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A HIERARCHY OF INTEGRAL OPERATORS

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1. Introduction. The complex form of the Gauss theorem leads to a representation formula for complex functions w in $W^{1,p}(\mathcal{D})$ for bounded domains $\mathcal{D}$ with smooth boundary. This formula generalizing the Cauchy formula for analytic functions was proved by Pompeiu [**12**] and is called the Cauchy-Pompeiu formula. The area integral appearing in this formula defines a weakly singular integral operator T which plays an important role in the theory of generalized analytic functions as well as in the study of Beltrami and generalized Beltrami equations. Its properties were extensively studied by I.N. Vekua [**15**]. If the density ρ of this integral belongs to $L_p(\mathcal{D})$ with $2 < p$, then the integral $T\rho$ has first order weak derivatives $\partial(T\rho)/\partial\bar{z} = \rho$ and $\partial(T\rho)/\partial z =: \Pi\rho$, where $\Pi\rho$ is a singular integral understood as a Cauchy principal value. Integrals of this type even in higher dimensions were investigated by Calderon and Zygmund [**6, 7**]. Because the Π operator for the whole complex plane $\mathbf{C}$ turns out to be unitary in $L_2(\mathbf{C})$, the Riesz theorem [**13**] describes some important properties of this operator.

Many papers dealing with complex first order partial differential equations are based on properties of the T and Π operators; see, for example, [**3, 4, 5, 10, 16, 17**]. Recently second order complex equations have been investigated by means of integral operators which originate from the T operator by integration; see [**2, 8, 9, 10**]. In the paper [**18**] a complex fourth order equation is handled with an integral operator which can be connected with the T operator by repeated integrations of the latter.

In this paper these ideas are carried further to produce a hierarchy of integral operators $T_{m,n}$, defined for pairs of integers (m, n) with $0 \leq m + n$, acting on certain $L_p(\mathcal{D})$ function spaces; the operator $T_{0,1}$ is the mentioned T operator while $T_{-1,1}$ is the Π operator and $T_{0,0}$ the identity operator. Whenever $0 < m + n$ the operators $T_{m,n}$ are regular or weakly singular, but for $m + n = 0$ they are singular operators with properties analogous to those of the Π operator. Dzhuraev [**8**]

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has studied certain special cases of these $T_{m,n}$ operators for bounded domains. (See in particular Chapters 2 and 3 of [**8**].) For example, the operator $S_{\mathcal{D},2n}$ of Dzhuraev is a constant multiple of our $T_{n,-n,\mathcal{D}}$ operator, while his $T_{\mathcal{D},n}$ and $\overline{T}_{\mathcal{D},n}$ operators are multiples of our $T_{0,n,\mathcal{D}}$ and $T_{n,0,\mathcal{D}}$ operators, respectively.

By an induction argument beginning with the classical Pompeiu formula, a higher order counterpart to the Pompeiu formula is developed. This higher order formula provides a representation of functions w in $C^{m+n}(\overline{\mathcal{D}})$ in terms of an area integral $T_{m,n}(\partial^{m+n}w/\partial z^m\partial\bar{z}^n)$, and some boundary integrals involving lower order derivatives.

Properties of the integrals $T_{m,n}w$, such as integrability, Hölder continuity, and differentiability, are investigated. For example, under certain conditions on w one has the useful formulas $\partial_z(T_{m,n}w) = T_{m-1,n}w$ and $\partial_{\bar{z}}(T_{m,n}w) = T_{m,n-1}w$. The singular operators $T_{-n,n}$ and $T_{n,-n}$ for $0 < n$ are shown to be unitary operators in $L_2(\mathbf{C})$.

The classical Riemann problem asks for an analytic function satisfying a "jump condition," $w^+ - w^- = f$, across the boundary of a domain in the plane; as is well known, the Cauchy integral of f supplies a solution w whenever f is Hölder continuous and the domain is sufficiently regular. An analogous problem, involving jumps also of higher order derivatives, is solved for so-called *polyanalytic functions*–solutions of $\partial^m w/\partial\bar{z}^m = 0$–by means of boundary integrals appearing in the higher order Pompeiu formula.

For the unit disk the Cauchy-Schwarz-Poisson-Pompeiu formula furnishes a solution to the Dirichlet problem for the inhomogeneous Cauchy-Riemann equation $\partial w/\partial\bar{z} = v$. A generalization of this formula is given which directly provides a solution to the higher order Dirichlet problem for the inhomogeneous equation $\partial^m w/\partial\bar{z}^m = v$.

As the operators T and Π have been widely used to study various boundary value problems, both linear and nonlinear, for first and second order complex partial differential equations, the operators $T_{m,n}$ should prove useful in the study of similar problems for higher order equations.

2. Definitions of kernels and operators. Let m and n be integers, with $m + n \geq 0$ but $(m,n) \neq (0,0)$; we introduce kernels $K_{m,n}$ as specified in three mutually exclusive cases:

$m \leq 0$:

$$K_{m,n}(z) := \frac{(-m)!(-1)^m}{(n-1)!\pi} z^{m-1} \bar{z}^{n-1};$$

$n \leq 0$:

$$K_{m,n}(z) := \frac{(-n)!(-1)^n}{(m-1)!\pi} z^{m-1} \bar{z}^{n-1};$$

$m, n \geq 1$:

$$K_{m,n}(z) := \frac{1}{(m-1)!(n-1)!\pi} z^{m-1} \bar{z}^{n-1} \cdot \left[\log |z|^2 - \sum_{k=1}^{m-1} \frac{1}{k} - \sum_{l=1}^{n-1} \frac{1}{l} \right].$$

(When $m = 1$ or $n = 1$, the summations in the last formula are taken as 0.)

Obviously each kernel $K_{m,n}$ is of class C^∞ in all the complex plane $\mathbf{C}$, except at the origin where in some cases there is a singularity. It is easy to verify the identities

$$K_{m,n}(z) = \overline{K_{n,m}(z)}, \tag{2.1}$$

$$K_{m,n} = \partial_z K_{m+1,n} = \partial_{\bar{z}} K_{m,n+1}; \tag{2.2}$$

moreover, simple calculations confirm that, for positive radii R,

$$\int_{|z|=R} K_{m,n}(z)\, dz = 0, \quad \text{if } m - n \neq -1, \tag{2.3}$$

$$\int_{|z|=R} K_{m,n}(z)\, d\bar{z} = 0, \quad \text{if } m - n \neq 1, \tag{2.4}$$

$$\iint_{|z| \leq R} |K_{m,n}(z)|\, dx\, dy < \infty, \quad \text{if } m + n > 0, \tag{2.5}$$

$$\lim_{R \to 0} \int_{|z|=R} |K_{m,n}(z)|\, |dz| = 0, \quad \text{if } m + n > 1. \tag{2.6}$$

For $\mathcal{D}$ a domain in the plane we formally define operators $T_{m,n,\mathcal{D}}$, acting on suitable complex valued functions w defined in $\mathcal{D}$, according to

$$T_{m,n,\mathcal{D}}w(z) := \iint_{\mathcal{D}} K_{m,n}(z-\zeta)w(\zeta)\,d\xi\,d\eta. \tag{2.7}$$

Observe that $T_{0,1,\mathcal{D}}$ and $T_{1,0,\mathcal{D}}$ are the familiar "T and $\overline{T}$ operators" analyzed in Vekua's well-known book [**15**],

$$\begin{aligned} T_{0,1,\mathcal{D}}w(z) &= T_{\mathcal{D}}w(z) \\ &= -\frac{1}{\pi}\iint_{\mathcal{D}} \frac{1}{\zeta - z} w(\zeta)\,d\xi\,d\eta, \\ T_{1,0,\mathcal{D}}w(z) &= \overline{T}_{\mathcal{D}}w(z) \\ &= -\frac{1}{\pi}\iint_{\mathcal{D}} \frac{1}{\bar\zeta - \bar z} w(\zeta)\,d\xi\,d\eta, \end{aligned}$$

while $T_{-1,1,\mathcal{D}}$ and $T_{1,-1,\mathcal{D}}$ are the so-called "Π and $\overline{\Pi}$ operators," defined as Cauchy principal value integrals,

$$\begin{aligned} T_{-1,1,\mathcal{D}}w(z) &= \Pi_{\mathcal{D}}w(z) \\ &= -\frac{1}{\pi}\iint_{\mathcal{D}} \frac{1}{(\zeta - z)^2} w(\zeta)\,d\xi\,d\eta, \\ T_{1,-1,\mathcal{D}}w(z) &= \overline{\Pi}_{\mathcal{D}}w(z) \\ &= -\frac{1}{\pi}\iint_{\mathcal{D}} \frac{1}{(\bar\zeta - \bar z)^2} w(\zeta)\,d\xi\,d\eta. \end{aligned}$$

We find it useful also to denote the identity operator as $T_{0,0,\mathcal{D}}$; thus

$$T_{0,0,\mathcal{D}}w(z) := w(z).$$

Then the operators $T_{m,n,\mathcal{D}}$ are defined for all integers m and n with $m + n \geq 0$.

In order to unify our formulas we find it convenient to introduce some novel notation. For integers m and n and complex numbers z we define

$$z^{(m,n)} := z^m \bar z^n.$$

For f an analytic function in an open set, with derivative f', the formulas

$$\partial_z f^{(m,n)} = m f^{(m-1,n)} f',$$
$$\partial_{\bar z} f^{(m,n)} = n f^{(m,n-1)} \overline{f'}$$

are easily verified.

For nonnegative integers m and n we may form the *bi-index* $\alpha := (m,n)$; then we define the differential operator $\partial_\alpha = \partial_{(m,n)}$, acting on complex valued functions w according to

$$\partial_\alpha w = \partial_{(m,n)} w := \frac{\partial^{m+n}}{\partial z^m \partial \bar z^n} w.$$

(This derivative might be taken in the classical or the Sobolev sense, depending on the situation.) For $\alpha = (m,n)$ we define also

$$T_\alpha := T_{m,n}, \qquad K_\alpha := K_{m,n}, \qquad \tilde\alpha := (n,m).$$

As is customary, we define the *magnitude* of the bi-index $\alpha = (m,n)$ as $|\alpha| := m+n$, and if $\beta = (s,t)$ is another bi-index, we say $\alpha \le \beta$ if and only if $m \le s$ and $n \le t$, and that $\alpha < \beta$ if and only if $\alpha \le \beta$ and $|\alpha| < |\beta|$.

3. Higher order Pompeiu formulas. For the purposes of this paper, we say that a domain $\mathcal{D}$ in the plane is *regular* if and only if it is bounded, with boundary Γ consisting of a finite disjoint collection of simple and piecewise smooth Jordan curves

$$\{\Gamma_j : 0 \le j \le J\},$$

where Γ_0 is assumed to border the unbounded component of the complement of $\mathcal{D}$, with the remaining boundary curves, $\Gamma_1, \dots, \Gamma_J$, lying in the interior of Γ_0.

