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A BOUNDARY VALUE PROBLEM FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS WITH IMPULSE EFFECTS

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ABSTRACT. A two-point boundary value problem for a system of first order ordinary differential equations with impulse effects is studied. The method of upper and lower solutions is employed to obtain the existence of a solution and a method of forced monotonicity is employed to obtain iterative improvement. The main result is illustrated with an application to the Liénard equation with periodic boundary conditions.

1. Introduction. Let $n \geq 1$, $m \geq 0$ be integers. Let $I = [a,b] \subset \mathbf{R}$, and let $a = t_0 < t_1 < \cdots < t_{m+1} = b$ be given. Let $f : I \times \mathbf{R}^n \to \mathbf{R}^n$, $r_k : I \times \mathbf{R}^n \to \mathbf{R}^n$, $k = 1, \dots, m$, be continuous. Let M and N be $n \times n$ matrices with real entries, and let $c \in \mathbf{R}^n$. We shall study the impulsive boundary value problem (BVP) for the system of first order differential equations,

$$y' = f(t,y), \quad t \in I \backslash \{t_1, \dots, t_m\}, \tag{1.1}$$

$$\Delta y(t_k) = r_k(t_k, y(t_k^-)), \quad k = 1, \dots, m, \tag{1.2}$$

$$My(a) + Ny(b) = c, \tag{1.3}$$

where $\Delta y(t) = y(t^+) - y(t^-)$. For simplicity, we shall sometimes denote $y(t^-)$ by $y(t)$ and we shall sometimes denote the boundary operator, $My(a) + Ny(b)$, by Ty; note that we shall consider an impulsive BVP with fixed moments.

Bainov et al. [**2, 3, 4**] have developed the theory of impulsive differential equations. An extensive literature exists and is documented in [**3**]. In the case of periodic systems, Bainov and Simeonov [**3**] have

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thoroughly developed the theory and have obtained a Green's function representation for the solution of the BVP, (1.1), (1.2), (1.3), where $Ty = c$ has the form $y(a) = y(b)$. In [**2**] and [**3**], they employ a monotone iterative technique relying on upper and lower quasi-solutions to estimate solutions of the impulsive BVP, (1.1), (1.2), (1.3).

Werner [**8**] has developed a method of forced monotonicity for the BVP without impulse effects. He defined partial orders on $C_n(I)$, the set of continuous, n-vector real-valued functions defined on I; these partial orders are constructed naturally, once the Green's matrix, $G(t,s)$, for an associated BVP, $y' - D(t)y = 0$, $t \in I$, $Ty = 0$, is characterized.

In this paper we shall employ Werner's method of forced monotonicity to study the impulsive BVP, (1.1), (1.2), (1.3). In Section 2, we shall employ the Green's matrix for an associated BVP, $y' - D(t)y = 0$, $t \in I$, $Ty = 0$, to invert the impulsive BVP, (1.1), (1.2), (1.3). We shall then apply the natural partial orders, as constructed by Werner, and obtain conditions for which an appropriate functional integral operator is monotone. We shall then apply the method of upper and lower solutions to obtain the existence of a solution of the impulsive BVP, (1.1), (1.2), (1.3). Finally, in Section 3, we shall apply the abstract results, developed in Section 2, to a BVP with periodic boundary conditions. We point out that the technical details not related to the impulse effect have been previously developed by Werner [**8**].

In [**7**], Murty et al. have applied the contraction mapping principle to study the impulsive BVP, (1.1), (1.2), with the general boundary conditions (1.3).

2. Partial orders and a method of forced monotonicity. Let $D(t)$ be an $n \times n$ matrix with entries in $C(I)$. Let $U(t)$ denote a fundamental matrix for the system, $y' - D(t)y = 0$. Assume throughout that the homogeneous BVP, $y' - D(t)y = 0$, $t \in I$, $Ty = 0$, is uniquely solvable. Define the Green's matrix, $G(t,s)$, for the BVP, $y' - D(t)y = 0$, $t \in I$, $Ty = 0$, by

$$G(t,s) = \begin{cases} U(t)AMU(a)U^{-1}(s) & a \le s < t \le b, \\ U(t)(AMU(a) - E)U^{-1}(s) & a < t \le s \le b, \end{cases}$$

where E denotes the $n \times n$ identity matrix, and $A = (MU(a) + NU(b))^{-1}$. That A exists follows by the unique solvability of the homogeneous problem. The following lemma can be verified directly, or see [**8**].

