x \mapsto x^-$. For $\mu \in \mathcal{M}(K)$, the adjoint μ^* of μ is defined by $\mu^* = \overline{\mu^-}$.

Now recall some definitions and basic facts about quasi-positive (indefinite) spaces which can be found in [**2**] and [**4**]. Let n be a nonnegative integer. A quasi-positive space with negative rank n, denoted by QP_n space, is a complex vector space V together with a nondegenerate Hermitian form $(\cdot,\cdot)$ from $V \times V$ to $\mathbf{C}$ such that:

1. there is an n-dimensional subspace $N \subseteq V$ which is negative, i.e., $(\xi,\xi) < 0$ for any nonzero $\xi \in N$;

2. there is no negative subspace of V of dimension greater than n.

It is known that $(V,(\cdot,\cdot))$ is a quasi-positive space with negative rank n if and only if there exists a finite subset $\{\xi_1^0,\dots,\xi_k^0\}$ of V for which the number of negative eigenvalues of the Hermitian matrix $(\xi_i^0,\xi_j^0)_{i,j=1,\dots,k}$ is n and for any proper subset $\{\xi_1,\dots,\xi_r\}$ of V, the number of negative eigenvalues of the Hermitian matrix $(\xi_i,\xi_j)_{i,j=1,\dots,r}$ is less than n.

Let $(V,(\cdot,\cdot))$ be a nondegenerate QP_n space, $n > 0$, and let N be an n-dimensional negative subspace of V. Then $N^\perp$ is positive definite, i.e., $(\xi,\xi) \geq 0$ for all $\xi \in N^\perp$, and V is the orthogonal direct sum of N and $N^\perp$. So we have a fundamental decomposition $V = N \oplus N^\perp$, and any $\xi \in V$ is given in the form $\xi = \xi_- + \xi_+$ where $\xi_- \in N$ and $\xi_+ \in N^\perp$. To this fundamental decomposition corresponds a positive definite inner product $[\cdot,\cdot]$ on V defined by

$$[\xi,\eta] = -(\xi_-,\eta_-) + (\xi_+,\eta_+), \quad \forall\, \xi,\eta \in V.$$

Thus V can also be regarded as an ordinary pre-Hilbert space with the scalar product $[\cdot,\cdot]$ and with the norm

$$\|\xi\| = \sqrt{[\xi,\xi]}.$$

For any $\xi,\eta \in V$ we have

$$|(\xi,\eta)| \leq \|\xi\| \cdot \|\eta\|.$$

It is also known [**2**] that if V becomes a Hilbert space under the inner product $[\cdot,\cdot]$, then any norm topologies corresponding to fundamental decompositions of V are mutually equivalent. A nondegenerate QP_n space $(V,(\cdot,\cdot))$ is called a Pontryagin space with negative rank n, denoted by π_n-space if V becomes a Hilbert space under the inner product $[\cdot,\cdot]$ corresponding to a fundamental decomposition of V.

If $(V,(\cdot,\cdot))$ is a quasi-positive space, we denote by $\mathrm{End}\,(V)$ the algebra of all linear operators on V and by $\mathcal{B}(V)$ the subalgebra of $\mathrm{End}\,(V)$ of all continuous operators on the normed space $(V,\|\cdot\|)$.

Finally, in accordance with [**1**] and [**7**], respectively [**5**], we define ω-continuous, respectively continuous, representations of hypergroups in quasi-positive spaces. By an ω-continuous representation of the hypergroup K in the quasi-positive space V, we mean a mapping $\mu \mapsto U_\mu$ of $\mathcal{M}(K)$ into $\mathrm{End}\,(V)$ such that:

1. $U_{\mu_1 * \mu_2} = U_{\mu_1} U_{\mu_2}$ for all $\mu_1, \mu_2 \in \mathcal{M}(K)$;
2. $U_{p_e} = I$;
3. $(U_\mu \xi, \eta) = (\xi, U_{\mu^*}\eta)$ for all $\mu \in \mathcal{M}(K)$, $\xi, \eta \in V$;
4. If $\xi, \eta \in V$, then the mapping $\mu \mapsto (U_\mu \xi, \eta)$ is continuous on $\mathcal{M}^+(K)$.