For $\mathcal{D}$ a regular domain and w a complex valued function in $C^1(\overline{\mathcal{D}})$, we recall (see [**15**, Chapter I]) the well-known Gauss formulas

$$\begin{aligned} \iint_{\mathcal{D}} w_{\bar z}\, dx\, dy &= -\frac{i}{2}\int_\Gamma w\, dz, \\ \iint_{\mathcal{D}} w_z\, dx\, dy &= \frac{i}{2}\int_\Gamma w\, d\bar z, \end{aligned} \tag{3.1}$$

as well as the *Pompeiu formulas*, valid for $z \in \mathcal{D}$,

$$w(z) = -\frac{1}{\pi}\iint_{\mathcal{D}} \frac{1}{\zeta - z} w_{\bar\zeta}(\zeta)\, d\xi\, d\eta + \frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta - z} w(\zeta)\, d\zeta, \tag{3.2}$$

$$w(z) = -\frac{1}{\pi}\iint_{\mathcal{D}} \frac{1}{\bar\zeta - \bar z} w_{\zeta}(\zeta)\, d\xi\, d\eta - \frac{1}{2\pi i}\int_\Gamma \frac{1}{\bar\zeta - \bar z} w(\zeta)\, d\bar\zeta. \tag{3.3}$$

Lemma 3.1. *Let $\mathcal{D}$ be a regular domain in the plane, with boundary Γ, and let $w \in C^1(\overline{\mathcal{D}})$. Then, for $m + n > 0$ and $z \in \mathbf{C} - \Gamma$,*

$$T_{m,n,\mathcal{D}} w(z) = T_{m,n+1,\mathcal{D}} w_{\bar z}(z) + \frac{i}{2}\int_\Gamma K_{m,n+1}(z - \zeta) w(\zeta)\, d\zeta, \tag{3.4}$$

$$T_{m,n,\mathcal{D}} w(z) = T_{m+1,n,\mathcal{D}} w_{z}(z) - \frac{i}{2}\int_\Gamma K_{m+1,n}(z - \zeta) w(\zeta)\, d\bar\zeta. \tag{3.5}$$

Proof. We prove only (3.4), the proof of (3.5) being similar. By virtue of (2.5), and the boundedness of $\mathcal{D}$ and regularity of w, the integrals $T_{m,n,\mathcal{D}} w(z)$ and $T_{m,n+1,\mathcal{D}} w(z)$ are absolutely convergent for all $z \in \mathbf{C}$. Fix $z \in \mathbf{C} - \Gamma$ and for $\varepsilon > 0$ let $\mathcal{D}_\varepsilon$ be the domain $\mathcal{D} - \{\zeta : |\zeta - z| \le \varepsilon\}$. For ε sufficiently small the domain $\mathcal{D}_\varepsilon$ is regular, with corresponding boundary Γ_ε, and we may apply the first formula of (3.1) to the product

$K_{m,n+1}(z-\cdot)w(\cdot)$ in the domain $\mathcal{D}_\varepsilon$ to obtain

$$\begin{aligned}\iint_{\mathcal{D}_\varepsilon} K_{m,n}(z-\zeta)w(\zeta)\,d\xi\,d\eta \\ &= -\iint_{\mathcal{D}_\varepsilon} \partial_{\bar\zeta}[K_{m,n+1}(z-\zeta)w(\zeta)]\,d\xi\,d\eta \\ &\quad + \iint_{\mathcal{D}_\varepsilon} K_{m,n+1}(z-\zeta)\partial_{\bar\zeta}w(\zeta)\,d\xi\,d\eta \\ &= \frac{i}{2}\int_{\Gamma_\varepsilon} K_{m,n+1}(z-\zeta)w(\zeta)\,d\zeta \\ &\quad + \iint_{\mathcal{D}_\varepsilon} K_{m,n+1}(z-\zeta)\partial_{\bar\zeta}w(\zeta)\,d\xi\,d\eta.\end{aligned}$$

We let $\varepsilon \to 0$, again noting (2.5), and obtain (3.4). □

Theorem 3.2 (Higher order Pompeiu formulas). *Let $\mathcal{D}$ be a regular domain in the plane, and let $w \in C^m(\overline{\mathcal{D}})$, $m \geq 1$. Consider a chain of bi-indices*

$$(0,0) = \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_m,$$

with $|\alpha_n| = n$, $0 \leq n \leq m$; thus,

$$\alpha_{n+1} = \alpha_n + \delta_n, \quad 0 \leq n \leq m-1,$$

where, for each n, $\delta_n = (1,0)$ or $\delta_n = (0,1)$. Then, for $z \in \mathcal{D}$,

$$\begin{aligned}w(z) &= T_{\alpha_m,\mathcal{D}}\partial_{\alpha_m}w(z) \\ &\quad + \sum_{n=0}^{m-1}\frac{1}{2}\int_\Gamma K_{\alpha_{n+1}}(z-\zeta)\partial_{\alpha_n}w(\zeta)\,d[(i\zeta)^{\tilde\delta_n}].\end{aligned} \tag{3.6}$$

Proof. We proceed by induction on m. First observe that the Pompeiu formulas (3.2) and (3.3) may be written respectively as

$$w(z) = T_{0,1,\mathcal{D}}w_{\bar z}(z) + \frac{1}{2}\int_\Gamma K_{0,1}(z-\zeta)w(\zeta)\,d(i\zeta),$$
$$w(z) = T_{1,0,\mathcal{D}}w_z(z) + \frac{1}{2}\int_\Gamma K_{1,0}(z-\zeta)w(\zeta)\,d(\overline{i\zeta}).$$

The first formula is (3.6) with $m = 1$ and $\alpha_1 = (0,1)$, $\delta_0 = (0,1)$, $\tilde{\delta}_0 = (1,0)$, while the second is (3.6) with $m = 1$ and $\alpha_1 = \delta_0 = (1,0)$, $\tilde{\delta}_0 = (0,1)$. Now assume (3.6) is valid for m and that $w \in C^{m+1}(\overline{\mathcal{D}})$. Using (3.4) of Lemma 3.1, applied to $\partial_{\alpha_m} w$, in (3.6) we write

$$\begin{aligned} T_{\alpha_m,\mathcal{D}}\partial_{\alpha_m} w(z) &= T_{\alpha_m+(0,1),\mathcal{D}}\partial_{\alpha_m+(0,1)} w(z) \\ &\quad + \frac{i}{2}\int_\Gamma K_{\alpha_m+(0,1)}(z-\zeta)\partial_{\alpha_m} w(\zeta)\,d\zeta, \end{aligned}$$

which gives (3.6) for $m+1$ whenever $\delta_{m+1} = (0,1)$. In an analogous way, we use (3.5) of Lemma 3.1 to derive (3.6) for $m+1$ whenever $\delta_{m+1} = (1,0)$. □

We examine in more detail formula (3.6) in the special case $\alpha_n = (0,n)$, $0 \le n \le m$. Then $\delta_n = (0,1)$, $\tilde{\delta}_n = (1,0)$, and for $w \in C^m(\overline{\mathcal{D}})$ we obtain

$$\begin{aligned} w(z) &= T_{0,m,\mathcal{D}}\frac{\partial^m w(z)}{\partial \bar{z}^m} + \sum_{n=0}^{m-1}\frac{i}{2}\int_\Gamma K_{0,n+1}(z-\zeta)\frac{\partial^n w(\zeta)}{\partial\bar{\zeta}^n}\,d\zeta \\ &= \frac{1}{(m-1)!\pi}\iint_{\mathcal{D}}\frac{(\bar{z}-\bar{\zeta})^{m-1}}{z-\zeta}\frac{\partial^m w(\zeta)}{\partial\bar{\zeta}^m}\,d\xi\,d\eta \\ &\quad - \sum_{n=0}^{m-1}\frac{1}{n!}\frac{1}{2\pi i}\int_\Gamma\frac{(\bar{z}-\bar{\zeta})^n}{z-\zeta}\frac{\partial^n w(\zeta)}{\partial\bar{\zeta}^n}\,d\zeta. \end{aligned} \tag{3.7}$$

Expanding the powers $(\bar{z}-\bar{\zeta})^n$, we find that this representation takes the form

$$w(z) = T_{0,m,\mathcal{D}}\frac{\partial^m w(z)}{\partial\bar{z}^m} + \sum_{n=0}^{m-1}\phi_n(z)\bar{z}^n, \tag{3.8}$$

where the functions $\phi_0, \phi_1, \dots, \phi_{m-1}$ are analytic in $\mathcal{D}$.

The formula (3.7) was derived also by Dzhuraev [**8**, Chapter 3] in the process of generalizing the Bergman kernel function to higher dimensions.

4. Existence and continuity of integrals. We investigate existence and continuity of the integral $T_{m,n,\mathcal{D}}w$, in domains $\mathcal{D}$ in the complex plane $\mathbf{C}$. The case $m+n=0$, when the integral must be viewed as a Cauchy principal integral, is delayed until the end of the section. When $m+n \geq 1$, we discuss first the case when $\mathcal{D}$ is a bounded domain in $\mathbf{C}$, and then we consider $\mathcal{D}=\mathbf{C}$. By the notation $M(\dots)$, we mean a nonnegative constant depending on the entities listed inside the parentheses.

Lemma 4.1. *Let $\mathcal{D}$ be a bounded domain, and suppose $m+n \geq 1$, with*

(a) $1 \leq p < 2$, *when* $m+n=1$,

(b) $1 \leq p \leq \infty$, *when* $m+n=2$, $mn \leq 0$,

(c) $1 \leq p < \infty$, *when* $m+n=2$, $mn > 0$,

(d) $1 \leq p \leq \infty$, *when* $m+n \geq 3$.

Then $K_{m,n} \in L^p(\mathcal{D})$, *and*

$$\|K_{m,n}\|_{p,\mathcal{D}} \leq M(m,n,p,\mathcal{D}). \tag{4.1}$$

Proof. Viewing the formulas for $K_{m,n}$ in Section 2, we observe that

$$|K_{m,n}(z)| \leq M(m,n)|z|^{m+n-2}(1+|\log|z||), \quad \text{if } mn>0, \tag{4.2}$$

$$|K_{m,n}(z)| \leq M(m,n)|z|^{m+n-2}, \quad \text{if } mn \leq 0. \tag{4.3}$$

We deduce (4.1) from these inequalities. □

Theorem 4.2. *Let $\mathcal{D}$ be a bounded domain, suppose $m+n \geq 1$, and let w be a complex valued function in $L^1(\mathcal{D})$. Then the integral $T_{m,n,\mathcal{D}}w(z)$ converges absolutely for almost all z in $\mathbf{C}$. Moreover, if*

(a) $1 \leq p < 2$, *when* $m+n=1$,

(b) $1 \leq p \leq \infty$, *when* $m+n=2$, $mn \leq 0$,

(c) $1 \leq p < \infty$, *when* $m+n=2$, $mn > 0$,

(d) $1 \leq p \leq \infty$, *when* $m+n \geq 3$,

then for any bounded domain Ω, $T_{m,n,\mathcal{D}}w \in L^p(\Omega)$ *with*

$$\|T_{m,n,\mathcal{D}}w\|_{p,\Omega} \leq M(m,n,p,\mathcal{D},\Omega)\|w\|_{1,\mathcal{D}}. \tag{4.4}$$

