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A FORCED PENDULUM EQUATION WITH MANY PERIODIC SOLUTIONS

RAFAEL ORTEGA

1. Introduction. Consider the periodic problem for the forced pendulum equation

$$x'' + A\sin x = p(t) \tag{1.1}$$

where $A > 0$ and $p(t)$ satisfies

$$p \in L^1(\mathbf{R}/T\mathbf{Z}), \quad \int_0^T p(t)\,dt = 0. \tag{1.2}$$

This problem has a long history that can be found in [**6**]. In particular, it is known that for each p verifying (1.2) there exist at least two T-periodic solutions that are geometrically different (this means that they do not differ by a multiple of 2π). Recently it was proved in [**3**] that for arbitrary A it is possible to find a certain forcing term $p(t)$ in the conditions of (1.2) and such that (1.1) has at least four different T-periodic solutions. The basic technique in [**3**] was singularity theory, and the result was of interest because A was arbitrary. We remark that if $A > (2\pi/T)^2$ the result is trivial. In fact, the autonomous equation with $p = 0$ has a closed orbit with minimal period T, and this orbit produces a continuum of different T-periodic solutions. In the present paper the following result is proved.

Theorem 1.1. *Given $A > 0$ and an integer $N \geq 1$ there exists $p(t)$ satisfying* (1.2) *and such that* (1.1) *has at least $2N$ T-periodic solutions that are geometrically different. In addition, there exists $\delta > 0$ such that if $\tilde{p}(t)$ satisfies* (1.2) *and $\|p - \tilde{p}\|_{L^1} < \delta$ then the conclusion also holds for $\tilde{p}$.*

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The proof of this result will be based on the perturbation method developed in [**5**]. This paper considers the equation

$$x'' + g(x) = \varepsilon p(t)$$

and assumes that the autonomous equation, $\varepsilon = 0$, has a closed orbit γ of period T. Then, imposing certain conditions on p, it is possible to obtain many bifurcations from γ. In this way the perturbed equation has many periodic solutions for small ε. The results in [**5**] are not directly applicable in our case because the pendulum equation has not closed orbits of period T when $A \le (2\pi/T)^2$. However, such an orbit exists if the phase space is the cylinder instead of the plane. From this orbit it is possible to create a continuum of T-periodic solutions of (1.1) with $p = p_0$, $p_0(t) = -2\pi \sum_{n=-\infty}^{+\infty} \delta'_{nT}(t)$. Here $\delta_{t_0}(t)$ is the Dirac measure at $t = t_0$, and the derivatives are understood in the sense of distributions. Of course, p_0 is not a function and satisfies (1.2) only in a generalized sense. At this point we apply the ideas of [**5**] to the perturbed equation

$$x'' + A \sin x = p_\varepsilon(t)$$

in such a way that p_ε satisfies (1.2) for some small ε and has many periodic solutions.

2. An outline of the construction. Assuming that (1.2) holds, we consider the change of variables

$$x = y + P(t)$$

where $P \in W^{2,1}(\mathbf{R}/T\mathbf{Z})$, $P'' = p$. This change transforms (1.1) in

$$y'' + A \sin(y + P(t)) = 0. \tag{2.1}$$

The sets of T-periodic solutions of both equations are in a one-to-one correspondence because the change is periodic in time. Even when $P(t)$ is not smooth, the equation (2.1) makes sense, but in such a case (2.1) is not equivalent to an equation of the kind (1.1).

Consider the equation (2.1) with $P(t) = 2\pi t/T$. In this case the function P is not periodic but the equation is not changed if P is replaced by $P_0(t) = 2\pi(t/T - [t/T])$, that is, a periodic and nonsmooth function. This example was first considered in [**1**] and it can be shown that, for $P = P_0$, there exists a continuum of T-periodic solutions $\{y_c\}_{c\in\mathbf{R}}$. The main idea in our construction will be to bifurcate simultaneously at many points of this continuum. To achieve this we shall consider the perturbed equation

$$y'' + A\sin(y + P_0(t) + \Psi(t,\varepsilon)) = 0$$

and impose conditions on Ψ that guarantee:

(i) the previous equation has at least $2N$ periodic bifurcations for $\varepsilon = 0$ of the form

$$y_i(t;\varepsilon) = y_{c_i}(t) + O(\varepsilon), \quad \varepsilon \longrightarrow 0$$

with $0 \le c_0 < c_1 < \cdots < c_{2N-1} < T$.

