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TRANSITION VECTOR MEASURES AND MULTIMEASURES AND PARAMETRIC SET-VALUED INTEGRALS

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ABSTRACT. In this paper we examine transition vector measures and multimeasures. First we prove a Radon-Nikodym theorem for transition vector measures and then we use it to establish the existence of a set-valued Radon-Nikodym derivative for transition multimeasures. Subsequently, we examine parametric set-valued integrals and obtain two results characterizing their measurable selectors (integral versions of Filippov's implicit function lemma). We conclude with a useful observation concerning transition measures.

1. Introduction. In a recent paper [**10**] we proved a Radon-Nikodym theorem for transition multimeasures (set-valued measures). The purpose of this paper is to extend the above mentioned results to the case where the dominating (control) measure is a transition measure too and then establish some properties of parametric set-valued integrals. Multimeasures and set-valued integrals are the natural generalization of classical single-valued measures and integrals and so it is interesting to know to what extent we can duplicate the existing theory on them. But, in addition, multimeasures and set-valued integrals are the appropriate analytical tools in various applied areas, like mathematical economics, statistics, optimization and optimal control. We refer to [**10, 11**] and [**12**] for a list of relevant references. We also mention that transition multimeasures are useful in the study of Markov temporary equilibrium processes in dynamic economies; see Blume [**2**].

In this paper, first we prove a Radon-Nikodym theorem for transition vector valued measures and then we use that result to prove a Radon-Nikodym theorem for transition multimeasures. Then, in Section 4,

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we turn our attention to parametric set-valued integrals and try to characterize their measurable selectors. This leads us to two integral versions of the implicit function lemma (*Filippov's lemma*), used in control theory and optimization, see Himmelberg [**6**, Section 7]. We close the paper with an observation concerning finite transition measures.

2. Preliminaries. In this section we fix our notation and recall some basic definitions and facts from the theory of measurable multifunctions and set-valued measures (multimeasures).

Let (Ω, Σ) be a measurable space and X a separable Banach space. We define $P_{f(c)}(X) = \{A \subseteq X\text{: nonempty, closed, (convex)}\}$ and $P_{(w)k(c)}(X) = \{A \subseteq X\text{: nonempty, (weakly-) compact, (convex)}\}$. Also if $A \subseteq X$ is nonempty, we set $|A| = \sup\{\|x\| : x \in A\}$ (the *norm* of the set A), $\sigma(x^*, A) = \sup\{(x^*, x) : x \in A\}$ for $x^* \in X^*$ (the *support function* of the set A) and $d(z, A) = \inf\{\|z - x\| : x \in A\}$ for $z \in X$ (the *distance function* from the set A).

A multifunction (set-valued function), is said to be measurable if, for all $z \in X$, $\omega \to d(z, F(\omega))$ is measurable. Other equivalent definitions of measurability can be found in Himmelberg [**6**] (cf. Theorems 3.5 and 5.6). For a measurable multifunction, we have that $\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field X (graph measurability of $F(\cdot)$). The converse is true, if there is a complete, σ-finite measure $\mu(\cdot)$ on (Ω, Σ) (see Himmelberg [**6**, Theorem 3.5(iii)]).

Let $\mu(\cdot)$ be a finite measure on (Ω, Σ), and let $F : \Omega \to 2^X \setminus \{\varnothing\}$. We define $S_F^1 = \{f \in L^1(\Omega, X) : f(\omega) \in F(\omega)\mu - \text{a.e.}\}$. This set may be empty. It is easy to see using Theorem 5.7 of Himmelberg [**6**], that for a graph measurable multifunction the set S_F^1 is nonempty if and only if $\omega \to \inf\{\|x\| : x \in F(\omega)\} \in L^1(\Omega)$. In particular, this is the case for an *integrably bounded* multifunction $F(\cdot)$, i.e., $\omega \to |F(\omega)| \in L^1(\Omega)$. Using the set S_F^1 we can define a set-valued integral for $F(\cdot)$ by setting $\int_\Omega F(\omega)\, d\mu(\omega) = \{\int_\Omega f(\omega)\, d\mu(\omega) : f \in S_F^1\}$.