Proof. Define F on $\mathbf{C}$ according to

$$F(z) = \iint_{\mathcal{D}} |K_{m,n}(z-\zeta)|\,|w(\zeta)|\,d\xi\,d\eta,$$

and let g be an arbitrary function in $L^q(\Omega)$, where $1/p+1/q=1$; then

$$\begin{aligned}(4.5)\qquad \iint_{\Omega} F(z)|g(z)|\,dx\,dy \\ = \iint_{\mathcal{D}} |w(\zeta)| \iint_{\Omega} |K_{m,n}(z-\zeta)|\,|g(z)|\,dx\,dy\,d\xi\,d\eta.\end{aligned}$$

In the cases listed under conditions (a)–(d) where $p=\infty$, we have $q=1$; we may apply Lemma 4.1 to a bounded domain large enough to contain the set $\{z-\zeta : z\in\Omega, \zeta\in\mathcal{D}\}$, and deduce that

$$\begin{aligned}\iint_{\Omega} |K_{m,n}(z-\zeta)|\,|g(z)|\,dx\,dy &\le \sup_{z\in\Omega,\zeta\in\mathcal{D}} |K_{m,n}(z-\zeta)|\cdot\|g\|_1 \\ &= M(m,n,\mathcal{D},\Omega)\|g\|_{q,\Omega}.\end{aligned}$$

In the cases where $1\le p<\infty$ we have $1<q\le\infty$, and Lemma 4.1 gives

$$\begin{aligned}&\iint_{\Omega} |K_{m,n}(z-\zeta)|\,|g(z)|\,dx\,dy \\ &\qquad \le \left(\iint_{\Omega} |K_{m,n}(z-\zeta)|^p\,dx\,dy\right)^{1/p} \|g\|_{q,\Omega} \\ &\qquad \le M(m,n,p,\mathcal{D},\Omega)\|g\|_{q,\Omega}.\end{aligned}$$

Thus, in all cases, we obtain from (4.5) that

$$\iint_{\Omega} F(z)|g(z)|\,dx\,dy \le \|w\|_{1,\mathcal{D}} M(m,n,p,\mathcal{D},\Omega)\|g\|_{q,\Omega}.$$

Therefore, $F\in L^p(\Omega)$ with $F(z)$ defined almost everywhere in Ω, and

$$\|F\|_{p,\Omega} \le M(m,n,p,\mathcal{D},\Omega)\|w\|_{1,\mathcal{D}}.$$

Since F dominates $T_{m,n,\mathcal{D}}w$, we have $T_{m,n,\mathcal{D}}w \in L^p(\Omega)$, with (4.4) holding. □

Theorem 4.3. *Let $\mathcal{D}$ be a bounded domain, suppose $m+n \geq 1$, and assume $w \in L^p(\mathcal{D})$ where*

(a) $2 < p \leq \infty$, *when* $m+n=1$,

(b) $1 \leq p \leq \infty$, *when* $m+n=2$, $mn \leq 0$,

(c) $1 < p \leq \infty$, *when* $m+n=2$, $mn > 0$,

(d) $1 \leq p \leq \infty$, *when* $m+n \geq 3$.

Then $T_{m,n,\mathcal{D}}w(z)$ exists as a Lebesgue integral for all z in $\mathbf{C}$, $T_{m,n,\mathcal{D}}$ is continuous in $\mathbf{C}$, and for $|z| \leq R$ with $R > 0$,

$$|T_{m,n,\mathcal{D}}w(z)| \leq M\|w\|_{p,\mathcal{D}}, \tag{4.6}$$

where $M = M(m,n,p,\mathcal{D})$ in cases (a) *and* (b), *$M = M(m,n,p,\mathcal{D},R)$ in* (c) *and* (d).

Proof. From the formula

$$T_{m,n,\mathcal{D}}w(z) = \iint_{\mathcal{D}} K_{m,n}(z-\zeta)w(\zeta)\,d\xi\,d\eta,$$

it follows that

$$|T_{m,n,\mathcal{D}}w(z)| \leq \|w\|_{p,\mathcal{D}}\|K_{m,n}(z-\cdot)\|_{q,\mathcal{D}}.$$

Observing inequalities (4.2) and (4.3), we see that conditions (a)–(d) imply

$$\|K_{m,n}(z-\cdot)\|_{q,\mathcal{D}} \leq \begin{cases} M(m,n,p,\mathcal{D}) & \text{in cases (a) and (b),} \\ M(m,n,p,\mathcal{D},R) & \text{in cases (c) and (d);} \end{cases}$$

thus we obtain (4.6).

In proving continuity of $T_{m,n,\mathcal{D}}w$, we may assume $p < \infty$ in all cases, as otherwise we may take p smaller. Setting $w \equiv 0$ outside $\mathcal{D}$, we have for z in $\mathbf{C}$,

$$T_{m,n,\mathcal{D}}w(z) = \iint_{\mathbf{C}} K_{m,n}(\zeta)w(z-\zeta)\,d\xi\,d\eta.$$

For z_1 and z_2 inside some disk $\mathbf{B}$, there is some larger disk $\mathbf{B}_0$ such that

$$\begin{aligned}|T_{m,n,\mathcal{D}}w(z_1) - T_{m,n,\mathcal{D}}w(z_2)| \\ \leq \iint_{\mathbf{B}_0} |K_{m,n}(\zeta)|\, |w(z_1-\zeta) - w(z_2-\zeta)|\, d\xi\, d\eta.\end{aligned}$$

Under conditions (a)–(d), Lemma 4.1 asserts that $\|K_{m,n}\|_{q,\mathbf{B}_0}$ is finite; then we obtain

$$\begin{aligned}|T_{m,n,\mathcal{D}}w(z_1) - T_{m,n,\mathcal{D}}w(z_2)| \\ \leq \|K_{m,n}\|_{q,\mathbf{B}_0} \left\{ \iint_{\mathbf{B}_0} |w(z_1-\zeta) - w(z_2-\zeta)|^p\, d\xi\, d\eta \right\}^{1/p},\end{aligned}$$

which tends to zero as $z_1 \to z_2$ in $\mathbf{B}$. (We require $p < \infty$ here.) □

Lemma 4.4. *For k and l integers, $k+l \geq -1$, and for nonzero complex numbers a and b, we have the following inequalities*:

$$|a^k \bar{a}^l - b^k \bar{b}^l| \leq M(k,l)|a-b| \cdot \begin{cases} 1/(|a||b|) & \text{if } k+l=-1, \\ 1/|a| & \text{if } k+l=0, \\ \sum_{j=0}^{k+l-1} |a|^j |b|^{k+l-1-j} & \text{if } k+l \geq 1. \end{cases} \tag{4.7}$$

Proof. We may assume $l \geq 0$, as the case $k \geq 0$ then follows by conjugation.

First suppose $k+l=-1$; then

$$a^k\bar{a}^l - b^k\bar{b}^l = a^{-l-1}b^{-l-1}[b^{l+1}\bar{a}^l - a^{l+1}\bar{b}^l]. \tag{4.8}$$

Using the identity

$$\begin{aligned}b^{l+1}c^l - a^{l+1}d^l = (b-a)\sum_{j=0}^{l} b^{l-j}a^j c^{l-j} d^j \\ + (c-d)\sum_{j=0}^{l-1} b^{l-j}a^{j+1}c^{l-1-j}d^j,\end{aligned}$$

and setting $c = \bar{a}$, $d = \bar{b}$, we obtain from (4.8) the estimate (4.7) for the case $k + l = -1$.

Now assume $k + l = 0$. For general k and l we may write

$$a^k \bar{a}^l - b^k \bar{b}^l = \frac{1}{a}[a^k \bar{a}^l](a - b) + b[a^{k-1}\bar{a}^l - b^{k-1}\bar{b}^l]$$

obtaining the estimate

$$|a^k \bar{a}^l - b^k \bar{b}^l| \leq |a|^{k+l-1}|a - b| + |b||a^{k-1}\bar{a}^l - b^{k-1}\bar{b}^l|. \tag{4.9}$$

Taking $k + l = 0$ and applying (4.7) to the pair $(k - 1, l)$, we then have

$$|a^k \bar{a}^l - b^k \bar{b}^l| \leq |a|^{-1}|a - b| + |b| M(k, l) \frac{|a - b|}{|a||b|},$$

and hence (4.7) for $k + l = 0$.

Next consider the case $k + l = 1$. First we note that switching a and b in (4.7) yields, for $k + l = 0$,

$$|a^k \bar{a}^l - b^k \bar{b}^l| \leq M(k, l)|a - b| \frac{1}{|b|}.$$

Applying (4.9) then, we obtain for $k + l = 1$,

$$|a^k \bar{a}^l - b^k \bar{b}^l| \leq |a - b| + |b| M(k, l)|a - b| \frac{1}{|b|},$$

which gives (4.7) when $k + l = 1$.

Finally, we suppose (4.7) holds for $k + l = p$ where $p \geq 1$, and we consider the case $k + l = p + 1$. From (4.9) and (4.7) for $(k - 1, l)$, we obtain

$$\begin{aligned} |a^k \bar{a}^l - b^k \bar{b}^l| \leq{}& |a|^{k+l-1}|a - b| \\ &+ |b| M(k, l)|a - b| \sum_{j=0}^{k+l-2} |a|^j |b|^{k+l-2-j}, \end{aligned}$$

and hence (4.7) for (k, l). $\square$

We next discuss Hölder continuity of the integral $T_{m,n,\mathcal{D}}w$ whenever $m+n \geq 1$ and $mn \leq 0$. The cases $mn > 0$ are more easily handled after our later discussion of differentiability of these integrals.

Theorem 4.5. *Suppose $m+n \geq 1$ and $mn \leq 0$, let $\mathcal{D}$ be a bounded domain in $\mathbf{C}$, and assume w is a complex valued function in $L^p(\mathcal{D})$; suppose also that*

(a) $2 < p < \infty$, *if* $m+n = 1$,

(b) $2 < p \leq \infty$, *if* $m+n = 2$,

(c) $1 \leq p \leq \infty$, *if* $m+n = 3$,

(d) $1 \leq p \leq \infty$, *if* $m+n \geq 4$.

For $z \in \mathbf{C}$, set

$$v(z) := T_{m,n,\mathcal{D}}w(z) = \iint_{\mathcal{D}} K_{m,n}(z-\zeta)w(\zeta)\,d\xi\,d\eta.$$

Then for $z_1, z_2 \in \mathbf{C}$, say with $|z_1|, |z_2| \leq R$,

$$(4.10)\quad |v(z_1)-v(z_2)| \leq M\|w\|_{p,\mathcal{D}} \cdot \begin{cases} |z_1-z_2| & \text{if } m+n \geq 2, \\ |z_1-z_2|^{(p-2)/p} & \text{if } m+n = 1, \end{cases}$$

where $M = M(m,n,p)$ in case (a), *$M = M(m,n,p,\mathcal{D})$ in cases* (b) *and* (c), *and $M = M(m,n,p,\mathcal{D},R)$ in case* (d).