(ii) $P_0 + \Psi(\cdot,\varepsilon)$ is smooth for some ε small.

The first condition will produce many different periodic solutions when ε is small, and the second condition allows us to transform the equation to one of the kind (1.1).

3. The autonomous pendulum equation. Consider the autonomous equation

$$x'' + A\sin x = 0. \tag{3.1}$$

We denote by $x_0(t)$ the solution of (3.1) satisfying:

(i) $x_0(t+T) = x_0(t) + 2\pi$ for all $t \in \mathbf{R}$,

(ii) $x_0'(t) > 0$ for all $t \in \mathbf{R}$,

(iii) $x_0(0) = 0$.

This solution exists and is unique. In fact, from the conservation of energy, (i) and (ii), we deduce that it must verify

$$x' = \sqrt{2(E + A\cos x)}$$

for some $E > A$. Also, it is easy to prove that the function

$$\tau(E) = \int_0^{2\pi} \frac{d\xi}{\sqrt{2(E + A\cos\xi)}}, \quad E > A$$

is smooth, strictly decreasing and $\tau(+\infty) = 0$, $\tau(A+0) = +\infty$. In consequence, there is a unique $E > A$ such that $\tau(E) = T$ and x_0 is a solution of the corresponding first order equation with initial condition $x(0) = 0$. The uniqueness of x_0 implies that it is an odd function. The Fourier expansion of x_0' in terms of cosines is denoted by

$$x_0'(t) \sim \sum_{n \geq 0} a_n \cos \frac{2n\pi t}{T}.$$

Lemma 3.1. *The set* $I = \{n \in \mathbf{N} : a_n \neq 0\}$ *is infinite.*

Proof. A trigonometric polynomial of period T and degree $N \geq 1$ can be written in the form $f(t) = \sum_{|n| \leq N} f_n e^{2ni\pi t/T}$ with $\bar{f}_n = f_{-n}$, $f_N \neq 0$. We use the notation $d(f) = N$ and remark that $d(f') = d(f)$, $d(fg) = d(g) + d(g)$. By a contradiction argument, assume that x_0' is a trigonometric polynomial and $d(x_0') = N \geq 1$. From the equation we deduce also that $\sin x_0$ is a trigonometric polynomial with degree N. Taking derivatives, $(\cos x_0)' = -x_0' \sin x_0$ and $d(\cos x_0) = 2N$. Taking derivatives again, $(\sin x_0)' = x_0' \cos x_0$ and $d(\sin x_0) = 3N$. A contradiction with the previous value of this degree. □

Remark. It is possible to express the period function $\tau(E)$ in terms of elliptic integrals and x_0 in terms of the Jacobi functions, see [**2**]. Using the Fourier expansion of *sn* and *cn* one can compute the expansion of x_0' and verify the validity of the previous lemma in a direct but more tedious way.

4. The perturbation result. In this section we consider the differential equation

$$y'' + A \sin\left(y + \frac{2\pi t}{T} + \psi(t)\right) = 0 \tag{4.1}$$

where $\psi \in L^2(\mathbf{R}/T\mathbf{Z})$. When ψ is smooth, the change of variables $x = y + 2\pi t/T + \psi(t)$ reduces (4.1) to the forced pendulum equation with $p = \psi''$. When $\psi = 0$ the function $x_0(t)$ of the previous section allows us to construct a continuum of T-periodic solutions of (4.1). These periodic solutions are defined as

$$y_c(t) = x_0(t+c) - \frac{2\pi t}{T}, \quad c \in \mathbf{R}.$$

The existence of such a continuum was first observed in [**1**]. We look for small perturbations of $y_0(t)$ when ψ is small.

Proposition 4.1. *Given $\nu > 0$ there exist positive constants C and c such that if the following conditions hold*

$$\|\psi\|_{L^2} \le c, \qquad \int_0^T x_0'''(t)\psi(t)\,dt = 0,$$
$$\int_0^T x_0''''(t)\psi(t)\,dt \ge \nu\|\psi\|_{L^2},$$

then (4.1) *has a T-periodic solution $y(t;\psi)$ satisfying*

$$\left| y(t;\psi) - x_0(t) + \frac{2\pi t}{T}\right| + \left| y'(t;\psi) - x_0'(t) + \frac{2\pi}{T}\right| \le C\|\psi\|_{L^2}$$

for all $t \in \mathbf{R}$.