A continuous representation of the hypergroup K in the space V is an ω-continuous representation with values in $\mathcal{B}(V)$ which satisfies

5. $\|U_\mu\| \leq \|\mu\|$ for all $\mu \in \mathcal{M}(K)$.

A vector $\xi \in V$ is called cyclic for the (ω-) continuous representation U if the linear span of $\{U_\mu \xi \mid \mu \in \mathcal{M}(K)\}$ is dense in V. The continuous representations U^1 and U^2 of K in the spaces V_1, respectively V_2, are called isometric equivalent (and denoted $U^1 \cong U^2$) if there exists an isometric intertwining operator τ of V_1 into V_2, that is,

$$\tau U^1_\mu = U^2_\mu \tau, \quad \forall\, \mu \in \mathcal{M}(K).$$

If U^1, U^2 are cyclic representations we say that U^1, U^2 are isometric equivalent if there are cyclic vectors ξ_1 of U^1 and ξ_2 of U^2 and an isometric intertwining operator τ with $\tau\xi_1 = \xi_2$.

1. Quasi-positive definite functions and indefinite spaces associated with them.

Definition. A function $\varphi \in \mathcal{C}(K)$ is called *Hermitian* if $\varphi(x^-) = \overline{\varphi(x)}$ holds for all x in K.

It is a straightforward calculation to prove the following.

Lemma. *Let φ be in $\mathcal{C}(K)$. The following statements are equivalent*:

(1) *φ is Hermitian.*

(2) $\varphi(x^- * y) = \overline{\varphi(y^- * x)}$ *for all* $x, y \in K$.

(3) $\int_K \varphi(x)d(\mu^* * \nu)(x) = \overline{\int_K \varphi(x)d(\nu^* * \mu)(x)}$ *for all* $\mu, \nu \in \mathcal{M}(K)$.

According to the above lemma, for any Hermitian function φ and finite sequence $\mu_1, \dots, \mu_k$ in $\mathcal{M}(K)$, the matrix $\Phi = (\int_K \varphi(z)d(\mu_i^* * \mu_j)(z))_{i,j=1,\dots,k}$ is Hermitian.

Definition. A Hermitian function $\varphi : K \to \mathbf{C}$ is said to have n *negative squares* if, for any choice of k and $\mu_1, \dots, \mu_k \in \mathcal{M}(K)$, the Hermitian matrix

$$\Phi = \left(\int_K \varphi(z)d(\mu_i^* * \mu_j)(z) \right)_{i \leq j \leq k}$$

is a complex matrix which has at most n negative eigenvalues (counted with multiplicity) and, for some choice of k and $\mu_1, \dots, \mu_k \in \mathcal{M}(K)$, the matrix Φ has exactly n negative eigenvalues.

We denote by $\mathcal{P}_n(K)$ the space of all complex valued functions on K with n negative squares. A Hermitian function ϕ on K is said to be quasi-positive definite if $\varphi \in \cup_{n=0}^{\infty} \mathcal{P}_n(K)$.

We remark that the definitions are compatible with the ones for groups and semigroups. Moreover, if we consider the restrictions of the elements of $\mathcal{P}_n(K)$ to the maximal subgroup of K, $G(K)$, these are quasi-positive definite functions (with n negative squares) on the group (semigroup) $G(K)$ in the sense of [**7**] (or [**1**]). The elements of $\mathcal{P}_0(K)$ are just the bounded positive-definite functions on the hypergroup K [**5**].