Proof. In all cases, Hölder's inequality gives

$$(4.11)\quad |v(z_1)-v(z_2)| \leq \|w\|_{p,\mathcal{D}}\|K_{m,n}(z_1-\cdot)-K_{m,n}(z_2-\cdot)\|_{q,\mathcal{D}}.$$

When $p=1$, then $q=\infty$ with

$$(4.12)\quad \begin{aligned} &\|K_{m,n}(z_1-\cdot)-K_{m,n}(z_2-\cdot)\|_{\infty,\mathcal{D}} \\ &\qquad = \sup_{\zeta\in\mathcal{D}} |K_{m,n}(z_1-\zeta)-K_{m,n}(z_2-\zeta)|, \end{aligned}$$

while $1 < p \leq \infty$ implies $1 \leq q < \infty$, with

$$(4.13)\quad \begin{aligned} &(\|K_{m,n}(z_1-\cdot)-K_{m,n}(z_2-\cdot)\|_{q,\mathcal{D}})^q \\ &\qquad = \iint_{\mathcal{D}} |K_{m,n}(z_1-\zeta)-K_{m,n}(z_2-\zeta)|^q\,d\xi\,d\eta; \end{aligned}$$

in both cases, the formulas of Section 2 lead to the estimate

$$\begin{aligned}|K_{m,n}(z_1-\zeta)-K_{m,n}(z_2-\zeta)|\\ &\le M(m,n)|(z_1-\zeta)^{m-1}\overline{(z_1-\zeta)}^{n-1}\\ &-(z_2-\zeta)^{m-1}\overline{(z_2-\zeta)}^{n-1}|,\end{aligned}$$

which with use of Lemma 4.4 yields

$$|K_{m,n}(z_1-\zeta)-K_{m,n}(z_2-\zeta)|\le M(m,n)|z_1-z_2| \cdot \begin{cases} |z_1-\zeta|^{-1}|z_2-\zeta|^{-1} & \text{if } m+n=1,\\ |z_1-\zeta|^{-1} & \text{if } m+n=2,\\ 1 & \text{if } m+n=3,\\ \sum_{j=0}^{m+n-3}|z_1-\zeta|^{j}|z_2-\zeta|^{m+n-3-j} & \text{if } m+n\ge 4.\end{cases} \tag{4.14}$$

When $m+n=1$, by a formula in [**15**, p. 39], we find that

$$\iint_{\mathcal{D}}|z_1-\zeta|^{-q}|z_2-\zeta|^{-q}\,d\xi\,d\eta\le M(p)|z_1-z_2|^{2-2q},\quad 1<q<2,$$

which we use with (4.11)–(4.14) to obtain (4.10) when $2<p<\infty$.

When $m+n=2$, we use the estimate

$$\iint_{\mathcal{D}}|z_1-\zeta|^{-q}\,d\xi\,d\eta\le M(p,\mathcal{D}),\quad 1\le q<2,$$

which with (4.11)–(4.14) yields (4.10) when $2<p\le\infty$.

In the cases $m+n=3$ and $m+n=4$, when $1\le p\le\infty$, (4.10) follows easily from (4.11)–(4.14). □

Our results concerning the operators $T_{m,n,\mathcal{D}}$ for bounded domains lead directly to corresponding results when $\mathcal{D}=\mathbf{C}$; for brevity, when $\mathcal{D}=\mathbf{C}$, we omit the subscript referring to the domain, adopting the notation

$$T_{m,n}w:=T_{m,n,\mathbf{C}}w.$$

Corollary 4.6. *Assume $m+n \geq 1$, and let w be a complex valued function in $L^1_{\text{loc}}(\mathbf{C})$ such that, for some $\delta > 0$,*

$$|w(z)| = O(|z|^{-m-n-\delta}), \quad \text{as } z \to \infty. \tag{4.15}$$

Then the integral $T_{m,n}w(z)$ converges absolutely for almost all z in $\mathbf{C}$ and, provided that p satisfies conditions (a)–(d) *of Theorem* 4.2, *$T_{m,n}w \in L^p_{\text{loc}}(\mathbf{C})$.*

Proof. By (4.15), there exists $K \geq 0$ such that, for all R sufficiently large,

$$|w(z)| \leq K|z|^{-m-n-\delta}, \quad \text{if } |z| \geq R. \tag{4.16}$$

Choosing such an R, for $|z| < R$ we may write

$$\begin{aligned} T_{m,n}w(z) = &\iint_{|\zeta|<R} K_{m,n}(z-\zeta)w(\zeta)\,d\xi\,d\eta \\ &+ \iint_{|\zeta|\geq R} K_{m,n}(z-\zeta)w(\zeta)\,d\xi\,d\eta. \end{aligned} \tag{4.17}$$

By Theorem 4.2, the first integral on the right converges absolutely for almost all z in the disk $|z| < R$, representing a function of class L^p in this disk; on the other hand, inequality (4.16), along with (4.2) and (4.3), guarantees that the second integral converges absolutely in the disk $|z| < R$, representing a function of class C^∞ there. Since R may be arbitrarily large, the theorem follows. □

Corollary 4.7. *Assume $m+n \geq 1$, let w be a complex valued function in $L^p_{\text{loc}}(C)$ where p satisfies conditions* (a)–(d) *of Theorem* 4.3, *and suppose that* (4.15) *holds for some $\delta > 0$. Then $T_{m,n}w(z)$ exists as a Lebesgue integral for all z in $\mathbf{C}$, and $T_{m,n}w$ is continuous in $\mathbf{C}$.*

Proof. We split $T_{m,n}w$ into two integrals as in the proof of Corollary 4.6, and apply Theorem 4.3. □

Corollary 4.8. *Assume $m+n \geq 1$ and $mn < 0$, let w be a complex valued function in $L^p_{\text{loc}}(\mathbf{C})$ where p satisfies conditions* (a)–(d) *of Theorem* 4.5*, and suppose that* (4.15) *holds for some* $\delta > 0$*. Then* $T_{m,n}w$ *is locally Lipschitz continuous in* $\mathbf{C}$ *when* $m+n \geq 2$*, and locally Hölder continuous in* $\mathbf{C}$ *with exponent* $(p-2)/p$ *when* $m+n=1$*.*

Proof. Again, we split $T_{m,n}w$ into two integrals; then we apply Theorem 4.5. □

Finally, we discuss the operators $T_{m,n}$ when $m+n=0$. In these cases the singularity of $K(z-\zeta)$ at $\zeta = z$ has the order $|z-\zeta|^{-2}$; consequently, the integral $T_{m,n,\mathcal{D}}w$ does not converge in the ordinary Lebesgue sense. We must view the integral as a *Cauchy principal value integral*,

$$T_{m,n,\mathcal{D}}w(z) := \lim_{\varepsilon\to 0} \iint_{\mathcal{D}_\varepsilon} K_{m,n}(z-\zeta)w(\zeta)\,d\xi\,d\eta, \tag{4.18}$$

where $\mathcal{D}_\varepsilon$ is the domain $\mathcal{D} - \{\zeta : |\zeta - z| \leq \varepsilon\}$, and the limit is taken in the norm of $L^p(\mathcal{D})$. These integrals can be analyzed with the well-known theory of Calderon and Zygmund [**6, 7**] concerning singular integrals, a summary and extension of which is found in the book of Stein [**14**]. We first consider the case $\mathcal{D} = \mathbf{C}$.

Theorem 4.9. *Assume* $m+n=0$*,* $(m,n) \neq (0,0)$*, and let* w *be a complex valued function in* $L^p(\mathbf{C})$ *where* $1 < p < \infty$*. Then* $T_{m,n}w$*, as defined by* (4.18) *with* $\mathcal{D} = \mathbf{C}$*, also belongs to* $L^p(\mathbf{C})$*, and*

$$\|T_{m,n}w\|_{p,\mathbf{C}} \leq M(p)\|w\|_{p,\mathbf{C}}. \tag{4.19}$$

Proof. First we consider the case $n > 0$; then $T_{m,n}w = T_{-n,n}w$, $n > 0$. By the formulas of Section 2,

$$K_{-n,n}(z) := (-1)^n \frac{n}{\pi} z^{-n-1}\bar{z}^{n-1} = \frac{\Omega(z)}{|z|^2},$$

where

$$\Omega(z) := (-1)^n \frac{n}{\pi}\left(\frac{\bar{z}}{z}\right)^n.$$

Obviously, Ω is homogeneous of degree zero. Moreover, letting $d\sigma(z)$ denote the arc length differential on $|z|=1$, we have

$$\begin{aligned}\int_{|z|=1}\Omega(z)\,d\sigma(z) &= \int_0^{2\pi}\Omega(e^{i\theta})\,d\theta\\ &= \int_{|z|=1}\Omega(z)\frac{dz}{iz}\\ &= (-1)^n\frac{n}{\pi i}\int_{|z|=1}\bar{z}^n z^{-n-1}\,dz\\ &= (-1)^n\frac{n}{\pi i}\int_{|z|=1}z^{-2n-1}\,dz\\ &= 0.\end{aligned}$$

Since also Ω is C^∞ on the boundary of the unit disk, Theorem 3 of [**14**], Chapter 4, applies and we conclude that $T_{-n,n}$ is a bounded operator mapping $L^p(\mathbf{C})$ into itself, with (4.19) being valid.

The case $n<0$ can be treated in a similar manner; but it is simpler to observe only that $\overline{T_{n,-n}\rho}=T_{-n,n}\bar{\rho}$, as follows from (2.1). $\square$

Corollary 4.10. *Assume $m+n=0$, $(m,n)\neq(0,0)$, let $\mathcal{D}$ be a domain in $\mathbf{C}$, and let w be a complex valued function in $L^p(\mathcal{D})$ where $1<p<\infty$. Then $T_{m,n,\mathcal{D}}w$, as defined by* (4.18), *belongs to $L^p(\mathbf{C})$, and*

$$\|T_{m,n,\mathcal{D}}w\|_{p,\mathbf{C}}\leq M(p)\|w\|_{p,\mathcal{D}}. \tag{4.20}$$

Proof. We set $w\equiv 0$ outside $\mathcal{D}$; then $T_{m,n,\mathcal{D}}w=T_{m,n,\mathbf{C}}w$, $\|w\|_{p,\mathcal{D}}=\|w\|_{p,\mathbf{C}}$, and the result follows from Theorem 4.9. $\square$

Corollary 4.11. *Assume $m+n=0$, let w be a complex valued function in $L^p_{\text{loc}}(\mathbf{C})$ where $1<p<\infty$, and assume that, for some $\delta>0$,*

$$|w(z)|=O(|z|^{-\delta}),\quad \text{as } z\to\infty.$$

Then $T_{m,n}w\in L^p_{\text{loc}}(\mathbf{C})$.

Proof. We may assume $(m, n) \neq (0, 0)$ as otherwise the result is trivial. As in the proof of Corollary 4.6, (4.17) holds for R sufficiently large. By Corollary 4.10, the first integral on the right of (4.17) represents a function in $L^p(\mathbf{C})$, while the second integral is of class C^∞ in the disk $|z| < R$; hence, the result follows. □

Remark. In all circumstances discussed in this section where $T_{m,n,\mathcal{D}}w$ is defined, it is clear from (2.1) that $T_{m,n,\mathcal{D}}\bar{w}$ also is defined, and in fact,

$$\overline{T_{m,n,\mathcal{D}}w} = T_{n,m,\mathcal{D}}\bar{w}. \tag{4.21}$$

5. Differentiability of integrals. Now we discuss differentiability of the integrals $T_{m,n,\mathcal{D}}w$. It is helpful first to examine some special cases, when $\mathcal{D} = \mathbf{C}$ and w satisfies certain regularity conditions.