This result will be obtained as a modification of the results in [**5**]. The proof is postponed to the end of the paper.

5. Proof of Theorem 1.1. We start with a multiplicity result for equation (4.1). To state this result, we need to consider the convolution operator generated by x_0'''. This operator associates to $\psi \in L^2(\mathbf{R}/T\mathbf{Z})$ the smooth function

$$F_\psi(\tau) = \int_0^T x_0'''(t-\tau)\psi(t)\,dt, \quad \tau \in \mathbf{R}.$$

Lemma 5.1. *Given an integer N and positive constants ρ, ν with $\rho < T/(2N+1)$ there exists $\varepsilon > 0$ such that* (4.1) *has at least $2N$ T-periodic* *solutions that are geometrically different for every* $\psi \in L^2(\mathbf{R}/T\mathbf{Z})$ *with* $\|\psi\|_{L^2} \leq \varepsilon$ *and satisfying the condition stated below*

$$(C_N) \qquad \begin{cases} F_\psi \text{ has } 2N \text{ zeros in } [0,T] \text{ satisfying} \\ \rho < \tau_1 < \tau_2 < \cdots < \tau_{2N} < T-\rho, \\ |\tau_i - \tau_j| \geq \rho, \quad i \neq j, \\ |F'_\psi(\tau_i)| \geq \nu \|\psi\|_{L^2}. \end{cases}$$

Proof. From the definition of x_0 in Section 3 we obtain positive constants β and γ, with $\gamma < 2\pi$, such that

$$(5.1) \qquad |x_0(t_1) - x_0(t_2)| \geq \beta, \quad \forall\, t_1, t_2 \in \mathbf{R}, \quad \text{with} \quad |t_1 - t_2| \geq \rho,$$

$$(5.2) \qquad |x_0(t_1) - x_0(t_2)| \leq \gamma, \quad \forall\, t_1, t_2 \in [\rho, T-\rho].$$

Let C and c be the constants given by Proposition 4.1, and define $\varepsilon = \min\{c, \beta/(4C), (2\pi - \gamma)/(4C)\}$. Assume that $\|\psi\|_{L^2} \leq \varepsilon$. If (C_N) holds, it follows from the perturbation result that the equation

$$(5.3) \qquad y'' + A\sin\left(y + \frac{2\pi t}{T} + \psi(t - \tau_i)\right) = 0$$

has a T-periodic solution y_i satisfying

$$\left| y_i(t) - x_0(t) + \frac{2\pi t}{T} \right| + \left| y'_i(t) - x'_0(t) + \frac{2\pi}{T} \right| \leq C\|\psi\|_{L^2}.$$

The functions $z_i(t) = y_i(t + \tau_i) + 2\pi\tau_i/T$ are T-periodic solutions of (4.1). In view of (5.1) and (5.2), they satisfy, $i \neq j$,

$$\begin{aligned} |z_i(0) - z_j(0)| &\geq |x_0(\tau_i) - x_0(\tau_j)| - 2C\|\psi\|_{L^2} \\ &\geq \beta - 2C\|\psi\|_{L^2} > 0, \\ |z_i(0) - z_j(0)| &\leq |x_0(\tau_i) - x_0(\tau_j)| + 2C\|\psi\|_{L^2} \\ &\leq \gamma + 2C\|\psi\|_{L^2} < 2\pi. \end{aligned}$$

As a consequence, all the solutions $z_i(t)$, $i = 1, \dots, 2N$ are different.

The Fourier expansion of x_0''' is of the form

$$x_0'''(t) \sim \sum_{n\geq 1} \alpha_n \cos\frac{2n\pi t}{T}$$

with infinitely many coefficients α_n different from zero. This fact follows from Lemma 3.1. The function P_0 is the T-periodic function, defined in Section 2, $P_0(t) = 2\pi(t/T - [t/T])$. □

Lemma 5.2. *Assume that* $\alpha_N \neq 0$. *Then, given* $\varepsilon > 0$, *there exists* $\psi \in L^2(\mathbf{R}/T\mathbf{Z})$ *such that*

(i) $\|\psi\|_{L^2} \leq \varepsilon$

(ii) $P_0 + \psi \in C^2(\mathbf{R}/T\mathbf{Z})$

and condition (C_N) *of Lemma* 5.1 *holds with* $\rho = T/(8N)$, $\nu = (N\pi/2)|\alpha_N|\sqrt{2/T}$.