Lemma 2.1. *Let* $h \in C_n(I)$, $c \in \mathbf{R}^n$. *u is a solution of the BVP,* $y' - D(t)y = h(t)$, $t \in I$, $Ty = c$ *if and only if*

$$u(t) = U(t)Ac + \int_a^b G(t,s)h(s)\,ds, \quad t \in I.$$

Let $PC_n(I)$ denote the set of piecewise continuous, n-vector, real-valued functions on I. Let $y \in PC_n(I)$, and let $y_{(k)}$ denote the restriction of y on $[t_k, t_{k+1}]$, $k = 0, \dots, m$. Let

$$B = \{y \in PC_n(I) : y_{(k)} \in C_n[t_k, t_{k+1}], k = 0, \dots, m\}.$$

Then B is a Banach space with norm,

$$\|y\| = \max_{k=0,\dots,m} \|y_{(k)}\|,$$

where $\|y_{(k)}\|$ denotes the $C_n[t_k, t_{k+1}]$ norm of $\|y_{(k)}\|$.

We first define an operator, K_1, by

$$K_1 y(t) = U(t)Ac + \int_a^b G(t,s)(f(s,y(s)) - D(s)y(s))\,ds, \quad t \in I.$$

Note that y is a solution of the BVP, $y' - D(t)y = f(t,y) - D(t)y$, $t \in I$, $Ty = c$, if and only if $y \in C_n(I)$ and $y(t) = K_1 y(t)$, $t \in I$. Now, define $K : B \to B$ by

(2.1)

$$\begin{aligned} Ky(t) = K_1 y(t) + U(t)\Bigg[&AMU(a)\sum_{k=1}^{j} U^{-1}(t_k) r_k(t_k, y(t_k)) \\ &+ (AMU(a) - E)\sum_{k=j+1}^{m} U^{-1}(t_k) r_k(t_k, y(t_k))\Bigg], \\ & t \in [t_j, t_{j=1}], \end{aligned}$$

$j = 0, \dots, m$, where, if $j = 0$, the first sum is zero and, if $j = m$, the second sum is zero.

Lemma 2.2. *$y(t)$ is a solution of the impulsive* BVP, (1.1), (1.2), (1.3), *if and only if* $y \in B$ *and* $y(t) = Ky(t)$, $t \in [t_k, t_{k+1}]$, $k = 0, \dots, m$.

Proof. Lemma 2.2 can be verified directly. We also refer the reader to [**3**, page 39], with $B_k = 0$, $k = 1, \dots, m$ and $M = -N = E$. We do note that the solution, u, of the nonhomogeneous impulsive BVP, $y' - D(t)y = h$, $t \in I\backslash\{t_1, \dots, t_m\}$, $\Delta y(t_k) = r_k$, $k = 1, \dots, m$, $Ty = c$, has the characterization, $u = u_1 + u_2$, where u_1 satisfies the BVP, $y' - D(t)y = h$, $t \in I$, $Ty = c$, and u_2 satisfies the impulsive BVP, $y' - D(t)y = 0$, $t \in I\backslash\{t_1, \dots, t_m\}$, $\Delta y(t_k) = r_k$, $k = 1, \dots, m$, $Ty = 0$. Thus, $K_1 y$ plays the role of u_1. To determine the characterization for u_2, we find the solution, v_j, of the impulsive BVP, $y' - D(t)y = 0$, $t \in I\backslash\{t_j\}$, $\Delta y(t_j) = r_j$, $Ty = 0$. Then

$$v_j(t) = \begin{cases} U(t)\alpha_1 & t < t_j, \\ U(t)\alpha_2 & t > t_j, \end{cases}$$

where $\alpha_1 = (AMU(a) - E)U^{-1}(t_j)r_j$ and $\alpha_2 = AMU(a)U^{-1}(t_j)r_j$. Thus, $u_2 = \sum_{j=1}^m v_j$ and is represented by the second term in the definition of K in (2.1). $\square$

Remark. Representations of the Green's matrix for multipoint point boundary value problems related to $y' + Dy = 0$ are well-known (see [**1**] or [**5**]). Hence, the abstract development in this paper can be readily extended to multipoint boundary value problems.