Lemma 5.1. *Assume that $m + n \geq 0$, that $w \in C(\mathbf{C})$, and that, for some $\delta > 0$,*

$$|w(z)| = O(|z|^{-m-n-\delta}), \quad \textit{as } z \to \infty. \tag{5.1}$$

(a) *If $m + n \geq 1$, then $T_{m,n}w(z)$ exists as a Lebesgue integral for all z in $\mathbf{C}$, and $T_{m,n}w$ is continuous in $\mathbf{C}$.*

(b) *If $m + n = 0$, $(m, n) \neq (0, 0)$, and w is locally Hölder continuous in $\mathbf{C}$, then for all z in $\mathbf{C}$ the limit*

$$T_{m,n}w(z) = \lim_{\varepsilon \to 0} \iint_{|\zeta - z| \geq \varepsilon} K_{m,n}(z - \zeta)w(\zeta)\, d\xi\, d\eta$$

exists as a limit in the norm of $\mathbf{C}$.

(c) *Suppose, moreover, that $w \in C^1(\mathbf{C})$; if*

$$|w_z(z)| = O(|z|)^{-m-n-1-\delta}, \quad \textit{as } z \longrightarrow \infty, \tag{5.2}$$

then

$$T_{m,n}w = T_{m+1,n}w_z, \quad m + n \geq 0, \tag{5.3}$$

and if

$$|w_{\bar z}(z)| = O(|z|)^{-m-n-1-\delta}, \quad \text{as } z \longrightarrow \infty, \tag{5.4}$$

then

$$T_{m,n}w = T_{m,n+1}w_{\bar z}, \quad m+n \geq 0. \tag{5.5}$$

Proof. (a) The result is a special case of Corollary 4.7.

(b) First we use (2.2) and (3.1) to observe that, for $0 < \varepsilon < R$,

$$\begin{aligned}\iint_{\varepsilon<|z|<R} K_{m,n}(z)\,dx\,dy &= \iint_{\varepsilon<|z|<R} \partial_z K_{m+1,n}(z)\,dx\,dy \\ &= \frac{i}{2}\bigg[\int_{|z|=R} K_{m+1,n}(z)\,d\bar z \\ &\qquad - \int_{|z|=\varepsilon} K_{m+1,n}(z)\,d\bar z\bigg].\end{aligned}$$

But $m+n=0$ and $(m,n)\neq(0,0)$ implies $(m+1)-n\neq 1$; thus (2.4) confirms that the last two integrals vanish. Hence, we conclude that

$$\begin{aligned}&\iint_{\varepsilon<|\zeta-z|<R} K_{m,n}(z-\zeta)\,d\xi\,d\eta \\ &\qquad = (-1)^{m+n}\iint_{\varepsilon<|z|<R} K_{m,n}(z)\,dx\,dy = 0.\end{aligned}$$

Therefore, fixing $R>0$, for $0<\varepsilon<R$ we may write

$$\begin{aligned}&\iint_{|\zeta-z|\geq\varepsilon} K_{m,n}(z-\zeta)w(\zeta)\,d\xi\,d\eta \\ &\qquad = \iint_{|\zeta-z|\geq R} K_{m,n}(z-\zeta)w(\zeta)\,d\xi\,d\eta \\ &\qquad\quad + \iint_{\varepsilon\leq|\zeta-z|\leq R} K_{m,n}(z-\zeta)[w(\zeta)-w(z)]\,d\xi\,d\eta.\end{aligned} \tag{5.6}$$

Recalling that $|K_{m,n}(z-\zeta)| = (\text{constant})|z-\zeta|^{-2}$ when $m+n=0$, and viewing also (5.1), we conclude that the first integral on the right of (5.6) is absolutely convergent; while as $\varepsilon \to 0$ the second integral converges to the absolutely convergent integral

$$\iint_{|\zeta - z| \le R} K_{m,n}(z-\zeta)[w(\zeta) - w(z)]\, d\xi\, d\eta,$$

by virtue of the Hölder continuity of w at z.

(c) First we consider $(m,n) = (0,0)$. To verify (5.3), we apply (3.3) to w and with $\mathcal{D}$ a large disk, say of radius R about 0; then, for $|z| < R$,

$$\begin{aligned} T_{0,0}w(z) = w(z) = &-\frac{1}{\pi} \iint_{|\zeta|<R} \frac{1}{\bar\zeta - \bar z} w_\zeta(\zeta)\, d\xi\, d\eta \\ &- \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{w(\zeta)}{\bar\zeta - \bar z}\, d\bar\zeta. \end{aligned}$$

As $R \to \infty$, (5.1) implies that the second integral tends to zero, while (5.2) shows that $T_{1,0}w_z$ exists as an ordinary Lebesgue integral; thus, we obtain

$$T_{0,0}w = T_{1,0}w_z.$$

In a similar way we can use (3.2) to verify (5.5) for $(m,n) = (0,0)$.

Next we verify (5.3) for the case $(m,n) \ne (0,0)$. We write

$$\begin{aligned} T_{m,n}w(z) &= \lim_{\varepsilon\to 0} \iint_{\varepsilon \le |\zeta - z| \le 1/\varepsilon} K_{m,n}(z-\zeta) w(\zeta)\, d\xi\, d\eta \\ &= \lim_{\varepsilon\to 0} \iint_{\varepsilon \le |\zeta - z| \le 1/\varepsilon} (\partial_\zeta \{-K_{m+1,n}(z-\zeta)w(\zeta)\} \\ &\qquad\qquad + K_{m+1,n}(z-\zeta) w_\zeta(\zeta))\, d\xi\, d\eta \\ &= \lim_{\varepsilon\to 0} \left[\frac{i}{2} \int_{|\zeta - z| = \varepsilon} K_{m+1,n}(z-\zeta) w(\zeta)\, d\bar\zeta \right. \\ &\qquad \left. - \frac{i}{2} \int_{|\zeta - z| = 1/\varepsilon} K_{m+1,n}(z-\zeta) w(\zeta)\, d\bar\zeta \right] \\ &\quad + T_{m+1,n} w_z(z). \end{aligned}$$

(Note that $T_{m+1,n}w_z(z)$ exists as a Lebesgue integral, by application of (a) to w_z.) When $m+n \ge 1$, (2.6) applied to the pair $(m+1,n)$, along with the continuity of w, confirms that the first line integral inside the large brackets tends to zero with ε. When $m+n=0$ and $(m,n)\neq(0,0)$, then $(m+1)-n\neq 1$, and we may use (2.4) to write this integral as

$$\frac{i}{2}\int_{|\zeta-z|=\varepsilon} K_{m+1,n}(z-\zeta)[w(\zeta)-w(z)]\,d\bar{\zeta},$$

which tends to zero as $\varepsilon\to 0$ by virtue of (4.2), (4.3), and the smoothness of w. Moreover, when $(m,n)\neq(0,0)$, we can use (5.1), (4.2), and (4.3) to show that the second line integral inside the brackets also tends to 0 with ε; thus we obtain (5.3).

When $(m,n)\neq(0,0)$, we can prove formula (5.5) in a similar manner; alternatively, we can also use (5.3) and the identity (4.21). □

Throughout the remainder of the paper, the letter ρ will refer to a complex valued function in $C_0^\infty(\mathbf{C})$, the space of functions of class C^∞ in $\mathbf{C}$ with compact support.

Lemma 5.2. *For $\rho\in C_0^\infty(\mathbf{C})$,*

$$\overline{T_{m,n}\rho}=T_{n,m}\bar{\rho},\quad \textit{if } m+n\geq 0, \tag{5.7}$$

$$T_{m,n}\rho=T_{m+1,n}\rho_z=T_{m,n+1}\rho_{\bar{z}},\quad \textit{if } m+n\geq 0, \tag{5.8}$$

$$\partial_z(T_{m,n}\rho)=T_{m-1,n}\rho,\quad \textit{if } m+n\geq 1, \tag{5.9}$$

$$\partial_{\bar{z}}(T_{m,n}\rho)=T_{m,n-1}\rho,\quad \textit{if } m+n\geq 1, \tag{5.10}$$

$$\partial_z(T_{m,n}\rho)=T_{m,n}\rho_z,\quad \textit{if } m+n\geq 0, \tag{5.11}$$

$$\partial_{\bar{z}}(T_{m,n}\rho)=T_{m,n}\rho_{\bar{z}},\quad \textit{if } m+n\geq 0. \tag{5.12}$$

Proof. Formula (5.7) is again (4.21), while (5.8) combines (5.3) and (5.5), valid because ρ satisfies the hypotheses required in Lemma 5.1. Not that in fact all of Lemma 5.1 applies to ρ.

Formulas (5.9) and (5.10) are clear if $m+n>1$, as in these cases the singularity at $\zeta = z$ of the kernels $K_{m-1,n}(z-\zeta)$ and $K_{m,n-1}(z-\zeta)$ is no worse than $O(|z-\zeta|^{-1}\log|z-\zeta|)$, allowing differentiation under the integral of $T_{m,n}\rho$. When $m+n=1$, we use (5.8) as well as (5.9) and (5.10) for $m+n>1$, to write

$$\partial_z(T_{m,n}\rho) = \partial_z(T_{m+1,n}\rho_z) = T_{m,n}\rho_z = T_{m-1,n}\rho,$$
$$\partial_{\bar z}(T_{m,n}\rho) = \partial_{\bar z}(T_{m,n+1}\rho_{\bar z}) = T_{m,n}\rho_{\bar z} = T_{m,n-1}\rho.$$

Finally, combining (5.8)–(5.10), we obtain (5.11) and (5.12) from

$$\partial_z T_{m,n}\rho = \partial_z T_{m+1,n}\rho_z = T_{m,n}\rho_z,$$
$$\partial_{\bar z} T_{m,n}\rho = \partial_{\bar z} T_{m,n+1}\rho_{\bar z} = T_{m,n}\rho_{\bar z}. \qquad \square$$

Remark. Iterating (5.11) and (5.12), we conclude that $T_{m,n}\rho \in C^\infty(\mathbf{C})$, with

$$\frac{\partial^{k+l}}{\partial z^k \partial \bar z^l} T_{m,n}\rho = T_{m,n}\left(\frac{\partial^{k+l}\rho}{\partial z^k \partial \bar z^l}\right). \tag{5.13}$$

Theorem 5.3. *Let $\mathcal{D}$ be a bounded domain in $\mathbf{C}$, let w be a complex valued function in $\mathcal{D}$, and set $w \equiv 0$ outside $\mathcal{D}$. Under either of the conditions*

(a) *$m+n \geq 2$ and $w \in L^1(\mathcal{D})$*

(b) *$m+n \geq 1$ and $w \in L^p(\mathcal{D})$ for some $p>1$,*

we have in $\mathbf{C}$ the Sobolev derivatives

$$\partial_z T_{m,n,\mathcal{D}} w = T_{m-1,n,\mathcal{D}} w, \tag{5.14}$$
$$\partial_{\bar z} T_{m,n,\mathcal{D}} w = T_{m,n-1,\mathcal{D}} w. \tag{5.15}$$

Moreover, for $w \in L^1(\mathcal{D})$ we have also in $\mathbf{C}$ the Sobolev derivatives

$$\partial_z T_{1,0,\mathcal{D}} w = \partial_{\bar z} T_{0,1,\mathcal{D}} w = w. \tag{5.16}$$

Proof. Under condition (a), let $\{\rho_l\}_{l=1}^\infty$ be a sequence of complex valued functions in $C_0^\infty(\mathcal{D})$ converging to w in the norm of $L^1(\mathcal{D})$. We apply Theorem 4.2 and Lemma 5.2 to conclude that, for any bounded domain Ω,

(5.17)
$$T_{m,n,\mathcal{D}}\rho_l \longrightarrow T_{m,n,\mathcal{D}}w \quad \text{in } L^1(\Omega),$$

(5.18)
$$\partial_z T_{m,n,\mathcal{D}}\rho_l = T_{m-1,n,\mathcal{D}}\rho_l \longrightarrow T_{m-1,n,\mathcal{D}}w \quad \text{in } L^1(\Omega),$$

(5.19)
$$\partial_{\bar z} T_{m,n,\mathcal{D}}\rho_l = T_{m,n-1,\mathcal{D}}\rho_l \longrightarrow T_{m,n-1,\mathcal{D}}w \quad \text{in } L^1(\Omega).$$

Therefore, (5.14) and (5.15) hold in Ω and throughout $\mathbf{C}$ since Ω is arbitrary. When $w \in L^1(\mathcal{D})$ and $(m,n) = (1,0)$ we replace (5.18) with $\partial_z T_{1,0,\mathcal{D}}\rho_l = \rho_l \to w$ in $L^1(\Omega)$ to conclude that the first equation of (5.16) holds; the second equation of (5.16) is obtained similarly.