Now let X be any Banach space. A set-valued set function $M : \Sigma \to P_f(X)$ is said to be a multimeasure (set-valued measure) if and only if, for every $x^* \in X^*$, $A \to \sigma(x^*, M(A))$ is a signed measure. For a multimeasure $M(\cdot)$ and $A \in \Sigma$ we define $|M|(A) = \sup_\pi \Sigma_k |M(A_k)|$, where the supremum is taken over all finite Σ-partitions $\pi = \{A_k\}_{k=1}^n$ of A. If $|M|(\Omega) < \infty$, then $M(\cdot)$ is said to be a bounded variation. By S_M we denote the set of all vector measures $m : \Sigma \to X$ such that $m(A) \in M(A)$ for all $A \in \Sigma$ (measure selectors of M). Finally we say that $M \ll \mu$ if, for every $A \in \Sigma$ with $\mu(A) = 0$, we have that $M(A) = \{0\}$.

Next, let (Ω, Σ) and $(T, \mathcal{T})$ be two measurable spaces and X a separable Banach space. A set map $m : \mathcal{T} \times \Omega \to X$ is said to be a transition vector measure if, (1) for all $A \in \mathcal{T}$, $\omega \to m(A, \omega)$ is measurable and (2) for all $\omega \in \Omega$, $A \to m(A, \omega)$ is a measure. Analogously a multivalued map $M : \mathcal{T} \times \Omega \to P_f(X)$ is a transition multimeasure if (1) for all $A \in \mathcal{T}$, $\omega \to M(A, \omega)$ is a measurable multifunction and (2) for all $\omega \in \Omega$, $A \to M(A, \omega)$ is a multimeasure. A transition vector measure $m(A, \omega)$ satisfying $m(A, \omega) \in M(A, \omega)$ for all $(A, \omega) \in \mathcal{T} \times \Omega$ is said to be a *transition selector of* M. The set of all transition selectors of $M(A, \omega)$ will be denoted by TS_M.

Let T be a Polish space, i.e., T is metrizable by some metric d such that (T, d) is complete and separable. By $C_b(T)$ we will denote the space of all bounded, continuous, $\mathbf{R}$-valued functions on T. Also, by $M^b(T)$ we will denote the space of bounded Borel measures on T. Note that since T is a Polish space, every such measure is regular, hence a Radon measure. The *narrow* topology on $M^b(T)$ is the weakest topology on $M^b(T)$ for which the maps $\mu \to \langle \mu, f \rangle = \int_T f(t)\mu(dt)$, $f \in C_b(T)$, are continuous, i.e., the narrow topology is the $w(M^b(T), C_b(T))$-topology, where the duality brackets for the pair $(M^b(T), C_b(T))$ are given by $\langle \mu, f \rangle = \int_T f(t)\mu(dt)$. We remark that if T is compact, then $C_b(T) = C(T)$ and $C(T)^* = M(T)$ (Riesz representation theorem), and so the narrow topology coincides with the weak*-topology. By $M_+^b(T)$ we will denote the cone of all positive bounded (Radon) measures in $M^b(T)$. It is well-known that the space $M_+^b(T)$ furnished with the narrow topology becomes a Polish space (see Dellacherie-Meyer [**3**, Theorem 60]). Given that every measure $\mu \in M^b(T)$ can be written uniquely as $\mu = \mu^+ - \mu^-, \mu^+, \mu^- \in M_+^b(T)$, we see immediately that $M^b(T)$ equipped with the narrow topology is a Souslin space, i.e., the continuous image of a Polish space.

3. Radon-Nikodym theorems. First we prove a Radon-Nikodym theorem for transition vector measures. So our result extends the classical Radon-Nikodym theorem to transition vector measures.

Theorem 1. *If* $(\Omega, \Sigma, \lambda)$ *is a finite measure space,* T *is a Polish space with its Borel* σ*-field* $B(T)$*,* X *is a Banach space with the Radon-Nikodym property* (RNP)*,* $m : B(T) \times \Omega \to X$ *is a transition vector measure of bounded variation,* $\mu : B(T) \times \Omega \to \mathbf{R}_+$ *is a finite transition measure, and for all* $\omega \in \Omega \backslash N$*,* $\lambda(N) = 0$*,* $m(\cdot, \omega) \ll \mu(\cdot, \omega)$*, then there exists a jointly measurable function* $f : T \times \Omega \to X$ *such that* $f(\cdot, \omega) \in L^1(T, \mu(\cdot, \omega); X)$*,* $\omega \in \Omega$*, and* $m(A, \omega) = \int_A f(t, \omega)\mu(dt, \omega)$ *for all* $(A, \omega) \in B(T) \times (\Omega \backslash N)$*.*