We now introduce some partial orderings on B.

i) For y, $z \in B$, $y = (y_1, \dots, y_n)^T$, $z = (z_1, \dots, z_n)^T$, define the relation, $\leq$, by $y \leq z$ if and only if $y_j(t) \leq z_j(t)$, $j = 1, \dots, n$, $t \in [t_k, t_{k+1}]$, $k = 0, \dots, m$. Then $\leq$ is a partial ordering on B and we shall call $\leq$ the natural partial ordering on B. It is readily seen that B is a partially ordered Banach space with respect to the natural partial ordering.

ii) Let $H : B \to B$ be an invertible linear operator. For $y, z \in B$, define the relation, $\leq_H$, by $y \leq_H z$ if and only if $Hy \leq Hz$. Then $\leq_H$ is a partial ordering on B and we shall say that $\leq_H$ is the partial ordering induced by H. Again, it is readily shown that B is a partially ordered Banach space with respect to $\leq_H$.

In applying these partial orders, now assume that there exist $n \times n$ matrices, H and J, with J invertible, such that $H(AMU(a))J^{-1} \geq 0$, elementwise, and $H(AMU(a)-E)J^{-1} \geq 0$, elementwise. Let $\leq_1$ denote the partial ordering induced by $HU^{-1}(t)$, and let $\leq_2$ denote the partial ordering induced by $JU^{-1}(t)$. Throughout the remainder of the paper, assume that f satisfies the monotone property,

$$y, z \in B, \quad y \leq_1 z \quad \text{implies} \quad f(t,y) - Dy \leq_2 f(t,z) - Dz, \tag{2.2}$$

and r_k, $k = 1, \dots, m$, satisfies the monotone property,

$$y, z \in B, \quad y \leq_1 z \quad \text{implies} \quad r_k(t,y) \leq_2 r_k(t,z), \quad k = 1, \dots, m. \tag{2.3}$$

Lemma 2.3. *Assume that f and r_k, $k = 1, \dots, m$, are continuous, and assume that* (2.2) *and* (2.3) *are satisfied. Then K is a monotone operator with respect to $\leq_1$ on B; that is, if $y \leq_1 z$, then $Ky \leq_1 Kz$.*

Proof. We show that if $y \leq_1 z$, then $HU^{-1}(t)Ky \leq HU^{-1}(t)Kz$. So, assume $y \leq_1 z$. For $t \in [t_j, t_{j+1}]$,

$$\begin{aligned}
&HU^{-1}(t)Ky(t)\\
&\quad = HU^{-1}(t)\Big[U(t)Ac\\
&\qquad + \int_a^t U(t)AMU(a)U^{-1}(s)(f(s,y(s)) - D(s)y(s))\,ds\\
&\qquad + \int_t^b U(t)(AMU(a)-E)U^{-1}(s)(f(s,y(s)) - D(s)y(s))\,ds\\
&\qquad + U(t)AMU(a)\sum_{k=1}^{j} U^{-1}(t_k)r_k(t_k, y(t_k))\\
&\qquad + U(t)(AMU(a)-E)\sum_{k=j+1}^{m} U^{-1}(t_k)r_k(t_k, y(t_k))\Big]\\
&\quad = HAc\\
&\qquad + \int_a^t HAMU(a)J^{-1}(JU^{-1}(s))(f(s,y(s)) - D(s)y(s))\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_t^b H(AMU(a)-E)J^{-1}(JU^{-1}(s))(f(s,y(s))-D(s)y(s))\,ds\\
&+HAMU(a)J^{-1}\sum_{k=1}^{j}JU^{-1}(t_k)r_k(t_k,y(t_k))\\
&+H(AMU(a)-E)J^{-1}\sum_{k=j+1}^{m}JU^{-1}(t_k)r_k(t_k,y(t_k))\\
\le{}& HAc+\int_a^t HAMU(a)J^{-1}(JU^{-1}(s))(f(s,z(s))-D(s)z(s))\,ds\\
&+\int_t^b H(AMU(a)-E)J^{-1}(JU^{-1}(s))(f(s,z(s))-D(s)z(s))\,ds\\
&+HAMU(a)J^{-1}\sum_{k=1}^{j}JU^{-1}(t_k)r_k(t_k,z(t_k))\\
&+H(AMU(a)-E)J^{-1}\sum_{k+j+1}^{m}JU^{-1}(t_k)r_k(t_k,z(t_k))\\
={}& HU^{-1}(t)Kz(t). \qquad \square
\end{aligned}
$$