Now we shall construct the quasi-positive space with negative rank n associated with a function $\phi \in \mathcal{P}_n(K)$. First we introduce in $\mathcal{C}(K)$ the translation operator E_μ, $\mu \in \mathcal{M}(K)$, by the formula

$$E_\mu g = \mu^* * g, \quad g \in \mathcal{C}(K).$$

We denote by $\mathcal{A}$ the complex linear space $\{E_\mu : \mathcal{C}(K) \to \mathcal{C}(K) \mid \mu \in \mathcal{M}(K)\}$. Since $E_{\mu * \nu} g = E_\nu(E_\mu(g))$ for any g in $\mathcal{C}(K)$, it follows that $\mathcal{A}$ is an algebra. For an arbitrary function $f \in \mathcal{C}(K)$ the subspace of $\mathcal{C}(K)$

$$T(f) = \{Af \mid A \in \mathcal{A}\}$$

is invariant under each operator $A \in \mathcal{A}$. The restriction of A to $T(f)$ will also be denoted by A.

Let now φ be a quasi-positive definite function. We consider an inner product $(\cdot,\cdot)_\varphi := (\cdot,\cdot)$ on $T(\varphi)$ by the formula

$$(g,h) = \int_K \varphi(z) d(\mu^* * \nu)(z)$$

where $g = E_\mu \varphi$ and $h = E_\nu \varphi$, $\mu, \nu \in \mathcal{M}(K)$.

We observe that

$$(g,h) = \int_K g(z)\, d\nu(z) = \int_K h(z)\, d\mu(z).$$

Indeed, we have

$$\begin{aligned}
\int_K g(z)\, d\nu(z) &= \int_K E_\mu \varphi(z)\, d\nu(z) \\
&= \int_K \left(\int_K \varphi(y^- * z)\, d\mu(y) \right) d\nu(z) \\
&= \int_K \left(\int_K \varphi(y * z)\, d\mu^*(y) \right) d\nu(z) \\
&= \int_K \varphi(t) d(\mu^* * \nu)(t).
\end{aligned}$$

Similarly, we obtain the second equality. These identities imply that the inner product is independent of the particular representations of g and h.

The function φ being Hermitian, it follows from the lemma that

$$(g,h) = \overline{(h,g)}.$$

It results that the vector space $T(\varphi)$ equipped with the inner product $(\cdot,\cdot)_\varphi := (\cdot,\cdot)$ is an inner product space in the sense of [**2**]. Accordingly, in view of the definition of $\mathcal{P}_n(K)$, when $\varphi \in \mathcal{P}_n(K)$, $T(\varphi)$ is a QP_n space.

Proposition. *Let g be an element of $T(\varphi)$. Then*

$$g(x) = (g, E_{p_x}\varphi), \quad \forall\, x \in K.$$

Proof. As g is in $T(\varphi)$, g has a representation of the form $g = E_\mu \varphi$. Then we have for x in K,

$$\begin{aligned}(g, E_{p_x}\varphi) &= \int_K \varphi(z) d(\mu^* * p_x)(z) \\ &= \int_K (\mu^* * \varphi)(z)\, dp_x(z) \\ &= \int_K E_\mu \varphi(z)\, dp_x(z) \\ &= E_\mu\varphi(x) = g(x). \qquad \square\end{aligned}$$

Remark . Let $\varphi \in \mathcal{P}_n(K)$, and let $T(\varphi)$ be the QP_n space constructed above. Let $T(\varphi) = P \oplus N$ be an orthogonal decomposition of $T(\varphi)$, where P is a pre-Hilbert space and N is an n-dimensional negative subspace. If $g \in P$, then $g_- = 0$ and, by the preceding proposition, we have

$$|g(x)|^2 = |(g, E_{p_x}\varphi)| \leq (g, g) \cdot \|E_{p_x}\varphi\|.$$

From this we see that, if $(g_n)_n$ is a Cauchy sequence in P, then $(g_n(x))_n$ is a complex Cauchy sequence for all x in K. It follows that there exists a function $g : K \to \mathbf{C}$ defined by

$$g(x) = \lim_{n\to\infty} g_n(x).$$

Let $\pi_n^+(\varphi)$ be the completion of P constructed by means of functions on K. Setting

$$\pi_n(\varphi) + \pi_n^+(\varphi) \oplus \pi_n^-(\varphi),$$

where $\pi_n^-(\varphi) = N$, $\pi_n(\varphi)$ becomes a π_n space. Since $g \mapsto (g, E_{p_x}\varphi)$ is continuous on $\pi_n(\varphi)$ and $T(\varphi)$ is dense in $\pi_n(\varphi)$, $g(x) = (g, E_{p_x}\varphi)$ holds for every $g \in \pi_n(\varphi)$.