Proof. Let $\omega \in \Omega \backslash N$. Then, since by hypothesis $m(\cdot, \omega) \ll \mu(\cdot, \omega)$ and by hypothesis X has the RNP, there exists $\hat{f}(\cdot, \omega) \in L^1(T, \mu(\cdot, \omega); X)$ such that, for all $A \in B(T)$,

$$m(A, \omega) = \int_A \hat{f}(t, \omega)\mu(dt, \omega).$$

We remark that since T is Polish, $B(T)$ is countably generated, and let $\mathcal{A} = \{C_m\}_{m \geq 1}$ be a countable field generating $B(T)$, i.e., $B(T) = \sigma(\mathcal{A})$. Let $\{\mathcal{P}_n\}_{n \geq 1}$ be a sequence of finite Borel-measurable partitions of T such that $\mathcal{P}_n$, $n \geq 1$, is the finest possible partition involving elements from the field generated by the family $\{T, C_m\}_{m=1}^n$. Let $\mathcal{P}_n = \{E_{n,k} : 1 \leq k \leq N_n\}$, $n \geq 1$. Note that $\mathcal{P}_n \leq \mathcal{P}_{n+1}$ in the sense that every element in $\mathcal{P}_n$ is the union of some elements in $\mathcal{P}_{n+1}$, and furthermore we have that $B(T) = \sigma(\cup_{n \geq 1} \mathcal{P}_n)$. Let $\hat{m}(A, \omega) = \int_A \hat{f}(t, \omega)\mu(dt, \omega)$ for all (A, ω) in $B(T) \times \Omega$. Then $\hat{m}(A, \omega) = m(A, \omega)$ for all $(A, \omega) \in B(T) \times (\Omega \backslash N)$. For every $n \geq 1$, define $f_n : T \times \Omega \to X$ as follows:

$$f_n(t, \omega) = \sum_{k=1}^{N_n} \frac{\hat{m}(E_{n,k}, \omega)}{\mu(E_{n,k}, \omega)} \chi_{E_{n,k}}(t)$$

with the usual convention that $0/0 = 0$ (see Diestel-Uhl [**4**]).

Clearly, for every $n \geq 1$, $(t, \omega) \to f_n(t, \omega)$ is $B(T) \times \Sigma$-measurable. Also, from Proposition 48.1 and Remark 48.3, of Parthasarathy [**13**], we know that for each $\omega \in \Omega$, $f_n(t, \omega) \to \hat{f}(t, \omega)$ in X for all $t \in T \backslash D(\omega)$, $\mu(D(\omega), \omega) = 0$ and $f_n(\cdot, \omega) \to \hat{f}(\cdot, \omega)$ in $L^1(T, \mu(\cdot, \omega); X)$. We should mention that the result of Parthasarathy [**13**] is for $\mathbf{R}$-valued measures, but his proof extends verbatim to the vector-valued case, thanks to the Neveu-Ionescu Tulcea convergence theorem for vector-valued conditional expectations (see, for example, Metivier [**8**, Theorem 11.2]).

For each $\omega \in \Omega$, on $B(T)$ we define the pseudometric

$$d(\omega)(A_1, A_2) = \mu(A_1 \Delta A_2, \omega).$$

Then, from Theorem B of Halmos [**5**], we know that

$$\begin{aligned}\operatorname{Gr} D &\subseteq \bigcap_{n\geq 1} \bigcup_{m\geq 1} \left\{(\omega, t) \in (\Omega\backslash N) \times T : \mu(C_m, \omega) \leq \frac{1}{n}, t \in C_m\right\} \\ &= \Gamma \in \Sigma \times B(T).\end{aligned}$$

Now define $f : T \times \Omega \to X$ by

$$f(t,\omega) = \begin{cases} \hat{f}(t,\omega) & \text{for } (\omega, t) \in \Gamma^c \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, since $\Gamma \in \Sigma \times B(T)$ and $f_n(t,\omega) \to \hat{f}(t,\omega)$ on $(\operatorname{Gr} D)^c \supseteq \Gamma^c$, we get that $(t,\omega) \to f(t,\omega)$ is jointly measurable on $T \times \Omega$ into X. Also, for all $A \in B(T)$ we have $\hat{m}(A,\omega) = \int_A f(t,\omega)\mu(dt,\omega)$ for all $\omega \in \Omega$. So $m(A,\omega) = \int_A f(t,\omega)\mu(dt,\omega)$ for all $\omega \in \Omega\backslash N$. $\square$

Now we will use this result to establish the existence of set-valued Radon-Nikodym derivatives for transition multimeasures with respect to transition measures. Our result extends Theorem 4.3 of [**10**], where the measure μ was independent of $\omega \in \Omega$ and the separable Banach space X was assumed to be reflexive. In addition, the proof is different.