Remark. The terms $AMU(a)$ and $AMU(a)-E$ which appear in the representation of $G(t,s)$ play a key role in Werner's [**8**] method of forced monotonicity. This method has been illustrated in the proof of Lemma 2.3. Since precisely these terms arise in the characterization of u_2, discussed in the proof of Lemma 2.2, Werner's method carries over naturally to the impulsive BVP, (1.1), (1.2), (1.3).

We now introduce a third partial order which governs the behavior of upper and lower solutions with respect to the boundary conditions. In particular, let $\le_3$ denote a partial order on $\mathbf{R}^n$ induced by HA.

Theorem 2.4. *Assume the hypotheses of Lemma* 2.3. *Assume that there exist an upper solution,* $v_1(x)$, *and a lower solution,* $w_1(x)$, *with respect to the impulsive* BVP, (1.1), (1.2), (1.3), *satisfying*

i) $w_1 \le_1 v_1$,

ii) $Tw_1 \le_3 c \le_3 Tv_1$,

iii) $w_1' - f(t, w_1) \leq_2 0 \leq_2 v_1' - f(t, v_1)$,

iv) $\Delta w_1(t_k) - r(t_k, w_1(t_k)) \leq_2 0 \leq_2 \Delta v_1(t_k) - r(t_k, v_1(t_k))$, $k = 1, \dots, m$.

Then the impulsive BVP, (1.1), (1.2), (1.3), *has a solution*, $y(x)$, *satisfying*

$$w_1 \leq_1 y \leq_1 v_1. \tag{2.4}$$

Further, define sequences, $\{w_l\}, \{v_l\}$, *by* $w_{l+1} = Kw_l, v_{l+1} = Kv_l$, $l = 1, 2, \dots$. *Then if* y *is a solution of* (1.1), (1.2), (1.3), *satisfying* (2.4), *then*

$$w_l \leq_1 w_{l+1} \leq_1 y \leq_1 v_{l+1} \leq_1 v_l, \tag{2.5}$$

for $l \geq 1$. *The sequence* $\{w_l\}$ *converges monotonically in* B, *with respect to* $\leq_1$, *to* w, $\{v_l\}$ *converges monotonically in* B, *with respect to* $\leq_1$, *to* v, *where* w *and* v *are solutions of the* BVP (1.1), (1.2), (1.3), *and* $w \leq_1 v$. *Finally, if* y *is a solution of the* BVP (1.1), (1.2), (1.3), *satisfying* (2.4), *then*

$$w \leq_1 y \leq_1 v.$$

Proof. Define $\Omega = \{z \in B : w_1 \leq_1 z \leq_1 v_1\}$. As outlined in [**5**, Chapter III], we shall show

$$w_1 \leq_1 w_2 \leq_1 v_2 \leq_1 v_1. \tag{2.6}$$

It will then follow by the monotonicity of K that $K(\Omega) \subseteq \Omega$ and the Schauder fixed point theorem applies to give the existence of a solution, y, of the impulsive BVP, (1.1), (1.2), (1.3), satisfying (2.4).

To that end, first note that v_1 is the solution of the impulsive BVP,

$$\begin{gathered} y' - Dy = v_1' - Dv_1, \quad t \in I \backslash \{t_1, \dots, t_m\}, \\ \Delta y(t_k) = \Delta v_1(t_k), \quad k = 1, \dots, m, \\ My(a) + Ny(b) = Mv_1(a) + Nv_1(b). \end{gathered}$$

Thus,

$$v_1(t) = U(t)ATv_1 + \int_a^b G(t,s)(v_1'(s) - D(s)v_1(s))\,ds$$
$$+ U(t)\Bigg[AMU(a)\sum_{k=1}^{j} U^{-1}(t_k)\Delta v_1(t_k)$$
$$+ (AMU(a) - E)\sum_{k=j+1}^{m} U^{-1}(t_k)\Delta v_1(t_k)\Bigg],$$

$t \in [t_j, t_{j+1}]$, $j = 0, \dots, m$. By a straightforward argument, completely analogous to that given in the proof of Lemma 2.3, $v_2 = Kv_1 \le_1 v_1$, since v_1 satisfies ii), iii) and iv). Similarly, $w_1 \le_1 Kw_1$. Since $w_1 \le_1 v_1$, (2.6) follows from Lemma 2.3.