Proof. The function $\chi(t) = (\varepsilon/2)\sqrt{(2/T)}\cos(2N\pi t/T)$ satisfies $\|\chi\|_{L^2} = \varepsilon/2$, $F_\chi(\tau) = \alpha_N(T/2)\chi(\tau)$, so that F_χ has the zeros $\tau_i^* = (i-1/2)T/(2N)$ and $|F_\chi'(\tau_i^*)| \geq N\pi|\alpha_N|\sqrt{(2/T)}\|\chi\|_{L^2}$. Since $\chi + P_0$ belongs to $L^2(\mathbf{R}/T\mathbf{Z})$ there exists a sequence $\phi_n \in L^2(\mathbf{R}/T\mathbf{Z})$ such that $\phi_n \in C^2(\mathbf{R}/T\mathbf{Z})$ and $\phi_n \to \chi + P_0$ in L^2. The function $\psi_n = \phi_n - P_0$ converges to χ in L^2. From the definition of F_ψ one deduces that $F_{\psi_n} \to F_\chi$ in $C^1(\mathbf{R}/T\mathbf{Z})$. In particular, the zeros of F_{ψ_n} tend to the zeros of F_χ and therefore ψ_n satisfies (i), (ii) and (C_N) when n is large. □

Proof of Theorem 1.1. Let $N \geq 1$ be such that $\alpha_N \neq 0$, and let ρ and ν be given as in the previous lemma. Select ε small enough so that Lemma 5.1 applies. According to Lemmas 5.2 and 5.1, there exists $\psi \in L^2(\mathbf{R}/T\mathbf{Z})$ such that $P_0 + \psi \in C^2(\mathbf{R}/T\mathbf{Z})$ and (4.1) has $2N$ T-periodic solutions. The equation (4.1) can be rewritten in the form

$$y'' + A\sin(y + P_0(t) + \psi(t)) = 0. \tag{5.4}$$

The change of variables $x = y + P_0(t) + \psi(t)$ transforms T-periodic solutions of (5.1) into T-periodic solutions of (1.1) with $p = (P_0 + \psi)''$. As a consequence, (1.1) will have $2N$ T-periodic solutions for such a p.

It remains to prove that these periodic solutions are preserved by small perturbations. Define $p = (P_0 + \psi)''$ and let $q \in L^1(\mathbf{R}/T\mathbf{Z})$ be such that $\int_0^T q = 0$ and $\|p - q\|_{L^1}$ is small. Let Q be the unique T-periodic solution of $Q'' = q$ with $\int_0^T Q = \int_0^T (P_0 + \psi)$ and define $\hat{\psi} = Q - P_0$. Then $\|\hat{\psi} - \psi\|_{L^2}$ is small and the pendulum equation

$$x'' + A \sin x = q(t)$$

can be transformed into

$$y'' + A \sin(y + P_0(t) + \hat{\psi}(t)) = 0.$$

Since F_ψ and $F_{\hat{\psi}}$ are close in the C^1-norm, we can apply Lemma 5.1 to the new equation to conclude that it also has $2N$ T-periodic solutions.

6. Proof of the perturbation result. This section follows along the lines of [**5**]. It is divided into several subsections.

I. *A Hill's equation.* The equation

$$z'' + [A \cos x_0(t)]z = 0 \tag{6.1}$$

is the linearization of the pendulum equation (3.1) at $x_0(t)$. Differentiating (3.1), we deduce that $p(t) = x_0'(t)$ is a positive T-periodic solution of (6.1). It satisfies the initial conditions

$$p(0) := \alpha > 0, \qquad p'(0) = 0. \tag{6.2}$$

The method of reduction of order allows us to obtain a second solution given by the formula

$$q(t) = p(t) \int_0^t \frac{ds}{p(s)^2}. \tag{6.3}$$

It satisfies

$$q(0) = 0, \qquad q'(0) = 1/\alpha. \tag{6.4}$$

As a consequence, the Wronskian $W(p, q)$ satisfies $W = 1$ and
(6.5)

$$p(t) = \alpha > 0, \qquad p'(t) = 0, \qquad q(T) := \beta > 0, \qquad q'(T) = 1/\alpha.$$