Theorem 2. *If $(\Omega, \Sigma, \lambda)$ is a finite, complete measure space, T is a Polish space with its Borel σ-field $B(T)$, X is a separable Banach space, $\mu : B(T) \times \Omega \to \mathbf{R}_+$ is a transition measure and $M : B(T) \times \Omega \to P_{wkc}(X)$ is a transition multimeasure such that for all $(A,\omega) \in B(T) \times (\Omega\backslash N)$, $\lambda(N) = 0$, $M(A,\omega) \subseteq \int_A W(t,\omega)\mu(dt,\omega)$, with $W : T \times \Omega \to P_{wkc}(X)$ measurable and for all $\omega \in \Omega\backslash N$, $W(\cdot,\omega)$ is $\mu(\cdot,\omega)$-integrably bounded, then there exists $F : T \times \Omega \to P_{wkc}(X)$ a measurable multifunction such that $F(\cdot,\omega)$ is $\mu(\cdot,\omega)$-integrably bounded* *and* $M(A,\omega) = \int_A F(t,\omega)\mu(dt,\omega)$ *for all* $(A,\omega) \in B(T) \times (\Omega \backslash N')$, $\mu(N') = 0$.

Proof. Let $D^* = \{x_n^*\}_{n\geq 1} \subseteq X^*$ be dense in X^* for the Mackey topology $m(X^*, X)$. The existence of such a sequence follows from the separability of X, cf. Wilansky [**14**, page 144]). Let $\hat{D} = \{\sum_{k=1}^n \lambda_k x_k^* : 1 \leq n, (\lambda_k, x_k^*) \in Q \times D^*\}$, i.e., the set of all rational linear combinations of elements in D^*). Clearly, $\hat{D}^*$ is countable and $m(X^*, X)$-dense in X^*. Note that, for every $x^* \in D^*$, $(A,\omega) \to \sigma(x^*, M(A,\omega))$ is an **R**-valued transition measure and $\sigma(x^*, M(\cdot,\omega)) \ll \mu(\cdot,\omega)$ for all $\omega \in \Omega\backslash N, \lambda(N) = 0$. Apply Theorem 1 to get $u(x^*) : T \times \Omega \to \mathbf{R}$, a jointly measurable function such that $u(x^*)(\cdot,\omega) \in L^1(T, \mu(\cdot,\omega); \mathbf{R})$ and $\sigma(x^*, M(A,\omega)) = \int_A u(x^*)(t,\omega)\mu(dt,\omega)$ for all $(A,\omega) \in B(T) \times (\Omega\backslash N)$.

Let $\hat{W}(t,\omega) = \overline{\text{conv}}[W(t,\omega) \cup (-W(t,\omega))]$. Then, from the Krein-Smulian theorem, see, for example, Diestel-Uhl [**4**, Theorem 11], we have that $\hat{W}(\cdot,\cdot)$ is $P_{wkc}(X)$-valued, with symmetric values, is jointly measurable, cf. Proposition 2.3 and Theorem 9.1 of Himmelberg [**6**]) and $\hat{W}(\cdot,\omega)$ is $\mu(\cdot,\omega)$-integrably bounded for all $\omega \in \Omega\backslash N$. Let $x^*, z^* \in \hat{D}^*$. We have, for all $(A,\omega) \in B(T) \times (\Omega\backslash N)$:

$$\begin{aligned}
\sigma(x^*, M(A,\omega)) - \sigma(z^*, M(A,\omega)) \\
&= \int_A (u(x^*)(t,\omega) - u(z^*)(t,\omega))\mu(dt,\omega) \\
&\leq \int_A \sigma(x^* - z^*, \hat{W}(t,\omega))\mu(dt,\omega) \\
&\Longrightarrow u(x^*)(t,\omega) - u(z^*)(t,\omega) \\
&\leq \sigma(x^* - z^*, \hat{W}(t,\omega))
\end{aligned}$$

for all $t \in T\backslash D_1(\omega)$, $\mu(D_1(\omega),\omega) = 0$. Similarly, since $\hat{W}(\cdot,\cdot)$ has symmetric values, we get that

$$u(z^*)(t,\omega) - u(x^*)(t,\omega) \leq \sigma(x^* - z^*, \hat{W}(t,\omega))$$

for all $t \in T\backslash D_2(\omega)$, $\mu(D_2(\omega),\omega) = 0$ and all $\omega \in \Omega\backslash N$. So we get

$$|(u(x^*)(t,\omega) - u(z^*)(t,\omega))| \leq \sigma(x^* - z^*, \hat{W}(t,\omega)) \tag{1}$$