(2.5) now follows immediately from Lemma 2.3. Finally the existence of a minimal solution, w, and the existence of a maximal solution, v in Ω, follow by application of Dini's theorem on each subinterval, $[t_j, t_{j+1}]$, $j = 0, \dots, m$. □

3. Periodic boundary conditions. Let $[a, b] = [0, \omega]$. We now consider the impulsive BVP, (1.1), (1.2), satisfying

$$y(0) = y(\omega). \tag{3.1}$$

Given the boundary conditions, (3.1), set $M = -N = E$.

Lemma 3.1. *Let D be an $n \times n$ matrix with real entries and with real eigenvalues, $\lambda_i \ne 0$, $i = 1, \dots, n$. Assume that D is diagonalizable by H; that is, assume $HDH^{-1} = \operatorname{diag}\{\lambda_i\}$. Set $J = -\operatorname{diag}\{\operatorname{sgn}(\lambda_i)\}H$. Then $HAMU(a)J^{-1} \ge 0$, elementwise, and $H(AMU(a) - E)J^{-1} \ge 0$, elementwise.*

Proof. Note that $HDJ^{-1} = \operatorname{diag}\{-|\lambda_i|\}$. Let $U(t) = e^{Dt}$ be the fundamental matrix for the system, $y' - Dy = 0$. Then $AMU(a) = (E - e^{D\omega})^{-1}$. Thus, $HAMU(a)H^{-1} = \operatorname{diag}\{(1 - e^{\lambda_i \omega})^{-1}\} = \operatorname{diag}\{\mu_i\}$ and $H(AMU(a) - E)H^{-1} = \operatorname{diag}\{\mu_i - 1\}$. Note that if $\lambda_i > 0$, then $\mu_i < 0$, and if $\lambda_i < 0$, then $\mu_i > 1 > 0$. Thus, $HAMU(a)J^{-1} =$

$-\text{diag}\,\{\mu_i \text{sgn}\,\lambda_i\} = \text{diag}\,\{|\mu_i|\}$, and $H(AMU(a)-E)J^{-1} = \text{diag}\,\{|\mu_i - 1|\}$. $\square$

To be more specific, let $y = (y_1, y_2)^T$, and consider the two-dimensional impulsive BVP,

$$\begin{cases} y_1' = y_2 - f(t, y_1), \\ y_2' = -g(t, y_1), \end{cases} \qquad t \in I\backslash\{t_1, \dots, t_m\}, \tag{3.2}$$

$$\Delta y_1(t_k) = 0, \qquad \Delta y_2(t_k) = r_k(t_k, y_1(t_k)), \quad k = 1, \dots, m, \tag{3.3}$$

(3.1), where each of $f, g : [0, \omega] \times \mathbf{R} \to \mathbf{R}$ is continuous, f has a continuous partial derivative with respect to each component, and g has a continuous partial derivative with respect to the second component. Moreover, we shall assume that f is ω-periodic in t. Assume for each $k = 1, \dots, m$, that $r_k : [0, \omega] \times \mathbf{R} \to \mathbf{R}$ is continuous and monotone decreasing with respect to the second component.

Theorem 3.2. *Assume that there exist upper and lower solutions,*

$$(w_1(t), w_1'(t) + f(t, w_1(t)))^T, (v_1(t), v_1'(t) + f(t, v_1(t)))^T \in B,$$

with respect to the impulsive BVP, (3.2), (3.3), (3.1), *satisfying*

i) $w_1(t) \le v_1(t)$,

ii) $w_1(0) = w_1(\omega)$, $w_1'(0) \ge w_1'(\omega)$, $v_1(0) = v_1(\omega)$, $v_1'(0) \le v_1'(\omega)$,

iii) $v_1''(t) \le -(d/dt)f(t, v_1) - g(t, v_1)$, $w_1''(t) \ge -(d/dt)f(t, w_1) - g(t, w_1)$,