Under condition (b), with also $m+n \geq 2$, (5.14) and (5.15) follow from case (a) since $\mathcal{D}$ is bounded; thus, we need to consider case (b) only when $m+n=1$. We may assume then that $p<\infty$, and that $\rho_l \to w$ in $L^p(\mathcal{D})$. Again (5.17) holds by Theorem 4.2, and (5.18) and (5.19) by Corollary 4.10 and Lemma 5.2; thus, we have (5.14) and (5.15). □

Remark. It is clear that formulas (5.14) and (5.15) may be iterated; for example, when $w \in L^p(\mathcal{D})$ with $1 < p < \infty$ and $\mathcal{D}$ a bounded domain, we obtain the formulas

(5.20)
$$\partial_z^k \partial_{\bar z}^l T_{m,n,\mathcal{D}}w = T_{m-k,n-l,\mathcal{D}}w, \quad \text{if } k+l \leq m+n.$$

(Again, when $(m-k, n-l) = (0,0)$ we must take $w \equiv 0$ outside $\mathcal{D}$.)

Corollary 5.4. *Assume $m+n \geq 1$, and let w be a measurable complex valued function in $\mathbf{C}$ such that, for some $\delta > 0$,*

(5.21)
$$|w(z)| = O(|z|^{-m-n-\delta}), \quad \text{as } z \longrightarrow \infty.$$

(a) *If $m+n \geq 2$ and $w \in L^1_{\text{loc}}(\mathbf{C})$, then in the sense of Sobolev derivatives in the entire plane $\mathbf{C}$,*

(5.22)
$$\partial_z T_{m,n}w = T_{m-1,n}w,$$

(5.23)
$$\partial_{\bar z} T_{m,n}w = T_{m,n-1}w$$

(b) *If* $m+n=1$ *and* $w \in L^p_{\text{loc}}(\mathbf{C})$ *for some* $p>1$*, then* (5.22) *and* (5.23) *again hold in the sense of Sobolev derivatives in* $\mathbf{C}$; *moreover, the formulas*

$$\partial_z T_{1,0} w = \partial_{\bar z} T_{0,1} w = w \tag{5.23}$$

are valid in $\mathbf{C}$ *even in the case* $p=1$.

Proof. (a) First note that, by Corollary 4.6, the integrals $T_{m,n}w$, $T_{m-1,n}w$, and $T_{m,n-1}w$ represent functions in $L^1_{\text{loc}}(\mathbf{C})$. By (5.21), we may choose a bounded domain $\mathcal{D}$ containing a large enough disk about the origin so that, for some constant $K \geq 0$,

$$|w(z)| \leq K|z|^{-m-n-\delta}, \qquad \forall\, z \in \mathbf{C} - \mathcal{D}; \tag{5.25}$$

then we write

$$T_{m,n}w = T_{m,n,\mathcal{D}}w + T_{m,n,\tilde{\mathcal{D}}}w, \tag{5.26}$$

where $\tilde{\mathcal{D}}$ is the domain complementary to $\mathcal{D}$ in $\mathbf{C}$. By Theorem 5.3, equations (5.14) and (5.15) hold in all of $\mathbf{C}$. Moreover, because of (5.25), $T_{m,n,\tilde{\mathcal{D}}}$ belongs to $C^\infty(\mathcal{D})$, and in $\mathcal{D}$ we have the formulas

$$\begin{aligned} \partial_z T_{m,n,\tilde{\mathcal{D}}}w &= T_{m-1,n,\tilde{\mathcal{D}}}w, \\ \partial_{\bar z} T_{m,n,\tilde{\mathcal{D}}}w &= T_{m,n-1,\tilde{\mathcal{D}}}w. \end{aligned} \tag{5.27}$$

Combining (5.27) with (5.14) and (5.15) then yields (5.22) and (5.23) in the domain $\mathcal{D}$, and hence in all of $\mathbf{C}$ since $\mathcal{D}$ may be chosen arbitrarily large.

(b) We prove the result for the derivative ∂_z, the proof for $\partial_{\bar z}$ being similar. We may assume that $p<\infty$. By Corollary 4.6, $T_{m,n}w \in L^1_{\text{loc}}(\mathbf{C})$; in fact, this assertion would be valid even if $p=1$. Moreover, Corollary 4.11 implies $T_{m-1,n}w \in L^p_{\text{loc}}(\mathbf{C})$, while if $(m-1,n)=(0,0)$ this statement is trivial even when $p=1$. Choosing $\mathcal{D}$ as in the proof of (a), we have again (5.26). By Theorem 5.3, (5.14) holds in $\mathcal{D}$, with $p=1$ allowed when $(m-1,n)=(0,0)$; moreover, the first formula of (5.27) again is valid in $\mathcal{D}$. Addition yields (5.22). $\square$

Remark. We may iterate formulas (5.22) and (5.23), as in the case of bounded domains. Consequently, if (5.21) holds and $w \in L^p_{\text{loc}}(\mathbf{C})$, where $p > 1$, we have again (5.20) with now $\mathcal{D} = \mathbf{C}$.

Finally, we can give an easy extension of Theorem 4.5 to the cases where $m \geq 1$ and $n \geq 1$.

Corollary 5.5. *Suppose $m \geq 1$ and $n \geq 1$. Let $\mathcal{D}$ be a bounded domain in $\mathbf{C}$, and assume that w is a complex valued function in $L^p(\mathcal{D})$; suppose also that*

(a) $2 < p \leq \infty$, *if* $m + n = 2$

(b) $1 < p \leq \infty$, *if* $m + n = 3$

(c) $1 \leq p \leq \infty$, *if* $m + n \geq 4$.

For $z \in \mathbf{C}$, set

$$v(z) := T_{m,n,\mathcal{D}} w(z) = \iint_{\mathcal{D}} K_{m,n}(z - \zeta) w(\zeta)\, d\xi\, d\eta.$$

Then for $z_1, z_2 \in \mathbf{C}$, say with $|z_1|, |z_2| \leq R$,

$$|v(z_1) - v(z_2)| \leq M \|w\|_{p,\mathcal{D}} |z_1 - z_2|, \tag{5.28}$$

where $M = M(m, n, p, \mathcal{D})$ in case (a), *$M = M(m, n, p, \mathcal{D}, R)$ in cases* (b) *and* (c).

Proof. By Theorems 4.3 and 5.3, $v \in C^1(\mathbf{C})$ with

$$\partial_z v = T_{m-1,n,\mathcal{D}} w, \qquad \partial_{\bar{z}} v = T_{m,n-1,\mathcal{D}} w,$$

where, for $|z| \leq R$,

$$|\partial_z v(z)|, |\partial_{\bar{z}} v(z)| \leq \|w\|_{p,\mathcal{D}} \cdot \begin{cases} M(m, n, p, \mathcal{D}) & \text{if } m+n = 2, \\ M(m, n, p, \mathcal{D}, R) & \text{if } m+n \geq 3. \end{cases}$$

Hence, the mean value theorem yields (5.28). □

6. Norms of strongly singular operators. It is well known (see [**15**, Chapter I] that the singular operators Π and $\bar{\Pi}$, acting on functions defined on all of $\mathbf{C}$ according to the formulas

$$\Pi w(z) = T_{-1,1}w(z) = -\frac{1}{\pi}\iint_{\mathbf{C}} \frac{1}{(\zeta - z)^2} w(\zeta)\, d\xi\, d\eta,$$
$$\bar{\Pi} w(z) = T_{1,-1}w(z) = -\frac{1}{\pi}\iint_{\mathbf{C}} \frac{1}{(\bar{\zeta} - \bar{z})^2} w(\zeta)\, d\xi\, d\eta,$$

are *isometries* on the space $L^2(\mathbf{C})$; that is to say, for complex valued functions w in the space $L^2(\mathbf{C})$, we have the identities

$$\|\Pi w\|_{L^2(\mathbf{C})} = \|\bar{\Pi} w\|_{L^2(\mathbf{C})} = \|w\|_{L^2(\mathbf{C})}. \tag{6.1}$$

We demonstrate now that in fact all operators $T_{m,-m}$, $m \in \mathbf{Z}$, are isometries on $L^2(\mathbf{C})$. Among the operators $T_{m,n}$, we call the operators $T_{m,-m}$ *strongly singular*, as the singularity of the kernels $K_{m,-m}(z-\zeta)$ at $\zeta = z$, having magnitude of the order $|z-\zeta|^2$, is nonintegrable. As already pointed out, the integral defining $T_{m,-m}w(z)$ must be interpreted as a Cauchy principal value.

We employ the inner product

$$\langle v, w\rangle := \iint_{\mathbf{C}} v\bar{w}\, dx\, dy,$$

defined for complex valued functions v and w in the space $L^2(\mathbf{C})$; the corresponding norm we write as

$$\|w\|_2 := \left[\iint_{\mathbf{C}} |w|^2\, dx\, dy\right]^{1/2}.$$

Our goal in this section is to establish the identity

$$\|T_{m,-m}w\|_2 = \|w\|_2, \qquad \forall\, w \in L^2(\mathbf{C}). \tag{6.2}$$

We require two lemmas, of interest in their own right.

Lemma 6.1. *Let m and n be integers; then for $w \in L^2(\mathbf{C})$,*

$$T_{m,-m}T_{k,-k}w = T_{m+k,-m-k}w. \tag{6.3}$$

In particular, when $k = -m$,

$$T_{m,-m}T_{-m,m}w = w. \tag{6.4}$$

Proof. Since $C_0^\infty(\mathbf{C})$ is dense in $L^2(\mathbf{C})$, and the operators $T_{m,-m}$ are bounded linear operators on $L^2(\mathbf{C})$ by Theorem 4.9, it is sufficient to establish (6.3) for functions ρ in $C_0^\infty(\mathbf{C})$.