for all $t \in T \backslash D_3(\omega)$, $\mu(D_3(\omega), \omega) = 0$ and all $\omega \in \Omega \backslash N$. Finally, note that if $x^*, z^* \in \hat{D}^*$ and $\beta \in \mathbf{Q}$, then for all $(A, \omega) \in B(T) \times (\Omega \backslash N)$, we have

$$\begin{aligned} \sigma(x^* + \beta z^*, M(A, \omega)) &\le \sigma(x^*, M(A, \omega)) + \beta\sigma(z^*, M(A, \omega)) \\ &\Longrightarrow \int_A u(x^* + \beta z^*)(t, \omega)\mu(dt, \omega) \\ &\le \int_A (u(x^*)(t, \omega) + \beta u(z^*)(t, \omega))\mu(dt, \omega) \\ &\Longrightarrow u(x^* + \beta z^*)(t, \omega) \\ &\le u(x^*)(t, \omega) + \beta u(z^*)(t, \omega) \end{aligned} \tag{2}$$

for all $t \in T \backslash D_4(\omega)$, $\mu(D_4(\omega), \omega) = 0$. Set $D(\omega) = \cup_{i=1}^4 D_i(\omega)$ and, as before, let $\Gamma = \operatorname{Gr} D = \{(\omega, t) \in (\Omega \backslash N) \times T : t \in D(\omega)\}$. We already know that $\Gamma \in \Sigma \times B(T)$. Then define $\hat{u}(x^*)(t, \omega)$ by

$$\hat{u}(x^*)(t, \omega) = \begin{cases} u(x^*)(t, \omega) & \text{for } (\omega, t) \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\hat{u}(x^*)(\cdot, \cdot)$ is jointly measurable. Also, from inequality (1) we see that, for all $(t, \omega) \in T \times (\Omega \backslash N)$

$$|\hat{u}(x^*)(t, \omega) - \hat{u}(z^*)(t, \omega)| \le \sigma(x^* - z^*, \hat{W}(t, \omega)).$$

Since $\hat{W}(\cdot, \cdot)$ is $P_{wkc}(X)$-valued, $\sigma(\cdot, \hat{W}(t, \omega))$ is $m(X^*, X)$-continuous. So, from the above inequality, we deduce that $x^* \to \hat{u}(x^*)(t, \omega)$ can be extended uniquely to an $m(X^*, X)$-continuous function defined on all of X^* and denoted by $\hat{u}_0(x^*)(t, \omega)$. Furthermore, it is clear from inequality (2) that $x^* \to \hat{u}_0(x^*)(t, \omega)$ is sublinear. So there exists $\hat{F} : T \times \Omega \to P_{wkc}(X)$ a measurable multifunction such that $\hat{F}(t, \omega) \subseteq \hat{W}(t, \omega)$, $\mu(\cdot, \omega)$-almost everywhere and $\hat{u}_0(x^*)(t, \omega) = \sigma(x^*, \hat{F}(t, \omega))$ for all $\omega \in \Omega \backslash N$.

Since $(t, \omega) \to \hat{u}_0(x^*)(t, \omega)$ is measurable, $(t, \omega) \to \sigma(x^*, \hat{F}(t, \omega))$ is measurable and so $(t, \omega) \to \hat{F}(t, \omega)$ is $\overline{B(T) \times \Sigma}$-measurable, where $\overline{B(T) \times \Sigma}$ is the universal σ-field corresponding to $B(T) \times \Sigma$. Then, from Theorem 5.6 of Himmelberg [**6**], we know that there exist $\hat{f}_n : T \times \Omega \to X$, $n \ge 1$, $\overline{B(T) \times \Sigma}$-measurable maps such that, for all $(t, \omega) \in$

$T \times \Omega$, $\hat{F}(t,\omega) = \mathrm{cl}\,\{\hat{f}_n(t,\omega)\}_{n\geq 1}$. Let $\hat{s}_{nm}(t,\omega) = \sum_{k=1}^{\hat{N}_m} \chi_{B^k_{nm}}(t,\omega)v_k$, with $B^k_{nm} \in \overline{B(T)\times\Sigma}$ and $v_k \in X$, be simple maps such that $s_{nm}(t,\omega) \to \hat{f}_n(t,\omega)$ as $m \to \infty$ for all (t,ω). Let $\gamma(\cdot)$ be the bounded measure on $(T\times\Omega, B(T)\times\Sigma)$ defined by $\gamma(D) = \int_\Omega \mu(D(\omega),\omega)\lambda(d\omega)$ for all $D \in B(T)\times\Sigma$. Then, from Theorem 13.13 of Halmos [**5**], we know that $B^k_m = \hat{B}^k_{nm}\Delta N^k_{nm}$ with $\hat{B}^k_{nm} \in B(T)\times\Sigma$ and N^k_{nm} is γ-null. Let $N_{nm} = \cup_{k=1}^{\hat{N}_m} N^k_{nm}$ and $N_0 = \cup_{n,m\geq 1} N_{nm}$. Clearly, N_0 is a γ-null set. Let $N_1 \in B(T)\times\Sigma$ be such that $\gamma(N_1) = 0$ and $N_0 \subseteq N_1$. Define