iv) $\Delta w_1(t_k) = 0 = \Delta v_1(t_k)$, $\Delta w_1'(t_k) - r(t_k, w_1) \ge 0 \ge \Delta v_1'(t_k) - r(t_k, v_1)$,

where the standard partial order on $[t_k, t_{k+1}]$, $k = 0, \dots, m$, *is employed in* i) *and* iii), *and the standard partial order on* $\mathbf{R}$ *is employed in* ii) *and* iv). *Assume that f and g are continuous, f has a continuous partial derivative with respect to each component, and g has a continuous partial derivative with respect to the second component. Moreover, assume that f is ω-periodic in t. Assume for each $k = 1, \dots, m$, that $r_k : [0, \omega] \times \mathbf{R} \to \mathbf{R}$ is continuous and monotone decreasing with respect* *to the second component. Then there exists a solution,* $y = (y_1, y_2)^T$ *of the impulsive* BVP, *(3.2), (3.3), (3.1), satisfying*

$$w_1(t) \le y_1(t) \le v_1(t), \quad t \in [t_k, t_{k+1}], \quad k = 0, \dots, m.$$

Proof. We apply Theorem 2.4 with

$$D = \begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix},$$

where $\lambda > 0$ is to be suitably chosen. First, apply Lemma 3.2 with

$$H = \begin{pmatrix} \lambda & 1 \\ \lambda & -1 \end{pmatrix}, \qquad J = \begin{pmatrix} -\lambda & -1 \\ \lambda & -1 \end{pmatrix},$$

and then apply Theorem 2.4 with $\le_1$ induced by He^{-Dt}, $\le_2$ induced by Je^{-Dt}, and $\le_3$ induced by HA.

Choose $\lambda > 0$ such that for $y = (y_1, y_2)^T$, $z = (z_1, z_2)^T \in B$ with $w_1(t) \le y_1(t) \le z_1(t) \le v_1(t)$, $t \in [t_k, t_{k+1}]$, $k = 0, \dots, m$, then

(3.4)
$$\lambda^2(z_1 - y_1)(t) + \lambda(f(t, z_1(t)) - f(t, y_1(t))) + (g(t, z_1(t)) - g(t, y_1(t))) \ge 0,$$

(3.5)
$$\lambda(v_1(\omega) - w_1(\omega)) + (v_1'(\omega) - w_1'(\omega)) + (f(\omega, v_1(\omega)) - f(\omega, w_1(\omega))) \ge 0,$$

(3.6)
$$\lambda^2(z_1 - y_1)(t) - \lambda(f(t, z_1(t)) - f(t, y_1(t))) + (g(t, z_1(t)) - g(t, y_1(t))) \ge 0,$$

(3.7)
$$\lambda(v_1(0) - w_1(0)) - [(v_1'(0) - w_1'(0)) + (f(0, v_1(0)) - f(0, w_1(0))] \ge 0,$$

$t \in [t_k, t_{k+1}]$, $k = 0, \dots, m$.

To see that $\lambda > 0$ can be selected, recall that f and g have continuous partial derivatives with respect to the second component. Let $M_1 \ge |f_2(t, c)|$ for $0 \le t \le \omega$ and $w_1(t) \le c \le v_1(t)$, where f_2 denotes the partial derivative of f with respect to the second component. Similarly, let $M_2 \geq |g_2(t,c)|$ for $0 \leq t \leq \omega$ and $w_1(t) \leq c \leq v_1(t)$. Then

$$\lambda^2(z_1 - y_1)(t) + \lambda(f(t, z_1(t)) - f(t, y_1(t))) + (g(t, z_1(t)) - g(t, y_1(t))) \\ \geq (z_1 - y_1)(t)(\lambda^2 - M_1\lambda - M_2) \geq 0$$

for $\lambda > 0$ and sufficiently large. Hence, (3.4) can be satisfied and (3.6) can be satisfied similarly. For (3.5), if $v_1(\omega) > w_1(\omega)$, then (3.5) can be satisfied similarly. If $v_1(\omega) = w_1(\omega)$, then $v_1(0) = w_1(0)$, and

$$v_1'(\omega) \leq w_1'(\omega) \leq w_1'(0) \leq v_1'(0) \leq v_1'(\omega).$$

In particular, $v_1'(\omega) = w_1'(\omega)$ and (3.5) is satisfied. (3.7) is addressed similarly.