First we verify the formula

$$T_{m,-m}T_{k,-k}\rho = T_{m+1,-m-1}T_{k-1,1-k}\rho. \tag{6.5}$$

We set $v := T_{k,-k}\rho$. By Lemma 5.2, $v \in C^\infty(\mathbf{C})$. If $k \neq 0$, then

$$v(z) := T_{k,-k}\rho(z) = \iint_{\operatorname{supp}\rho} K_{k,-k}(z-\zeta)\rho(\zeta)\,d\zeta,$$

and for $z \notin \operatorname{supp}\rho$,

$$\begin{aligned} v_z(z) &= \iint_{\operatorname{supp}\rho} \partial_z K_{k,-k}(z-\zeta)\rho(\zeta)\,d\xi\,d\eta \\ &= \iint_{\operatorname{supp}\rho} K_{k-1,-k}(z-\zeta)\rho(\zeta)\,d\xi\,d\eta; \end{aligned}$$

from these representations and (4.3) we conclude that, as $z \to \infty$,

$$|v(z)| = O(|z|^{-2}), \qquad |v_z(z)| = O(|z|^{-3}). \tag{6.6}$$

If $k = 0$, then $v = T_{0,0}\rho = \rho$, and these estimates are obvious. Therefore, in all cases we may apply Lemmas 5.1 and 5.2 to deduce that

$$T_{m,-m}T_{k,-k}\rho = T_{m,-m}v = T_{m+1,-m}v_z = T_{m+1,-m}\partial_z T_{k,-k}\rho.$$

Also by Lemma 5.2,

$$\partial_z T_{k,-k}\rho = T_{k,-k}\rho_z = \partial_{\bar z}T_{k,1-k}\rho_z = \partial_{\bar z}T_{k-1,1-k}\rho;$$

thus,

$$T_{m,-m}T_{k,-k}\rho = T_{m+1,-m}\partial_{\bar z}T_{k-1,1-k}\rho = T_{m+1,-m}u_{\bar z}$$

where $u := T_{k-1,1-k}\rho$. As we obtained (6.6), we have as $z \to \infty$ that

$$|u(z)| = O(|z|^{-2}), \qquad |u_{\bar{z}}(z)| = O(|z|^{-3});$$

thus, we may again apply Lemma 5.1 to obtain, as desired,

$$T_{m,-m}T_{k,-k}\rho = T_{m+1,-m}u_{\bar{z}} = T_{m+1,-m-1}u = T_{m+1,-m-1}T_{k-1,1-k}\rho.$$

Next we take the conjugate of (6.5), apply (4.21), and replace ρ by $\bar{\rho}$ to derive also

$$T_{-m,m}T_{-k,k}\rho = T_{-m-1,m+1}T_{1-k,k-1}\rho;$$

then replacing m by $-m$ and k by $-k$ yields

$$T_{m,-m}T_{k,-k}\rho = T_{m-1,1-m}T_{k+1,-k-1}\rho. \tag{6.7}$$

Iterating (6.5) and (6.7), we conclude that, for any integer l,

$$T_{m,-m}T_{k,-k}\rho = T_{m+l,-m-l}T_{k-l,-k+l}\rho. \tag{6.8}$$

Finally, taking $l = k$ in (6.8) gives (6.3) for ρ. □

In the next lemma we require the integration by parts formulas

$$\langle \rho_z, \sigma\rangle = -\langle \rho, \sigma_{\bar{z}}\rangle, \qquad \langle \rho_{\bar{z}}, \sigma\rangle = -\langle \rho, \sigma_z\rangle, \tag{6.9}$$

valid, for example, if $\rho \in C_0^1(\mathbf{C})$ and $\sigma \in C^1(\mathbf{C})$.

Lemma 6.2. *For $m + n \geq 0$ and $\rho, \sigma \in C_0^\infty(\mathbf{C})$,*

$$\langle T_{m,n}\rho, \sigma\rangle = (-1)^{m+n}\langle \rho, T_{n,m}\sigma\rangle. \tag{6.10}$$

Proof. First we assume $m + n \geq 1$. We compute

$$\begin{aligned}\langle T_{m,n}\rho, \sigma\rangle &= \iint_{\mathbf{C}} T_{m,n}\rho(z)\overline{\sigma(z)}\,dx\,dy \\ &= \iint_{\mathbf{C}} \overline{\sigma(z)} \iint_{\mathbf{C}} K_{m,n}(z-\zeta)\rho(\zeta)\,d\xi\,d\eta\,dx\,dy.\end{aligned}$$

Let $B(0,R)$ be a ball containing both $\operatorname{supp}\rho$ and $\operatorname{supp}\sigma$; since the singularity of $K_{m,n}(z-\zeta)$ is integrable (see (2.5)), we have

$$\begin{aligned}\iint_{\mathbf{C}}\iint_{\mathbf{C}}&|K_{m,n}(z-\zeta)|\,|\rho(\zeta)|\,|\sigma(z)|\,d\xi\,d\eta\,dx\,dy\\&\le\|\rho\|_\infty\|\sigma\|_\infty\iint_{|z|\le R}\iint_{|\zeta|\le R}|K_{m,n}(z-\zeta)|\,d\xi\,d\eta\,dx\,dy\\&<\infty;\end{aligned}$$

thus, we may interchange orders of integration to obtain

$$\begin{aligned}\langle T_{m,n}\rho,\sigma\rangle&=\iint_{\mathbf{C}}\rho(\zeta)\iint_{\mathbf{C}}K_{m,n}(z-\zeta)\overline{\sigma(z)}\,dx\,dy\,d\xi\,d\eta\\&=\iint_{\mathbf{C}}\rho(\zeta)\iint_{\mathbf{C}}\overline{K_{n,m}(z-\zeta)\sigma(z)}\,dx\,dy\,d\xi\,d\eta\\&=(-1)^{m+n}\iint_{\mathbf{C}}\rho(\zeta)\iint_{\mathbf{C}}\overline{K_{n,m}(\zeta-z)\sigma(z)}\,dx\,dy\,d\xi\,d\eta\\&=(-1)^{m+n}\langle\rho,T_{n,m}\sigma\rangle.\end{aligned}$$

Next assume that $m+n=0$. Using Lemma 5.2, formulas (6.9), and the result for $m+n>0$, we write

$$\begin{aligned}\langle T_{m,n}\rho,\sigma\rangle&=\langle T_{m+1,n}\rho_z,\sigma\rangle\\&=-\langle\rho_z,T_{n,m+1}\sigma\rangle=\langle\rho,\partial_{\bar z}T_{n,m+1}\sigma\rangle\\&=\langle\rho,T_{n,m}\sigma\rangle=(-1)^{m+n}\langle\rho,T_{n,m}\sigma\rangle.\qquad\square\end{aligned}$$

Theorem 6.3. *For any integer m, the operator $T_{m,-m}$ is unitary on $L^2(\mathbf{C})$, with $T_{-m,m}$ both the inverse and adjoint operator to $T_{m,-m}$. In particular, for v and w in $L^2(\mathbf{C})$,*

$$\langle T_{m,-m}v,w\rangle=\langle v,T_{-m,m}w\rangle,\tag{6.11}$$

$$\|T_{m,-m}w\|_2=\|w\|_2.\tag{6.12}$$

Proof. By Lemma 6.2, formula (6.11) holds for v and w in $C_0^\infty(\mathbf{C})$; then, since $C_0^\infty(\mathbf{C})$ is dense in $L^2(\mathbf{C})$ and the operators $T_{m,-m}$ are bounded, the result is valid also for v and w in $L^2(\mathbf{C})$. Formula (6.11) shows that $T_{-m,m}$ is the adjoint operator of $T_{m,-m}$, while (6.4) shows that it is the inverse; thus $T_{m,-m}$ is unitary. We obtain (6.12) by taking $w = T_{m,-m}v$ in (6.11) and then replacing v with w. □

7. Boundary value problems. Expanding on some of the ideas of the earlier sections of this paper, we investigate two boundary value problems involving a higher order version of the complex Cauchy-Riemann equation,

$$\frac{\partial^m w}{\partial \bar{z}^m} = 0, \tag{7.1}$$

and its corresponding inhomogeneous version,

$$\frac{\partial^m w}{\partial \bar{z}^m} = v. \tag{7.2}$$

(Solutions of (7.1) are called *polyanalytic functions* and have been widely studied. The book of Balk [**1**] is a good reference and bibliographical source.) We will solve a generalized Riemann problem for (7.1) and a generalized Hilbert problem for (7.2) when the domain is the unit disk. It is possible to use the machinery we have developed to study more general boundary value problems, but because of space limitations we restrict our attention here to these simpler problems; we believe they are of some interest as natural extensions of well-known problems for (7.1) and (7.2) when $m = 1$.

Let $\mathcal{D}$ be a regular domain with boundary Γ, and suppose $w \in C^m(\overline{\mathcal{D}})$ and solves (7.1) in $\mathcal{D}$; then, according to Theorem 3.2 and in particular formula (3.7), w has in $\mathcal{D}$ the representation

$$w(z) = \sum_{n=0}^{m-1} \frac{1}{n!} \frac{1}{2\pi i} \int_\Gamma \frac{(\bar{z} - \bar{\zeta})^n}{\zeta - z} \frac{\partial^n w(\zeta)}{\partial \bar{\zeta}^n}\, d\zeta. \tag{7.3}$$

This formula of course generalizes the Cauchy integral formula, corresponding to the case $m = 1$ when w is analytic.

Now let $f_0, f_1, \dots, f_{m-1}$ be a collection of complex valued functions, each Hölder continuous on Γ. Then we may form the sum of integrals

$$w(z) = \sum_{n=0}^{m-1} \frac{1}{n!} \frac{1}{2\pi i} \int_\Gamma \frac{(\bar{z} - \bar{\zeta})^n}{\zeta - z} f_n(\zeta)\, d\zeta. \tag{7.4}$$

We find that this sum solves a generalized Riemann boundary value problem for (7.1), as described in the statement of the next theorem. As is customary, we let $\mathcal{D}^+$ and $\mathcal{D}^-$ represent the interior and exterior of $\mathcal{D}$, respectively, and for a point ζ on $\partial\mathcal{D}$ we define the limits

$$w^+(\zeta) := \lim_{z\to\zeta, z\in\mathcal{D}^+} w(z),$$
$$w^-(\zeta) := \lim_{z\to\zeta, z\in\mathcal{D}^-} w(z).$$

We recall a well-known result concerning the Cauchy integral,

$$g(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta,$$

whenever the domain $\mathcal{D}$ is regular with boundary Γ of class C^1 and with nonzero tangent vector. If f is Hölder continuous on Γ with exponent α, $0 < \alpha < 1$, then g is Hölder continuous with the same exponent in $\mathcal{D}^+\cup\Gamma$ and $\mathcal{D}^-\cup\Gamma$, g vanishes at infinity, and g satisfies on Γ the jump condition $g^+ - g^- = f$. (For a proof, see, for example, [**11**, Chapter 2].)