$$s_{nm}(t,\omega) = \begin{cases} \hat{s}_{nm}(t,\omega) & \text{if } (t,\omega) \notin N_1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $s_{nm}(\cdot,\cdot)$ is $B(T)\times\Sigma$-measurable and $s_{nm}(t,\omega) \to \hat{f}_n(t,\omega)$ as $m \to \infty$ for all $(t,\omega) \in N_1^c$. So if, for every $n \geq 1$, we define

$$f_n(t,\omega) = \begin{cases} \hat{f}_n(t,\omega) & \text{if } (t,\omega) \notin N_1 \\ 0 & \text{otherwise,} \end{cases}$$

then this function is $B(T)\times\Sigma$-measurable. Set $F(t,\omega) = \mathrm{cl}\,\{f_n(t,\omega)\}_{n\geq 1}$. This multifunction is $B(T)\times\Sigma$-measurable, cf. Theorem 5.6 of Himmelberg [**6**], and furthermore $F(t,\omega) = \hat{F}(t,\omega)$ γ-almost everywhere on $T\times\Omega$. Thus, for all $\omega \in \Omega\backslash N'$, $\lambda(N') = 0$ and all $A \in B(T)$ we have

$$\begin{aligned} \int_A \sigma(x^*, \hat{F}(t,\omega))\mu(dt,\omega) &= \int_A \sigma(x^*, F(t,\omega))\mu(dt,\omega) \\ &\Rightarrow \sigma(x^*, M(A,\omega)) \\ &= \int_A \sigma(x^*, F(t,\omega))\mu(dt,\omega) \\ &= \sigma\bigg(x^*, \int_A F(t,\omega)\mu(dt,\omega)\bigg). \end{aligned}$$

Since $\int_A F(t,\omega)\mu(dt,\omega) \in P_{wkc}(X)$, we conclude that

$$M(A,\omega) = \int_A F(t,\omega)\mu(dt,\omega). \qquad \square$$

4. Parametric set-valued integrals. In this section we use the Radon-Nikodym type theorems obtained above to characterize the measurable selectors of parametric set-valued integrals.

Our first result can be viewed as an integral version of the implicit function theorems of Filippov type (see Himmelberg [**6**, Section 7]). It is well known that such results are useful in the analysis of control systems. Similarly, our integral version can be useful in the study of systems with transition measures as controls (relaxed systems).

So our theorem says that any measurable selector of a parametric set-valued integral can be expressed as the integral of a function which is measurable with respect to the parameter and takes values in the multivalued integrand.

Theorem 3. *If* $(\Omega, \Sigma, \lambda)$ *is a finite, complete measure space,* T *is a Polish space with its Borel* σ*-field* $B(T)$*,* X *is a separable reflexive Banach space,* $\mu : B(T)\times\Omega \to \mathbf{R}_+$ *is a transition measure,* $F : T\times\Omega \to P_{wkc}(X)$ *is a measurable multifunction such that, for all* $\omega \in \Omega\backslash N$*,* $\lambda(N) = 0$*,* $F(\cdot,\omega)$ *is* $\mu(\cdot,\omega)$*-integrably bounded and* $x : \Omega \to X$ *is a measurable function such that, for all* $\omega \in \Omega\backslash N$*,* $\lambda(N) = 0$*,* $x(\omega) \in \int_A F(t,\omega)\mu(dt,\omega)$ *for some* $A \in B(T)$*, then there exists* $f : T\times\Omega \to X$ *a measurable function such that for all* $\omega \in \Omega\backslash N$*,* $\lambda(N) = 0$*,* $f(\cdot,\omega) \in L^1(T,\mu(\cdot,\omega);X)$*,* $x(\omega) = \int_A f(t,\omega)\mu(dt,\omega)$ *and* $f(\cdot,\omega) \in F(\cdot,\omega)$*,* $\mu(\cdot,\omega)$*-almost everywhere.*

