ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 3, Summer 1997

COMPLEMENTED COPIES OF l_1 IN $L^\infty(\mu, X)$

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An active field of research in recent years has been the study of the inclusion, as a subspace or complemented subspace, of classical Banach sequence spaces such as c_0, l_1 or l_∞ in Banach spaces $L^p(\mu, X)$ of Bochner p-integrable (essentially bounded for $p = \infty$) functions over a finite measure space (Ω, Σ, μ) with values in a Banach space X. The following problem, originally posed by Labuda, is mentioned in [**4**, p. 389]: When does $L^\infty(\mu, X)$ contain a complemented copy of l_1? Natural conjectures such as "if (and only if) X has a (complemented) copy of l_1," were disproved by an example due to Montgomery-Smith [**4**, p. 389]: there is a Banach space X with separable dual such that $L^\infty(\mu, X)$ contains a complemented copy of l_1. The aim of this paper is to answer this question for the case when X is a Banach lattice.

Theorem. *Let X be a Banach lattice. The following are equivalent*:

(1) $L^\infty(\mu, X)$ *contains a complemented subspace isomorphic to* $L^1[0, 1]$.

(2) $L^\infty(\mu, X)$ *contains a complemented subspace isomorphic to* l_1.

(3) $l_\infty(X)$ *contains all* l_1^n *uniformly complemented.*

(4) X *contains all* l_1^n *uniformly complemented.*

Before proving this theorem, let us recall a few notions from the local theory of Banach spaces. A normed space X is said to be an $\mathcal{S}_p$-space, $1 \leq p \leq \infty$, if it contains all l_p^n uniformly complemented, i.e., if there is some $\lambda \geq 1$ such that, for every $n \in \mathbf{N}$ there are operators $J_n \in L(l_p^n, X)$ and $P_n \in L(X, l_p^n)$, satisfying

$$P_n J_n = \mathrm{id}_{l_p^n}; \qquad \|P_n\| \, \|J_n\| \leq \lambda.$$

We may assume throughout that $\|P_n\| \leq \lambda$ and $\|J_n\| \leq 1$, for all $n \in \mathbf{N}$.

The terminology an notations are standard except, perhaps, the following one: if (A_n) is a sequence of pairwise disjoint measurable

Received by the editors on August 31, 1994.

sets of finite nonzero measure, we write

$$[A_n] := \left\{ \sum_{n=1}^{\infty} \chi_{A_n}(\cdot)x_n, (x_n) \in l_\infty(X) \right\}.$$

Of course, $[A_n]$ is a complemented subspace of $L^\infty(\mu, X)$ isometrically isomorphic to $l_\infty(X)$.

Lemma. $(\oplus_{n=1}^{\infty} l_1^n)_\infty$ *contains a* 1*-complemented subspace isometrically isomorphic to* $L^1[0,1]$.

Proof. Since $C[0,1]$ is a separable $\mathcal{S}_\infty$-space, it follows from [**6**, Theorem II.5.11] that there are a positive number λ, an increasing sequence (X_n) of subspaces of $C[0,1]$ whose union is dense in $C[0,1]$, and linear isomorphisms $T_n : l_\infty^n \to X_n$ such that $\|T_n\| \leq 1$ and $\|T_n^{-1}\| \leq \lambda$ for every $n \in \mathbf{N}$. Now $C[0,1]^* = L^1(\nu)$ for some measure ν; hence, we may take $\lambda = 1$ [**6**, Theorem II.4.11]. Define the operator

$$T : x = (x_n) \in \left(\bigoplus_{n=1}^{\infty} l_\infty^n \right)_1 \longrightarrow T(x) = \sum_{n=1}^{\infty} T_n x_n \in C[0,1].$$