In this context, we now specify the meanings of the inequalities $\leq_1$, $\leq_2$ and $\leq_3$.

$$HDH^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

Thus,

$$He^{-Dt} = \begin{pmatrix} \lambda e^{-\lambda t} & e^{-\lambda t} \\ \lambda e^{\lambda t} & -e^{\lambda t} \end{pmatrix},$$

and $y \leq_1 z$ if and only if

$$\lambda y_1(t) + y_2(t) \leq \lambda z_1(t) + z_2(t),$$
$$\lambda y_1(t) - y_2(t) \leq \lambda z_1(t) - z_2(t),$$

$t \in [t_k, t_{k+1}]$, $k = 0, \dots, m$. Note that $y \leq_1 z$ implies that $y_1(t) \leq z_1(t)$, $t \in [t_k, t_{k+1}]$, $k = 0, \dots, m$. Similarly, $y \leq_2 z$ if and only if

$$-\lambda y_1(t) - y_2(t) \leq -\lambda z_1(t) - z_2(t),$$
$$\lambda y_1(t) - y_2(t) \leq \lambda z_1(t) - z_2(t),$$

$t \in [t_k, t_{k+1}]$, $k = 0, \dots, m$. Note that $y \leq_2 z$ implies that $y_2(t) \geq z_2(t)$, $t \in [t_k, t_{k+1}]$, $k = 0, \dots, m$. Finally, $HAH^{-1} = \operatorname{diag}\{\mu_i\}$ where $\mu_1 = (1 - e^{\lambda\omega})^{-1}$ and $\mu_2 = (1 - e^{-\lambda\omega})^{-1}$. Hence, $y \leq_3 z$ if and only if

$$-\lambda y_1 - y_2 \leq -\lambda z_1 - z_2,$$
$$\lambda y_1 - y_2 \leq \lambda z_1 - z_2.$$

We now show that each of (2.2) and (2.3) is satisfied so that Lemma 2.3 applies. Let $y = (y_1, y_2)^T$ and $z = (z_1, z_2)^T$. For (2.2), recall $y \leq_1 z$ implies $y_1(t) \leq z_1(t)$, $t \in [t_k, t_{k+1}]$, $k = 0, \dots, m$. Since

$$(y_2 - f(t, y_1), -g(t, y_1))^T - Dy = (-f(t, y_1), -g(t, y_1) - \lambda^2 y_1)^T,$$

(2.2) becomes

$$\begin{aligned} \lambda f(t, y_1) + g(t, y_1) + \lambda^2 y_1 &\leq \lambda f(t, z_1) + g(t, z_1) + \lambda^2 z_1, \\ -\lambda f(t, y_1) + g(t, y_1) + \lambda^2 y_1 &\leq -\lambda f(t, z_1) + g(t, z_1) + \lambda^2 z_1. \end{aligned}$$

These inequalities are valid by (3.4) and (3.6). For (2.3),

$$(0, r_k(t_k, y_1))^T \leq_2 (0, r_k(t_k, z_1))^T$$

reduces to $0 \leq r_k(t_k, y_1) - r_k(t_k, z_1)$, which is valid since r_k is monotone decreasing in the second component. Thus, Lemma 2.3 applies and the operator K is monotone with respect to $\leq_1$.

We now show that the hypotheses of Theorem 2.4 are satisfied. To see that

$$(w_1, w_1' + f(t, w_1))^T \leq_1 (v_1, v_1' + f(t, v_1))^T,$$

we require

$$\lambda w_1(t) + w_1'(t) + f(t, w_1(t)) \leq \lambda v_1(t) + v_1'(t) + f(t, v_1(t)),$$

and

$$\lambda w_1(t) - w_1'(t) - f(t, w_1(t)) \leq \lambda v_1(t) - v_1'(t) - f(t, v_1(t)).$$

To obtain the first inequality, employ condition iii) and (3.4) to obtain

$$\begin{aligned} (3.8) \quad (v_1'' - w_1'')(t) &+ (d/dt)(f(t, v_1(t)) - f(t, w_1(t))) \\ &\leq \lambda^2 (v_1 - w_1)(t) + \lambda(f(t, v_1(t)) - f(t, w_1(t))). \end{aligned}$$

Multiply (3.8) by $e^{-\lambda t}$ and integrate from t to ω. One obtains

$$\begin{aligned} e^{-\lambda\omega}[(\lambda(v_1(\omega) - w_1(\omega)) &+ (v_1'(\omega) - w_1'(\omega)) \\ &+ (f(\omega, v_1(\omega)) - f(\omega, w_1(\omega)))] \\ &\leq e^{-\lambda t}[(\lambda(v_1(t) - w_1(t)) + (v_1'(t) - w_1'(t)) \\ &\qquad + (f(t, v_1(t)) - f(t, w_1(t)))]. \end{aligned}$$

Thus, the first inequality follows from (3.5).