Proof. From Proposition 3.1 of [**9**], we know that, for all $(A,\omega) \in B(T)\times\Omega$, $M(A,\omega) = \int_A F(t,\omega)\mu(dt,\omega) \in P_{wkc}(X)$. Furthermore, for all $x^* \in X^*$, $\sigma(x^*, M(A,\omega)) = \int_A \sigma(x^*, F(t,\omega))\mu(dt,\omega)$, which shows that $M : B(T)\times\Omega \to P_{wkc}(X)$ is a transition multimeasure. Apply Proposition 4.2 of [**10**] to get $m : B(T)\times\Omega \to X$ a transition selector of $M(\cdot,\cdot)$, i.e., m belongs in TS_M, such that for all $\omega\in\Omega$, $m(A,\omega) = x(\omega)$. Then apply Theorem 1 of this paper to get $f : T\times\Omega \to X$ a measurable map such that for all $\omega \in \Omega\backslash N$, $\lambda(N) = 0$ and all $C \in B(T)$, we have

$$\begin{aligned} x(\omega) = m(C,\omega) &= \int_C f(t,\omega)\mu(dt,\omega) \\ &\Longrightarrow \int_C f(t,\omega)\mu(dt,\omega) \in \int_C F(t,\omega)\mu(dt,\omega) \\ &\Longrightarrow \int_C (x^*, f(t,\omega))\mu(dt,\omega) \\ &\leq \int_C \sigma(x^*, F(t,\omega))\mu(dt,\omega) \quad \text{for all } x^* \in D^* \end{aligned}$$

$$\Longrightarrow (x^*, f(t,\omega)) \le \sigma(x^*, F(t,\omega))$$
$$\text{for all } (t,x^*) \in (T\backslash Z) \times D^* \quad \text{with } \mu(Z,\omega) = 0.$$

Since both functions $x^* \to (x^*, f(t,\omega))$ and $x^* \to \sigma(x^*, F(t,\omega))$ are $m(X^*, X)$-continuous and D^* is $m(X^*, X)$-dense in X^*, then we get that

$$(x^*, f(t,\omega)) \le \sigma(x^*, F(t,\omega)) \quad \text{for all } (t,x^*) \in (T\backslash Z) \times X^*$$
$$\text{with } \mu(Z,\omega) = 0$$
$$\Longrightarrow f(t,\omega) \in F(t,\omega), \quad \mu(\cdot,\omega) - \text{a.e.} \qquad \square$$

If X is finite dimensional, then we can weaken our hypothesis on the multifunction $F(t,\omega)$ and allow it to have nonconvex values. So we have the following result:

Theorem 4. *If $(\Omega, \Sigma, \lambda)$ is a finite complete measure space, T is a Polish with its Borel σ-field $B(T)$, X is a finite dimensional Banach space, $F : T \times \Omega \to P_k(X)$ is a measurable multifunction such that, for all $\omega \in \Omega\backslash N$, $\lambda(N) = 0$, $F(\cdot,\omega)$ is $\mu(\cdot,\omega)$-integrably bounded and if t belongs in an atom of $\mu(\cdot,\omega)$, then $F(t,\omega)$ is also convex and $x : \Omega \to X$ is a measurable function such that for all $\omega \in \Omega\backslash N$, $\lambda(N) = 0$, $x(\omega) \in M(A,\omega)$ for some $A \in B(T)$, then there exists $f : T \times \Omega \to X$ a measurable function such that, for all $\omega \in \Omega\backslash N^*$, $\lambda(N^*) = 0$, $f(\cdot,\omega) \in L^1(T, \mu(\cdot,\omega); X)$, $x(\omega) = \int_A f(t,\omega)\mu(dt,\omega)$ and $f(t,\omega) \in F(t,\omega)\mu(\cdot,\omega)$ almost everywhere.*

Proof. Let $M(A,\omega) = \int_A F(t,\omega)\mu(dt,\omega)$ for $(A,\omega) \in B(T) \times \Omega$. From the properties of the set-valued integral, see, for example, Klein Thompson [**7**, Chapter 18], we have that for all $(A,\omega) \in B(T) \times \Omega$, $M(A,\omega) \in P_{kc}(X)$. Furthermore, for every $x^* \in X^*$ we have $\sigma(x^*, M(A,\omega)) = \int_A \sigma(x^*, F(t,\omega))\mu(dt,\omega)$ which shows that $(A,\omega) \to M(A,\omega)$ is a transition multimeasure. Apply Proposition 4.2 of [**10**] to get $m : B(T) \times \Omega \to X$ a transition selector of $M(\cdot,\cdot)$ such that $m(A,\omega) = x(\omega)$ for all $\omega \in \Omega\backslash N$, $\lambda(N) = 0$. From the definition of the set-valued integral, see Section 2, we know that $m(A,\omega) = \int_A \hat{f}(t,\omega)\mu(dt,\omega)$ for all $\omega \in \Omega\backslash N'$, $\mu(N') = 0$. In fact, $\hat{f}(\cdot,\omega) = dm(\cdot,\omega)/d\mu(\cdot,\omega)$ (the Radon-Nikodym derivative of $m(\cdot,\omega)$