We see that T is well-defined and $\|T\| \leq 1$, because $\|T(x)\| \leq \sum_{n=1}^{\infty} \|T_n x_n\| \leq \|x\|$. On the other hand, for each $n \in \mathbf{N}$, we also consider the operators,

$$U_n : x \in X_n \to U_n(x) = (0, \dots, 0, T_n^{-1}(x), 0, \dots) \in \left(\bigoplus_{n=1}^{\infty} l_\infty^n \right),$$

where $T_n^{-1}(x)$ occupies the nth position. Again they are well defined and $\|U_n\| \leq 1$ because $\|U_n(x)\| \leq \|T_n^{-1}(x)\| \leq \|x\|$. Moreover, TU_n is the identity operator on X_n. Now we can apply [**5**, Proposition 1] to obtain that T^* is an isomorphism from $C[0,1]^*$ into $(\oplus_n l_\infty^n)_1^* = (\oplus_n l_1^n)_\infty$, its inverse S has norm $\|S\| \leq \lambda \leq 1$, and there exists a projection P from $(\oplus_n l_\infty^n)_1^*$ onto $T^*(C[0,1]^*)$ with $\|P\| \leq \lambda \|T\| \leq 1$.

On the other hand, Lebesgue decomposition theorem plus Radon-Nikodym theorem tell us that $L^1[0,1]$ is isometrically isomorphic to a 1-complemented subspace of $C[0,1]^*$ and, therefore, isometrically isomorphic to a 1-complemented subspace of $(\oplus_n l_1^n)_\infty$. □

Remark. This proof shows in general that if X is a separable $\mathcal{S}_p$-space, then X^* is isomorphic to a complemented subspace of the l_∞-sum of the sequence of finite dimensional spaces $(l^n_{p'})$, where $1/p' + 1/p = 1$.

Proof of Theorem. (1) $\Rightarrow$ (2). It is trivial.

(2) $\Rightarrow$ (3). By (2), we see that $L^\infty(\mu, X)$ is an $\mathcal{S}_1$-space. Denote by $S_{\aleph_0}(X)$ the subspace of $L^\infty(\mu, X)$ formed by all functions $\varphi : \Omega \to X$ that can be written as

$$\varphi(\cdot) = \sum_{m=1}^{\infty} \chi_{A_m}(\cdot) x_m,$$

where (x_n) is a bounded sequence from X and (A_n) is a sequence of nonempty and pairwise disjoint subsets of Σ with positive measure covering Ω. By the proof of Pettis measurability theorem, we know that $S_{\aleph_0}(X)$ is dense in $L^\infty(\mu, X)$. On the other hand, the property of being an $\mathcal{S}_1$-space is inherited by dense subspaces (just consider the proof of [**6**, Proposition I.1.7] taking into account the fact that the sums are finite). It follows that $S_{\aleph_0}(X)$ is an $\mathcal{S}_1$-space.

Now, suppose that X_n is a λ-complemented subspace of $S_{\aleph_0}(X)$ which is λ-isomorphic to l_1^n, with basis $f_i = \sum_{m=1}^{\infty} \chi_{A_m(i)}(\cdot) x_m(i)$, $i = 1, \dots, n$. Let us arrange the family of pairwise disjoint measurable subsets,

$$\{A_{m_1}(1) \cap A_{m_2}(2) \cap \cdots \cap A_{m_n}(n) : m_1, m_2, \dots, m_n \in \mathbf{N}\}$$

in a sequence $(B_m) \subset \Sigma$. Then X_n is included, and still λ-complemented, in $[B_m]$. Since $[B_m]$ is isometrically isomorphic to $l_\infty(X)$, we have that $l_\infty(X)$ is an $\mathcal{S}_1$-space.

(3) $\Rightarrow$ (4). Obviously, $l_\infty(X)$ is isometrically isomorphic to $l_\infty(l_\infty(X))$. Hence, using (3), we can find operators $J_n \in L(l_1^n, l_\infty(X))$ and $P_n \in L(l_\infty(X), l_1^n)$ and $\lambda \geq 1$ such that

$$P_n J_n = \mathrm{id}_{l_1^n}, \qquad \sup_n \|P_n\| \leq \lambda, \qquad \sup_n \|J_n\| \leq 1.$$

Define the following two operators

$$\begin{aligned} J : x = (x_n) \in \left(\bigoplus_{n=1}^{\infty} l_1^n \right)_\infty &\longrightarrow J(x) \\ &= (J_n(x_n)) \in l_\infty(l_\infty(X)). \\ P : x = (x_n) \in l_\infty(l_\infty(X)) &\longrightarrow P(x) \\ &= (P_n(x_n)) \in \left(\bigoplus_{n=1}^{\infty} l_1^n \right)_\infty . \end{aligned}$$

Then the composition PJ is the identity operator in $(\oplus_n l_1^n)_\infty$. In other words, $l_\infty(X)$ contains a complemented copy of $(\oplus_n l_1^n)_\infty$ and, by the lemma above, a complemented copy of l_1.