In conclusion, it results that, for $\varphi \in \mathcal{P}_n(K)$, there exists a Pontryagin space $\pi_n(\varphi)$ of functions on K such that $T(\varphi)$ is dense in $\pi_n(\varphi)$ and $g(x) = (g, E_{p_x}, \varphi)$ for all $x \in K$, $g \in \pi_n(\varphi)$.

2. Relation between functions with n negative squares and representations in QP_n spaces.

Theorem 1. *Let U be a continuous representation of the hypergroup K in the QP_n space $(V, (\cdot, \cdot))$. Then, for every $\xi \in V$, the function $\varphi : K \to \mathbf{C}$,*

$$\varphi(x) = (\xi, U_{p_x}\xi)$$

has at most n negative squares and it has n negative squares if ξ is cyclic. Moreover, in this case $\varphi \in P_n(K)$.

Proof. Since U is a continuous representation, it is clear that $\varphi \in \mathcal{C}(K)$. Next, let $\{\mu_i\}_{i=1,2,\dots,k}$ be a finite sequence in $\mathcal{M}(K)$. Then, for all $i, j \in \{1, 2, \dots, k\}$, we have

$$\begin{aligned}\int_K \varphi(z) d(\mu_i^* * \mu_j)(z) &= \int_K (\xi, U_{p_z}\xi) d(\mu_i^* * \mu_j)(z) \\ &= (\xi, U_{\mu_i^* * \mu_j}\xi) = (U_{\mu_i}\xi, U_{\mu_j}\xi).\end{aligned}$$

Since $(V, (\cdot, \cdot))$ is a QP_n space and $\{U_{\mu_i}(\xi)\}_{i=1,2,\dots,k} \subset V$, it follows that the matrix $((U_{\mu_i}\xi, U_{\mu_j}\xi))_{1\le i,j\le k}$ has at most n negative eigenvalues, so for any choice of k and $\mu_1, \dots, \mu_k \in \mathcal{M}(K)$ the matrix $(\int_K \varphi(z) d(\mu_i^* * \mu_j)(z))_{1\le i,j\le k}$ has at most n negative eigenvalues.

If ξ is cyclic, the linear span of $\{U_\mu \xi \mid \mu \in \mathcal{M}(K)\}$ is dense in the QP_n space V, and it is known [**4**] that it contains a negative subspace of dimension n.

Let $e_i = \sum_{l=1}^{r_i} c_l^{(i)} U_{\mu_l^{(i)}}\xi$, $i = 1, 2, \dots, n$, be an orthonormal basis of this subspace, i.e.,

$$\begin{aligned}(e_i, e_j) &= \sum_{l,q} c_l^{(i)} \overline{c_q^{(j)}} (U_{\mu_l^{(i)}}\xi, U_{\mu_q^{(j)}}\xi) \\ &= \sum_{l,q} c_l^{(i)} \overline{c_q^{(j)}} \int_K \varphi(z) d(\mu_l^{(i)*} * \mu_q^{(j)})(z) \\ &= -\delta_{ij}, \quad i, j \in \{1, 2, \dots, n\}.\end{aligned}$$

It results that the matrix $(\int_K \varphi(z) d(\mu_l^{(i)} * \mu_q^{(j)})(z))$ of order $r_1 + r_2 + \dots + r_n$ has n negative eigenvalues, so $\varphi \in \mathcal{P}_n(K)$.