Lemma 6.1. *In the previous notations*

$$\int_0^T q(t)p(t)^2 \sin x_0(t)\, dt = -\beta\alpha. \tag{6.6}$$

Proof. Since x_0 is a primitive of p, integrating by parts

$$I := \int_0^T qp^2 \sin x_0 = [-qp\cos x_0]_0^T + \int_0^T (qp)' \cos x_0.$$

From (6.2), (6.4) and (6.5), $I = -\beta\alpha + \int_0^T (qp' + pq')\cos x_0$. Since p and q are solutions of (6.1),

$$\begin{aligned} I &= -\beta\alpha - \int_0^T A^{-1}(q'p'' + p'q'') \\ &= -\beta\alpha - \int_0^T A^{-1}(p'q')' \\ &= -\beta\alpha. \end{aligned}$$

II. *The linear nonhomogeneous equation.* We first consider the equation

$$z'' + [A\cos x_0(t)](z + \Psi(t)) = 0, \tag{6.7}$$

where $\Psi \in L^2(\mathbf{R}/T\mathbf{Z})$. The Fredholm alternative implies that (6.7) has T-periodic solutions if and only if

$$\int_0^T A[\cos x_0(t)]\Psi(t)p(t)\, dt = 0. \tag{6.8}$$

When (6.8) holds, the formula of variation of constants shows that there exists a unique solution of (6.7) that is T-periodic and verifies $z(0) = 0$. It is given by the formula

$$h_1(t) = -\frac{\alpha}{\beta}Bq(t) + \int_0^t [p(t)q(s) - p(s)q(t)]A\cos x_0(s)\Psi(s)\, ds, \tag{6.9}$$

where

$$B := \int_0^T A[\cos x_0(t)]\Psi(t)q(t)\,dt. \tag{6.10}$$

Lemma 6.2. *According to the previous notation, if* (6.8) *holds,*

$$\int_0^T A\sin x_0(t)(h_1(t)+\Psi(t))p^2(t)\,dt = \int_0^T p'''(t)\Psi(t)\,dt.$$

Proof. It is enough to prove the identity when Ψ is smooth, say $\Psi \in C^1(\mathbf{R}/T\mathbf{Z})$. The general case follows by an approximation argument.

$$\begin{aligned}\int_0^T A\sin x_0(h_1+\Psi)p^2 &= [-(h_1+\Psi)pA\cos x_0]_0^T \\ &\quad + \int_0^T [(h_1+\Psi)p]'A\cos x_0 \\ &= \int_0^T (h_1+\Psi)p'A\cos x_0 + \int_0^T (h_1+\Psi)'pA\cos x_0 \\ &= -\int_0^T h_1''p' - \int_0^T (h_1+\Psi)'p'' \\ &= -\int_0^T (h_1'p')' - \int_0^T \Psi'p'' \\ &= \int_0^T \Psi p'''.\end{aligned}$$

We now consider the more general equation

$$z'' + \alpha(t)z + \beta(t) = 0 \tag{6.11}$$

where $\alpha \in L^\infty(0,T)$, $\beta \in L^1(0,T)$. □

Lemma 6.3. *Assume that* $\|\alpha\|_{L^\infty} \le A$, $\|\beta\|_{L^1} \le k$. *Then there exists* $K > 0$, *depending only on* A *and* k, *such that for each solution of* (6.11), *the following estimate holds*

$$|z(t)| + |z'(t)| \le K[|z(0)| + |z'(0)| + 1], \quad \forall\, t \in [0,T].$$

The proof is elementary.

III. *A quantitative version of the implicit function theorem.* Let $F = F(x, y)$ be a function defined on

$$\Omega = \{(x, y) \in \mathbf{R}^N \times \mathbf{R} : |x| < 1, |y| < 1\}.$$

(i) Assume that F is C^2, $F(0, 0) = 0$ and

$$F_y(0, 0) \geq \mu > 0, \qquad |F_x|, |F_y|, |F_{xy}|, |F_{yy}| \leq M \quad \text{on } \Omega.$$

Then there exist ε, δ and C (depending only on μ and M) such that the solutions of

$$F(x, y) = 0, \qquad |x| \leq \delta, \qquad |y| \leq \varepsilon$$

are of the form $(x, \varphi(x))$ where φ is a C^1 function defined on $|x| \leq \delta$ and such that $|\varphi(x)| \leq C|x|$.