Theorem 7.1. *Let $\mathcal{D}$ be a regular domain in the plane, of class C^1 with everywhere nonzero tangent, let $f_0, f_1, \dots, f_{m-1}$ be Hölder continuous on Γ, and let w be specified by* (7.4). *Then w is a C^∞ solution of* (7.1) *in $\mathcal{D}^+$ and $\mathcal{D}^-$, and the derivatives*

$$\frac{\partial^n w}{\partial\bar z^n}, \quad 0 \le n \le m-1, \tag{7.5}$$

are continuous in $\mathcal{D}^+\cup\Gamma$ and $\mathcal{D}^-\cup\Gamma$, satisfying on Γ the jump conditions

$$\left(\frac{\partial^n w}{\partial\bar z^n}\right)^+ - \left(\frac{\partial^n w}{\partial\bar z^n}\right)^- = f_n, \quad 0 \le n \le m-1. \tag{7.6}$$

Proof. It is clear from (7.4) that w is C^∞ in $\mathcal{D}^+$ and $\mathcal{D}^-$. Moreover, it is a consequence of the Lebesgue bounded convergence theorem that any integral of the form

$$\int_\Gamma \frac{(\bar z - \bar\zeta)^n}{\zeta - z} f(\zeta)\,d\zeta, \quad n \ge 1,$$

with f continuous on Γ, is continuous in the entire plane $\mathbf{C}$; indeed, writing $\zeta = \zeta(t)$ for some real parameter t, $a \leq t \leq b$, with $d\zeta = \zeta'(t)\,dt$, we find that the integrand is bounded in absolute value by the expression

$$|z - \zeta(t)|^{n-1} |f(\zeta(t))| |\zeta'(t)|,$$

which by continuity considerations has a bound independent of t and z as long as z remains in any bounded region. Consequently, we infer from (7.4) and standard results for the Cauchy integral that w is continuous in $\mathcal{D}^+ \cup \Gamma$ and $\mathcal{D}^- \cup \Gamma$, and that on Γ,

$$w^+ - w^- = f_0.$$

Differentiation of (7.4) in $\mathcal{D}^+ \cup \mathcal{D}^-$ yields

$$\partial_{\bar{z}} w(z) = \sum_{n=0}^{m-2} \frac{1}{n!} \frac{1}{2\pi i} \int_\Gamma \frac{(\bar{z} - \bar{\zeta})^n}{\zeta - z} f_{n+1}(\zeta)\, d\zeta;$$

thus, likewise $\partial_{\bar{z}} w$ is continuous in $\mathcal{D}^+ \cup \Gamma$ and $\mathcal{D}^- \cup \Gamma$, and satisfies on Γ

$$(\partial_{\bar{z}} w)^+ - (\partial_{\bar{z}} w)^- = f_1.$$

We continue with this procedure, obtaining in the last steps the formulas

$$\frac{\partial^{m-1} w(z)}{\partial \bar{z}^{m-1}} = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\zeta - z} f_{m-1}(\zeta)\, d\zeta,$$
$$\left(\frac{\partial^{m-1} w}{\partial \bar{z}^{m-1}} \right)^+ - \left(\frac{\partial^{m-1} w}{\partial \bar{z}^{m-1}} \right)^- = f_{m-1},$$

and finally, $\partial^m w / \partial \bar{z}^m = 0$ in $\mathcal{D}^+ \cup \Gamma$ and $\mathcal{D}^- \cup \Gamma$. □

Now we turn our attention to a boundary value problem for the inhomogeneous equation (7.2), when the domain $\mathcal{D}$ is the unit disk. We seek a solution w of (7.2) with derivatives satisfying the conditions

$$\text{(7.7)} \qquad \begin{aligned} &\operatorname{Re} \frac{\partial^n w}{\partial \bar{z}^n} = 0 \quad \text{on } \partial\mathcal{D}, \\ &\operatorname{Im} \left. \frac{\partial^n w}{\partial \bar{z}^n} \right|_{z=0} = 0, \quad \text{for } 0 \leq n \leq m-1. \end{aligned}$$

We solve this problem by use of integral operators S_m, $m = 1, 2, 3, \dots$; these operators act on appropriate functions v defined in $\mathcal{D}$, as specified by the formula

$$(7.8) \quad S_m v(z) := \frac{-1}{2\pi(m-1)!} \cdot \iint_{\mathcal{D}} (2\mathrm{Re}\,(z-\zeta))^{m-1} \left[\frac{\zeta+z}{\zeta-z} \frac{v(\zeta)}{\zeta} + \frac{1+z\bar{\zeta}}{1-z\bar{\zeta}} \frac{\overline{v(\zeta)}}{\bar{\zeta}} \right] d\xi\, d\eta.$$

Theorem 7.2. *Let v be a complex valued function in $L^p(\mathcal{D})$, where $\mathcal{D}$ is the unit disk and $2 < p \le \infty$. Then, for each $m \ge 1$, $S_m v(z)$ exists as an ordinary Lebesgue integral for all z in $\overline{\mathcal{D}}$ and $S_m v \in C(\overline{\mathcal{D}})$. Moreover,*

$$(7.9) \qquad \mathrm{Re}\, S_m v = 0, \qquad \mathrm{Im}\, S_m v(0) = 0,$$

and in the sense of weak derivatives, in $\mathcal{D}$ we have the formulas

$$(7.10) \qquad \partial_{\bar{z}}(S_m v) = \begin{cases} S_{m-1} v & \text{if } m \ge 2, \\ v & \text{if } m = 1. \end{cases}$$

Proof. First consider the case $m \ge 2$. Since $|z-\zeta|/|1-z\bar{\zeta}| \le 1$ for $|z| \le 1$ and $|\zeta| < 1$, when $|z| \le 1$ we obtain from (7.8) an estimate of the form

$$(7.11) \qquad \begin{aligned} |S_m v(z)| &\le M(m) \iint_{\mathcal{D}} \frac{|v(\zeta)|}{|\zeta|}\, d\xi\, d\eta \\ &\le M(m) \|v\|_{p,\mathcal{D}} \left[\iint_{\mathcal{D}} |\zeta|^{-q}\, d\xi\, d\eta \right]^{1/q} \\ &< \infty, \end{aligned}$$

where $1/p + 1/q = 1$ and we require $1 \le q < 2$, implying that $2 < p \le \infty$. Continuity of $S_m v$ follows from the Lebesgue dominated convergence theorem, as we have obtained an integrable bound on the absolute value of the integrand of (7.8) independent of z. Next, when

$m = 1$, we may write (7.8) as

$$
\begin{aligned}
S_1 v(z) = &-\frac{1}{\pi}\iint_{\mathcal{D}} \frac{1}{\zeta - z} v(\zeta)\, d\xi\, d\eta \\
&+ \frac{1}{2\pi}\iint_{\mathcal{D}} \frac{v(\zeta)}{\zeta}\, d\xi\, d\eta \\
&- \frac{z}{\pi}\iint_{\mathcal{D}} \frac{1}{1 - z\bar{\zeta}}\overline{v(\zeta)}\, d\xi\, d\eta \\
&- \frac{1}{2\pi}\iint_{\mathcal{D}} \frac{\overline{v(\zeta)}}{\bar{\zeta}}\, d\xi\, d\eta.
\end{aligned}
\tag{7.12}
$$

The first integral on the right of (7.12) is $T_{0,1,\mathcal{D}}v(z)$; by Theorem 4.3 it exists as a Lebesgue integral and is continuous in $\mathbf{C}$. The second and fourth integrals are constants; they exist in the Lebesgue sense because of the estimate of (7.11). In the third integral, when z stays inside $\mathcal{D}$ but a positive distance away from $\partial\mathcal{D}$ the integrand is bounded in absolute value by a constant multiple of $|v(\zeta)|$, and hence exists and is continuous with respect to z. But for $z \neq 0$, a few calculations confirm that this third integral is $\overline{-T_{0,1,\mathcal{D}}v(1/\bar{z})}$, and hence, again by Theorem 4.3, exists and is continuous in $\mathbf{C} - \{0\}$. Therefore the sum (7.12) is continuous in $\overline{\mathcal{D}}$.

Now, to verify the first equation of (7.9), we observe that when $z \in \partial\mathcal{D}$ we may use the relation $z = 1/\bar{z}$ to confirm that the integrand of (7.8) is purely imaginary. To check the second equation of (7.9), we merely set $z = 0$ in (7.8) and observe that the integrand is real.

Finally, the formulas (7.10) are easily verified when $m \geq 2$, as the differentiation

$$\partial_{\bar{z}}(2\mathrm{Re}\,(z-\zeta))^{m-1} = (m-1)(2\mathrm{Re}\,(z-\zeta))^{m-2}$$

is readily checked, and differentiation under the integral with respect to $\bar{z}$ in (7.8) can be justified by the absolute convergence of the resulting integrals, which we have already confirmed. When $m = 1$, we view again (7.12); the last three integrals on the right are analytic in $\mathcal{D}$, thereby yielding zero when differentiated with respect to $\bar{z}$, while the first integral has the derivative v, as stated in Theorem 5.3. □

Corollary 7.3. *Let $v \in L^p(\mathcal{D})$, with $\mathcal{D}$ the unit disk and $2 < p \leq \infty$, and set $w := S_m v$; then the derivatives $\partial^n w/\partial\bar{z}^n$ exist in the weak* *sense in $\mathcal{D}$ for $0 \le n \le m$, and are continuous in $\overline{\mathcal{D}}$ for $0 \le n \le m-1$. Moreover, w solves in $\mathcal{D}$ the inhomogeneous equation* (7.2) *and satisfies also conditions* (7.7). *Indeed, w is the only function defined on $\overline{\mathcal{D}}$ with these properties.*

Proof. By Theorem 7.2 we have in $\mathcal{D}$ the formulas

$$\begin{aligned}
&\frac{\partial w}{\partial \bar z} = S_{m-1}v,\\
&\frac{\partial^2 w}{\partial \bar z^2} = S_{m-2}v, \dots , \frac{\partial^{m-1} w}{\partial \bar z^{m-1}} = S_1 v,\\
&\frac{\partial^m w}{\partial \bar z^m} = v,
\end{aligned}$$

and all these derivatives up to order $m-1$ are continuous in $\overline{\mathcal{D}}$. The boundary conditions (7.7) follow from (7.9) applied to $S_n v$, $0 \le n \le m-1$.

Now suppose there are two functions w_1 and w_2 with these properties. Then the difference $u := w_1 - w_2$ solves in $\mathcal{D}$ the homogeneous equation (7.1), besides also satisfying conditions (7.7). Setting $\phi := \partial^{m-1}u/\partial \bar z^{m-1}$, we have $\phi_{\bar z} = 0$ in $\mathcal{D}$ implying that ϕ is analytic, with $\operatorname{Re}\phi = 0$ on $\partial\mathcal{D}$, $\operatorname{Im}\phi(0) = 0$; thus $\phi \equiv 0$. Repeating this argument, we obtain successively that

$$0 \equiv \frac{\partial^{m-1}u}{\partial \bar z^{m-1}} \equiv \frac{\partial^{m-2}u}{\partial \bar z^{m-2}} \equiv \cdots \equiv \frac{\partial u}{\partial \bar z} \equiv u,$$

and thus $w_1 \equiv w_2$. □

Corollary 7.4. *Let $v \in L^p(\mathcal{D})$, with $\mathcal{D}$ the unit disk and $2 < p \le \infty$. Then for positive integers k and l,*

$$S_k(S_l v) = S_{k+l}v. \tag{7.13}$$

Proof. Setting $m := k+l$, we observe that each side of (7.13) has the properties of the function w described in the statement of Corollary 7.3; since w is uniquely determined, (7.13) holds. □

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