with respect to $\mu(\cdot,\omega)$). Then, as in the proof of Theorem 1, we can show that there exists $f : T \times \Omega \to X$ a $B(T) \times \Sigma$-measurable function such that, for $\omega \in \Omega \backslash N'$, $\lambda(N') = 0$, $f(t,\omega) \in F(t,\omega)$, $\mu(\cdot,\omega)$ almost everywhere and $m(A,\omega) = \int_A f(t,\omega)\mu(dt,\omega)$. So for $\omega \in \Omega\backslash N^*$, $N^* = N \cup N'$, $\lambda(N^*) = 0$ we have $x(\omega) = \int_A f(t,\omega)\mu(dt,\omega)$. □

We conclude this paper with a useful observation concerning transition measures.

Proposition 5. *Assume that* $(\Omega, \Sigma, \lambda)$ *is a finite measure space and* T *a Polish space.*

(i) *If* $\mu : \Omega \to M_+^b(T)$ *is measurable (where* $M_+^b(T)$ *is equipped with the narrow topology), then* $\hat{\mu} : B(T) \times \Omega \to \mathbf{R}_+$ *defined by* $\hat{\mu}(A,\omega) = \mu(\omega)(A)$ *is a transition measure*;

(ii) *If* T *is compact and* $\mu : B(T) \times \Omega \to \mathbf{R}_+$ *is a finite transition measure, then* $\hat{\mu} : \Omega \to M_+^b(T)$ *defined by* $\hat{\mu}(\omega)(\cdot) = \mu(\cdot,\omega)$ *is measurable when* $M_+^b(T)$ *is equipped with the narrow topology.*

Proof. (i) Let $V \subseteq T$ be open. We claim that $\varphi_V : \omega \to \hat{\mu}(V,\omega) = \mu(\omega)(V)$ is measurable. Indeed note that $\varphi_V = \theta_2 \circ \theta_1$ where $\theta_1 : \Omega \to M_+^b(T)$ is defined by $\theta_1(\omega) = \mu(\omega)$ and $\theta_2 : M_+^b(T) \to \mathbf{R}_+$ is defined by $\theta_2(\lambda) = \lambda(V)$. By hypothesis $\theta_1(\cdot)$ is measurable while from the Portmanteau Theorem, see, for example, Ash [**1**, Theorem 4.5.1], $\theta_2(\cdot)$ is lower semicontinuous on $M_+^b(T)$ furnished with the narrow topology. Therefore, $\theta_2 \circ \theta_1 = \varphi_V$ is measurable. Then, exploiting the regularity of $\mu(\omega)(\cdot) = \hat{\mu}(\cdot,\omega)$, see Ash [**1**, Corollary 4.3.7], we know that, given $A \in B(T)$, we can find V_n open, $A \subseteq V_n$ such that $0 \leq \hat{\mu}(\omega, V_n) - \hat{\mu}(\omega, A) \leq 1/n$. So $\hat{\mu}(A,\omega) = \lim \hat{\mu}(V_n,\omega)$, which shows that $\omega \to \hat{\mu}(A,\omega)$ is measurable for every $A \in B(T) \Rightarrow \hat{\mu}(\cdot,\cdot)$ is a transition measure.

(ii) Let $f \in C(T)$. Then $\omega \to \int_T f(t)\mu(dt,\omega)$ is measurable implies $\omega \to \langle f, \hat{\mu}(\omega)\rangle$ is measurable, where $\langle\cdot,\cdot\rangle$ denotes the duality brackets for the pair $(C(T), M(T))$ (Riesz representation theorem; see Ash [**1**, Theorem 4.3.13]). Since $f \in C(T)$ was arbitrary, $\omega \to \hat{\mu}(\omega)$ is w^*-measurable, i.e., measurable from Ω into $M_+(T)$ with the narrow topology (note that since T is compact the weak* and narrow topologies coincide on $M(T)$). □

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