(ii) Assume that $N = 1$ and F is C^3, $F(0, y) = 0$, $|y| < 1$, $F_x(0, 0) = 0$ and

$$F_{xy}(0, 0) \geq \mu > 0, \qquad |F_{xx}|, |F_{xy}|, |F_{xxy}|, |F_{xyy}| \leq M \quad \text{on } \Omega.$$

Then there exist ε, δ and C, depending only on μ and M, such that the solutions of

$$F(x, y) = 0, \qquad |x| \leq \delta, \qquad |y| \leq \varepsilon$$

are of one of the following forms $(0, y)$ or $(x, \varphi(x))$ where φ is a C^1 function on $[-\delta, \delta]$ with $|\varphi(x)| \leq C|x|$.

IV. *Proof of Proposition 4.1.* From now on we consider the equation

$$y'' + A \sin\left(y + \frac{2\pi t}{T} + \varepsilon\Psi(t)\right) = 0 \tag{6.12}$$

where ε is a real parameter and $\Psi \in L^2(\mathbf{R}/T\mathbf{Z})$ satisfies

$$\|\Psi\|_{L^2} = 1, \qquad \int_0^T x_0'''(t)\Psi(t)\, dt = 0, \qquad \int_0^T x_0''''(t)\Psi(t)\, dt \geq \nu. \tag{6.13}$$

It will be sufficient to prove the existence of a T-periodic solution of (6.12), $y(t,\varepsilon)$ with $|\varepsilon| \le \varepsilon_0$, such that

$$(6.14)\qquad \begin{aligned} y(t,\varepsilon) &= x_0(t) - \frac{2\pi t}{T} + O(\varepsilon),\\ y'(t,\varepsilon) &= x_0'(t) - \frac{2\pi}{T} + O(\varepsilon), \quad \varepsilon \longrightarrow 0, \end{aligned}$$

where $\varepsilon_0 > 0$ and the previous asymptotic expansions are uniform with respect to Ψ satisfying (6.13).

Let $y(t;\xi,\eta,\varepsilon)$ be the solution of (6.12) with initial conditions

$$y(0) = \xi, \qquad y'(0) = \eta + \alpha.$$

(α is given by (6.2)). Define

$$\begin{aligned} F(\xi,\eta,\varepsilon) &= y(T;\xi,\eta,\varepsilon) - \xi,\\ G(\xi,\eta,\varepsilon) &= y'(T;\xi,\eta,\varepsilon) - \alpha - \eta. \end{aligned}$$

The solutions of $F = G = 0$ correspond in an obvious way to the initial conditions of the T-periodic solutions of (6.12). Since $\{y_c(t)\}_{c\in\mathbf{R}}$ is a continuum of T-periodic solutions for $\varepsilon = 0$, we obtain

$$(6.15)\qquad F(y_c(0), y_c'(0) - \alpha, 0) = G(y_c(0), y_c'(0) - \alpha, 0) = 0$$

and, in particular, $F = G = 0$ at $(0,0,0)$.

As a first step in the proof we shall compute the derivatives of F and G at the origin and obtain

$$(6.16)\qquad \begin{aligned} &F_\xi = 0, \qquad G_\xi = 0, \qquad F_\eta = \alpha\beta, \qquad G_\eta = 0,\\ &F_\varepsilon = \alpha B, \qquad G_\varepsilon = 0, \quad \text{at} \quad (\xi,\eta,\varepsilon) = (0,0,0). \end{aligned}$$

(α, β and B are defined by (6.2), (6.5) and (6.10)).

Once these derivatives are computed and, since $F_\eta > 0$, we apply the implicit function theorem to solve $F = 0$ with respect to $\eta = H(\xi,\varepsilon)$ to obtain

$$F(\xi, H(\xi,\varepsilon), \varepsilon) = 0.$$

It follows from (6.16) that

$$(6.17)\qquad H_\xi = 0, \qquad H_\varepsilon = -\frac{B}{\beta} \quad \text{at} \quad (\xi,\varepsilon) = (0,0).$$