To obtain the second inequality, employ condition iii) and (3.6) to obtain

$$(3.9)\quad \begin{aligned}(v_1'' - w_1'')(t) + (d/dt)(f(t, v_1(t)) - f(t, w_1(t))\\ \le \lambda^2(v_1 - w_1)(t) - \lambda(f(t, v_1(t)) - f(t, w_1(t)).\end{aligned}$$

Multiply (3.9) by $e^{\lambda t}$, integrate from 0 to t, and employ (3.7). Thus, condition i) of Theorem 2.4 is satisfied.

Since f is ω-periodic in t, and $w_1(0) = w_1(\omega)$, $v_1(0) = v_1(\omega)$, the condition, $Tw_1 \le_3 0 \le_3 Tv_1$, reduces to $-(w_1'(0) - w_1'(\omega)) \le 0 \le -(v_1'(0) - v_1'(\omega))$. This is precisely the second requirement in condition ii) of Theorem 3.2.

Condition iii) in Theorem 2.4 reduces to

$$\begin{aligned}v_1''(t) + (d/dt)f(t, v_1(t)) + g(t, v_1(t))\\ \le 0 \le w_1''(t) + (d/dt)f(t, w_1(t))\\ + g(t, w_1(t)),\end{aligned}$$

$t \in [t_k, t_{k+1}]$, $k = 0, \dots, m$, which is precisely condition iii) of Theorem 3.2. To see this, set $(y_1, y_2)^T = (w_1(t), w_1'(t) + f(t, w_1(t)))^T$. Then

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} - \begin{pmatrix} y_2 - f(t, y_1) \\ -g(t, y_1) \end{pmatrix} = \begin{pmatrix} 0 \\ w_1''(t) + (d/dt)f(t, w_1(t)) + g(t, w_1(t)) \end{pmatrix}.$$

Finally, if $\Delta w_1(t_k) = 0 = \Delta v_1(t_k)$, $k = 1, \dots, m$, then condition iv) in Theorem 2.4 reduces to

$$\Delta w_1'(t_k) - r(t_k, w_1(t_k)) \ge 0 \ge \Delta v_1'(t_k) - r(t_k, v_1(t_k)),$$

which is precisely condition iv) in Theorem 3.2.

This completes the proof of Theorem 3.2. □

Remark 1. The impulse effects given by (3.3) appear to be restrictive in the sense that solutions of the impulsive BVP, (3.2), (3.3), (3.1), will in fact be continuous on I. However, in comparing conditions ii) and iv) in Theorem 3.2, noting that in condition ii) we require that the upper and lower solutions satisfy equality in the first component, and noting the similarity in the specific meanings of $\leq_2$ and $\leq_3$ in this problem, the hypothesis $\Delta y_1(t_k) = 0$, $k = 1, \dots, m$, is a natural assumption.

Remark 2. Since Theorem 2.4 applies, the iterative improvement developed there applies to the impulsive BVP, (3.2), (3.3), (3.1).

Example. Consider the second order scalar ordinary differential equation,

$$u'' + h(u)u' + g(u) = 0, \quad t \in I \backslash \{t_1, \dots, t_m\}, \tag{3.10}$$

$$\Delta u(t_k) = 0, \qquad \Delta u'(t_k) = r_k(t_k, u(t_k)), \quad k = 1, \dots, m, \tag{3.11}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega). \tag{3.12}$$

Set $y_1 = u$, $y_2 = u' + \int_0^u h(s)\,ds$. Then with $f(t,y) = \int_0^y h(s)\,ds$, the impulsive BVPs, (3.2), (3.3), (3.1) and (3.10), (3.11), (3.12) are equivalent.

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