Theorem 2. *Let φ be in $\mathcal{P}_n(K)$. Then there exist a QP_n space $(V, (\cdot, \cdot))$, an ω-continuous representation U of the hypergroup K in the* *space V and a cyclic vector ξ such that*

$$\varphi(x) = (\xi, U_{px}\xi), \quad \forall\, x \in K.$$

Proof. If $\varphi \in \mathcal{P}_n(K)$, we can construct the inner product space $T(\varphi)$ as in Section 1. Since φ has n negative squares, it results that $T(\varphi)$ is a QP_n space. We take $V := T(\varphi)$ and we define $U : \mathcal{M}(K) \to \mathrm{End}\,(V)$ by

$$\mu \longmapsto U_\mu = E_\mu.$$

A short computation shows that

$$(E_\mu f, g) = (f, E_{\mu} * g), \quad \forall\, \mu \in \mathcal{M}(K), \quad f, g \in T(\varphi).$$

So it is clear that the mapping U has properties 1, 2 and 3 of the definition of ω-continuous representation. It remains to prove only 4, i.e., for any $f, g \in T(\varphi)$ the function $\mu \mapsto (U_\mu f, g)$ is continuous. Let f, g be two elements in $T(\varphi)$, i.e.,

$$f = E_\nu \varphi, \qquad g = E_\theta \varphi.$$

Then

$$(U_\mu f, g) = (E_\mu f, g) = \int_K \varphi(z) d(\nu^* * \mu * \theta).$$

If $(\mu\beta)_{\beta\in\mathcal{D}}$ is a net in $\mathcal{M}^+(K)$ converging to μ, then, by the continuity of the convolution and definition of the cone topology, it results that the net $(\int_K \varphi(z)d(\nu^* * \mu_\beta * \theta)(z))_\beta$ converges to $\int_K \varphi(z)d(\nu^* * \mu * \theta)$. This shows that the map $\mu \mapsto (U_\mu f, g)$ is continuous for every f, g in V. It is also clear that $\varphi \in V$ is a cyclic vector for U. By taking $\xi = \varphi$ and using the proposition of Section 1, the proof is complete. $\square$

For $U : K \to \mathcal{B}(V)$ a cyclic continuous representation of the hypergroup K in the QP_n space V, with a cyclic vector ξ, the function φ of $\mathcal{P}(K)$, defined as follows,

$$\varphi(x) = (\xi, U_{p_x}\xi), \quad x \in K$$

is called the characteristic function of the representation U.

Theorem 3. *For $i = 1, 2$, let $U^i : K \to \mathcal{B}(V_i)$ be a cyclic continuous representation of K in a QP_n space V_i with characteristic function φ_i. Then*

$$U^1 \cong U^2 \quad \textit{if and only if} \quad \varphi_1 = \varphi_2.$$

Proof. Suppose that $U^1 \cong U^2$, and let τ be an intertwining isometric operator. Then

$$\varphi_1(x) = (\xi_1, U^1_{p_x}\xi_1) = (\tau\xi_1, \tau U^1_{p_x}\xi_1) = (\xi_2, U^2_{p_x}\xi_2) = \varphi_2(x).$$

Conversely, let τ be defined on the linear span of $\{U^1_{p_x}\xi_1 \mid x \in K\}$ by

$$\tau\left(\sum_{l=1}^{p} b_l U^1_{p_{x_l}}\xi_1\right) = \sum_{l=1}^{p} b_l U^2_{p_{x_l}}\xi_2.$$

Since $\varphi_1 = \varphi_2$, it results that, for any element $u = \sum_{l=1}^{p} b_l U^1_{p_{x_l}}\xi_1$, $v = \sum_{q=1}^{r} c_q U^1_{p_{x_q}}\xi_1$ of the linear span $\{U^1_{p_x}\xi_1 \mid x \in K\}$, we have

$$\begin{aligned}
(u, v) &= \left(\sum_{l=1}^{p} b_l U^1_{p_{x_l}}\xi_1, \sum_{q=1}^{r} c_q U^1_{p_{x_q}}\xi_1\right) \\
&= \sum_{l,q} b_l \overline{c_q} (\xi_1, U_{p^*_{x_l} * p_{x_q}}\xi_1) \\
&= \sum_{l,q} b_l \overline{c_q} \int_K (\xi_1, U^1_{p_z}\xi_1) d(p^*_{x_l} * p_{x_q})(z) \\
&= \sum_{l,q} b_l \overline{c_q} \int_K (\xi_2, U^2_{p_z}\xi_2) d(p^*_{x_l} * p_{x_q})(z) \\
&= (\tau u, \tau v),
\end{aligned}$$

which shows that τ is an isometric linear map on the span $\{U^1_{p_x}\xi_1 \mid x \in K\}$. Since $(\mathrm{Sp}\,\{U^1_{p_x}\xi_1 \mid x \in K\})^\perp = (\mathrm{Sp}\,\{U^2_{p_x}\xi_2 \mid x \in K\})^\perp = \{0\}$, it follows that τ is an isometric isomorphism of $\mathrm{Sp}\,\{U^1_{p_x}\xi_1 \mid x \in K\}$ onto $\mathrm{Sp}\,\{U^2_{p_x}\xi_2 \mid x \in K\}$. By the density of these spaces, it follows that τ can be extended continuously to an isometric isomorphism of V_1 onto V_2 with

$$\tau\xi_1 = \xi_2 \quad \text{and} \quad \tau U^1_\mu = U^2_\mu \tau.$$

3. Examples of quasi-positive functions. For a function $f \in \mathcal{C}(K)$ we define the rank of f by $\operatorname{rk}(f) = \dim T(f)$.

Remark 1. It is obvious to see that $\operatorname{rk}(f) = 0$ if and only if $f = 0$.

Remark 2. A function f has the rank 1 if and only if it is proportional to a multiplicative bounded function. (We recall that a complex valued function χ on K is said to be a multiplicative function if χ is continuous and not identically zero and has the property $\chi(x * y) = \chi(x) \cdot \chi(y)$ for all $x, y \in K$.)

Indeed, let $f \in \mathcal{C}(K)$ be with $\operatorname{rk}(f) = 1$. It follows that $\{E_\mu\}$, $\mu \neq 0$, is a basis for $T(f)$. Then, for every x, y in K, we have

$$f(x * y) = c(x)(\mu^* * f)(y).$$

If we put here $y = e$ and then $x = e$, we obtain

$$f(x) = c(x)(\mu^* * f)(e), \quad \forall\, x \in K,$$

and, respectively,

$$f(y) = c(e)(\mu^* * f)(y), \quad \forall\, y \in K.$$

By taking into account these three equalities, we obtain

$$\begin{aligned} f(x * y) &= c(x)c(e)^{-1} f(y) = f(x)[(\mu^* * f)(e)]^{-1} c(e)^{-1} f(y) \\ &= [(\mu^* * f)(e)c(e)]^{-1} f(x) f(y). \end{aligned}$$

Now it is enough to define $\chi : K \to \mathbf{C}$

$$\chi(x) = [(\mu^* * f)(e)c(e)]^{-1} f(x).$$

With the previous calculation, it is clear that χ is a multiplicative function proportional to f.

Conversely, let $f = c\chi$ where χ is a multiplicative bounded function and $c \in \mathbf{C}$. For every Af in $T(f)$, there exists $\mu \in \mathcal{M}(K)$ such that $Af = E_\mu f$.

Then, for arbitrary x in K, we have

$$\begin{aligned} Af(x) &= (\mu^* * f)(x) = (\mu^* * c\chi)(x) \\ &= c\int_K \chi(y^- * x)\, d\mu^*(y) \\ &= c\chi(x)\int_K \chi(y)\, d\mu(y) \\ &= \left(c\int_K \chi(y)\, d\mu(y)\right)\chi(x). \end{aligned}$$

Proposition. *Let $\mu \to U_\mu$ be a continuous representation of K on a finite dimensional QP_n space $(V, (\cdot,\cdot))$. Then, for any $\xi, \eta \in V$, the function*

$$f(\xi) = (\xi, U_{p_x}\eta)$$

is of finite rank.