Next we define $J(\xi,\varepsilon)=G(\xi,H(\xi,\varepsilon),\varepsilon)$. The uniqueness in the implicit function theorem reduces $F=G=0$ to $J=0$ (in a neighborhood of the origin). Applying (6.15), we obtain

$$H(y_c(0),0)=y_c'(0)-\alpha,\qquad J(y_c(0),0)=0$$

so that $J(\xi,0)=0$ for all ξ. Also $J_\xi=J_\varepsilon=0$ at $(0,0)$ thanks to (6.16) and (6.17). We are now in the position of the classical bifurcation theorem as soon as $J_{\xi\varepsilon}(0,0)\neq 0$. In fact, we shall prove

$$J_{\xi\varepsilon}(0,0)=\frac{1}{\alpha^2}\int_0^T p'''\Psi, \tag{6.18}$$

so that $J_{\xi\varepsilon}\geq(1/\alpha^2)\nu$ thanks to (6.13). As a consequence, there exists a function $\xi=\varphi(\varepsilon)$, $|\varepsilon|\leq\varepsilon_0$ such that $J(\varphi(\varepsilon),\varepsilon)=0$ and $\varphi(\varepsilon)=O(\varepsilon)$. The solutions $y(t,\varepsilon)=y(t;\varphi(\varepsilon),H(\varphi(\varepsilon),\varepsilon),\varepsilon)$ are T-periodic and satisfy

$$y(t,\varepsilon)=y_0(t)+O(\varepsilon),\qquad y'(t,\varepsilon)=y_0'(t)+O(\varepsilon),\quad \varepsilon\longrightarrow 0. \tag{6.19}$$

This asymptotic expansion is justified using the theorem of differentiability with respect to initial conditions and parameters together with the bounds

$$\varphi(\varepsilon)=O(\varepsilon),\qquad H(\varphi(\varepsilon),\varepsilon)=O(\varepsilon). \tag{6.20}$$

Even if we assume that (6.16) and (6.18) have already been checked, the proof is not concluded. It remains to show the uniformity of ε_0 and (6.19) with respect to Ψ. For this purpose, we shall apply the quantitative versions of the implicit function theorem given in III. First we apply III.1 to deduce that the domain of definition of H is uniform in Ψ. This is done by obtaining uniform bounds of F_ξ, F_η, F_ε, $F_{\xi\eta}$, $F_{\varepsilon\eta}$, $F_{\eta\eta}$ for all $(\xi,\eta,\varepsilon)\in\mathbf{R}^3$. Notice that $F_\eta(0,0,0)=\alpha\beta$ is independent of Ψ. Next we apply III.2 to J after obtaining uniform bounds of $J_{\varepsilon\varepsilon}$, $J_{\xi\varepsilon}$, $J_{\varepsilon\xi\xi}$, $J_{\xi\varepsilon\varepsilon}$ on some neighborhood of $(0,0)$ independent of Ψ. This proves the uniformity of ε_0 and (6.20). Finally, we deduce that (6.19) is also uniform because there are uniform bounds in $C^1[0,T]$ of y_ξ, y_η and y_ε.

Proof of (6.16). The functions $y_\xi(t;0,0,0)$, $y_\eta(t;0,0,0)$ are solutions of (6.1) with certain initial conditions that imply

$$y_\xi = \alpha^{-1}p, \qquad y_\eta = \alpha q.$$

On the other hand, $y_\varepsilon(t;0,0,0)$ is a solution of (6.7) with trivial initial conditions. From (6.13),

$$0 = \int_0^T x_0'''\Psi = \int_0^T p''\Psi = -\int_0^T \{A\cos x_0\}p\Psi,$$

and therefore (6.8) holds. From (6.9), y_ε can be expressed in the form $y_\varepsilon = h_1 + (\alpha B/\beta)q(t)$. The derivatives of F and G at the origin are

$$\begin{aligned}
&F_\xi = y_\xi(T) - 1, \qquad G_\xi = y_\xi'(T),\\
&F_\eta = y_\eta(T), \qquad G_\eta = y_\eta'(T) - 1\\
&F_\varepsilon = y_\varepsilon(T) = h_1(T) + \frac{\alpha B}{\beta}q(t) = \frac{\alpha B}{\beta}q(T),\\
&G_\varepsilon = y_\varepsilon'(T) = h_1'(T) + \frac{\alpha B}{\beta}q'(T) = 0
\end{aligned}$$

and we use (6.5) to deduce (6.16). □

Proof of (6.18). From the chain rule, we obtain

$$J_{\xi\varepsilon} = G_{\xi\varepsilon} + G_{\xi\eta}H_\varepsilon + G_{\eta\varepsilon}H_\xi + G_{\eta\eta}H_\varepsilon H_\xi + G_\eta H_{\xi\varepsilon}$$

and (6.16) and (6.17) lead to

$$J_{\xi\varepsilon} = G_{\xi\varepsilon} - \frac{B}{\beta}G_{\eta\xi} \quad \text{at } (0,0).$$