We note that $l_\infty(X)$ is a Banach lattice with the natural order inherited from X and it can be lattice-identified with a sublattice of $l_\infty(X^{**})$ [**8**, Proposition 1.4.5]. Thus, if $l_\infty(X)$ contains a complemented copy of l_1 by [**1**, Theorem 14.21], we have that l_1 is lattice-isomorphic to a sublattice Y of $l_\infty(X^{**})$ and, therefore, there is a positive projection in $l_\infty(X^{**})$ whose range is exactly Y [**8**, Proposition 2.3.11]. This means that $l_\infty(X^{**})$ contains a complemented copy of l_1 and, therefore, is an $\mathcal{S}_1$-space. By local reflexivity, we obtain that $l_1(X^*)$ is an $\mathcal{S}_\infty$-space.

At this point we recall two results due to Maurey and Pisier [**7**]. The first one is that a Banach space is an $\mathcal{S}_\infty$-space if and only if it has no finite cotype, and the second one is that $L^1(\mu, X)$ has cotype q if and only if X has cotype q.

Using these results, it follows that $l_1(X^*)$ is an $\mathcal{S}_\infty$-space if and only if X^* is an $\mathcal{S}_\infty$-space. Again, by local reflexivity, we finally have that X is an $\mathcal{S}_1$-space.

(4) $\Rightarrow$ (1). Suppose that X contains all l_1^n uniformly complemented. Of course, X is 1-complemented in $l_\infty(X)$, hence $l_\infty(X)$ also contains all l_1^n uniformly complemented. Using the same arguments as in the beginning of (3) $\Rightarrow$ (4), we obtain that $l_\infty(X)$ contains a complemented copy of $(\oplus_n l_1^n)_\infty$.

The result follows now from a chain of complemented inclusions. Namely, by the lemma, $L^1[0,1]$ is isomorphic to a complemented subspace of $(\oplus_n l_1^n)_\infty$; we have proved that $(\oplus_n l_1^n)_\infty$ is isomorphic to a complemented subspace of $l_\infty(X)$ and, as we noted, using any $[A_n]$, $l_\infty(X)$ is isomorphic to a complemented subspace of $L^\infty(\mu, X)$. $\square$

Remarks. (1) By local reflexivity, we have that $c_0(X)$ is an $\mathcal{S}_1$-space if and only if $l_1(X^*)$ is an $\mathcal{S}_\infty$-space. This leads us to think of a natural way of coping with (3) $\Rightarrow$ (4), but, as we show, it has some troubles. Suppose that X_n is a λ-complemented subspace of $l_\infty(X)$ that is λ-isomorphic to l_1^n, with projection P. Let S be an ε-net in the unit sphere of X_n. For each $s \in S$, there is a $k_s \in \mathbf{N}$ such that $|s(k_s)| > 1-\varepsilon$. Then we can find $m \in \mathbf{N}$ such that $R(x) = \chi_{[1,m]}x$, $x \in l_\infty(X)$, is nearly an isometry from X_n onto $R(X_n)$. If T is the natural embedding of $l_\infty^m(X)$ into $l_\infty(X)$, then RPT looks like a good projection. However, we note that P might vanish on $c_0(X)$.

(2) The space $X = (\oplus_n l_1^n)_2$ is a reflexive separable Banach space such that $L^\infty([0,1], X)$ contains a complemented copy of l_1. This gives an example slightly stronger than the one due to Montgomery-Smith, mentioned in the introduction.

(3) The hypothesis that X is a lattice is only used in (3) $\Rightarrow$ (4), in order to find ways to extend operators which have l_1 as range space, from $l_\infty(X)$ to $l_\infty(X^{**})$. Therefore, our theorem is also true for other classes of Banach spaces such as Banach spaces complemented in their biduals.

Acknowledgments. We would like to thank José Mendoza (Universidad Complutense de Madrid) for many valuable conversations on the topic of this paper and to La Consejería de Educación y Ciencia de la Junta de Andalucía for partially supporting this research.

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