Proof. First, we observe that if μ is in $\mathcal{M}(K)$, we have

$$\begin{aligned} E_\mu f(x) &= \int_K f(y^- * x)\, d\mu^*(y) \\ &= \int_K \left(\int_K f(z) d(p_{y^-} * p_x)(z)\right) d\mu^*(y) \\ &= \int_K \left(\int_K (\xi, U_{p_z}\eta) d(p_{y^-} * p_x)(z)\right) d\mu^*(y) \\ &= \int_K (\xi, U_{p_{y^-} * p_x}\eta)\, d\mu^*(y) \\ &= \int_K (U_{p_y}\xi, U_{p_x}\eta)\, d\mu^*(y). \end{aligned}$$

Let now $e_1, e_2, \ldots, e_m$ be a basis for the finite m-dimensional QP_n space V. For every y in K and $\xi \in V$, we have

$$U_{py}\xi = \sum_{i=1}^m a_i(y) e_i.$$

Then, using the surjectivity of U we obtain for every $\mu \in \mathcal{M}(K)$ and $x \in K$,

$$\begin{aligned} E_\mu f(x) &= \int_K (U_{p_y}\xi, U_{p_x}\eta)\, d\mu^*(y) \\ &= \int_K \left(\sum_{i=1}^m a_i(y) e_i, U_{p_x}\eta \right) d\mu^*(y) \\ &= \sum_{i=1}^m \left(\sum_K a_i(y)\, d\mu^*(y) \right) (e_i, U_{p_x}\eta) \\ &= \sum_{i=1}^m \left(\int_K a_i(y)\, d\mu^*(y) \right) (U_{\mu_i^*}\xi, U_{p_x}\eta) \\ &= \sum_{i=1}^m \left(\int_K a_i(y)\, d\mu^*(y) \right) E_{\mu_i} f(x), \end{aligned}$$

where $U_{\mu_i^*}\xi = e_i$, $i = 1, \dots, m$.

Therefore, $T(f)$ is a finite dimensional space. □

Theorem (Examples of quasi-positive definite functions). *Let ψ be in $\mathcal{P}_0(K)$ with rank n. Then, for any θ in $\mathcal{P}_0(K)$, the difference*

$$\varphi = \theta - \psi$$

is a quasi-positive definite function, having at most n negative squares.

Proof. The proof is adapted from the group case [**7**]. First, it is clear that ψ is in $\mathcal{P}_0(K)$ and has rank n, then $-\psi$ is in $\mathcal{P}_n(K)$. Let $\theta \in \mathcal{P}_0(K)$ be arbitrary and $\varphi = \theta - \psi \in \mathcal{C}(K)$. Therefore, φ is the sum of two functions of $\mathcal{P}_0(K)$, respectively $\mathcal{P}_n(K)$. By virtue of Theorem 2 (Section 2), there exist ω-continuous representations U^1, U^2 of K in the quasi-positive V_1 (pre-Hilbert), V_2 (QP_n space) and $\xi_1 \in V_1$, $\xi_2 \in V_2$ such that θ and $-\psi$ are given in the form

$$\theta(x) = (\xi_1, U_{p_x}\xi_1), \quad \forall\, x \in K,$$

and, respectively,

$$-\psi(x) = (\xi_2, U_{p_x}\xi_2), \quad \forall\, x \in K.$$

If U is the product representation of K in the QP_n space $V_1 \times V_2$, a short computation shows

$$\varphi(x) = (\gamma, U_{p_x}\gamma)_{V_1 \times V_2}, \quad \forall\, x \in K,$$

where $\gamma = (\xi_1, \xi_2) \in V_1 \times V_2$.

As in the proof of Theorem 1 (Section 2), it follows that φ has at most n negative squares. □

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