To compute $G_{\xi\eta}$ and $G_{\xi\varepsilon}$ we notice that $y_{\xi\eta}$ and $y_{\xi\varepsilon}$ are solutions of certain equations of the kind (6.11) with $\alpha = A\cos x_0$ and $\beta = -A(\sin x_0)y_\xi y_\eta$ or $\beta = -A(\sin x_0)y_\xi(y_\varepsilon + \Psi)$. Solving these equations and using (6.5), one obtains

$$\begin{aligned}
G_{\xi\eta} &= y_{\xi\eta}'(T) = \frac{1}{\alpha}\int_0^T A(\sin x_0)p^2 q,\\
G_{\xi\varepsilon} &= y_{\xi\varepsilon}'(T) = \frac{1}{\alpha^2}\int_0^T A(\sin x_0)p^2\left\{h_1 + \frac{\alpha B}{\beta}q + \Psi\right\},
\end{aligned}$$

and Lemmas 6.1 and 6.2 lead to

$$G_{\xi\eta} = -A\beta, \qquad G_{\xi\varepsilon} = -AB + \frac{1}{\alpha^2}\int_0^T p'''\Psi. \qquad \square$$

Uniform bounds of F_ξ, $F_{\eta,\dots}$, $F_{\eta\eta}$, $J_{\varepsilon\varepsilon}, \dots, J_{\xi\varepsilon\varepsilon}$. We use the notation

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial\xi^{\alpha_1}\partial\eta^{\alpha_2}\partial\varepsilon^{\alpha_3}}, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3).$$

First we prove that $\partial^\alpha y(t;\xi,\eta,\varepsilon)$ is bounded in $C^1[0,T]$ if $1 \le |\alpha| \le 3$, $\alpha \neq (0,0,3)$ and this bound is independent of ξ,η,ε and Ψ with $\|\Psi\|_{L^2} = 1$. When $|\alpha| = 1$, $\partial^\alpha y$ is the solution of an equation of the kind (6.11) with $\alpha(t) = A\cos(y(t;\xi,\eta,\varepsilon) + 2\pi t/T + \varepsilon\Psi(t))$ and $|\beta(t)| \le A|\Psi(t)|^{\alpha_3}$. We apply Lemma 6.3 to deduce that $\partial^\alpha y$ is bounded in C^1. In the same way, we obtain the bound for $|\alpha| = 2$ and finally for $|\alpha| = 3$, $\alpha_3 \neq 3$. As a consequence, we deduce that $\partial^\alpha F$, $\partial^\alpha G$ with $|\alpha| \le 3$, $\alpha_3 \neq 3$ are uniformly bounded. When $|\xi|, |\eta|, |\varepsilon|$ are small, the solution $y(t;\xi,\eta,\varepsilon)$ is close to $y_0(t)$ in $C^1[0,T]$ uniformly with respect to Ψ, $\|\Psi\|_{L^2} = 1$. This follows from a variant of the theorem of continuous dependence or from Gronwall's inequality. The derivative $y_\eta(t;\xi,\eta,\varepsilon)$ is the solution of

$$z'' + A\cos\left(y(t;\xi,\eta,\varepsilon) + \frac{2\pi t}{T} + \varepsilon\Psi(t)\right)z = 0,$$
$$z(0) = 0, \qquad z'(0) = 1,$$

and, if $|\xi|$, $|\eta|$, $|\varepsilon|$ are small, this linear equation is close in $(0,T)$ to (6.1) in the L^2-sense. The continuous dependence theorem and (6.16) allow us to assume that

$$F_\eta(\xi,\eta,\varepsilon) = y_\eta(T;\xi,\eta,\varepsilon) \ge \frac{\alpha\beta}{2}$$

in a neighborhood of the origin that may be small but independent of Ψ. From the previous estimates and implicit differentiation, it is easy to obtain bounds on $\partial^\beta H$, $\beta = (\beta_1, \beta_2)$, $|\beta| \le 3$, $\beta_2 < 3$. The bounds on the derivative of J follow by the chain rule.

Remark. It is possible to obtain another proof of Proposition 4.1 using the alternative method and the ideas in [**4**, p. 290]. In some sense that approach is simpler because it works directly with the equation instead of studying the Poincaré map. The counterpart is the need of a functional setting of infinite dimensions.

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Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain