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A NEW REFINEMENT OF THE ARITHMETIC MEAN–GEOMETRIC MEAN INEQUALITY

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In what follows, we denote by A_n and G_n the weighted arithmetic and geometric means of the positive real numbers $x_1, \dots, x_n$, that is,

$$A_n = \sum_{i=1}^{n} p_i x_i \quad \text{and} \quad G_n = \prod_{i=1}^{n} x_i^{p_i},$$

where $p_1, \dots, p_n$ are nonnegative real numbers with $\sum_{i=1}^{n} p_i = 1$.

The famous arithmetic mean–geometric mean inequality $G_n \le A_n$ has found much attention among many mathematicians, and numerous proofs, refinements, extensions and related results of what E.F. Beckenbach and R. Bellman call "probably the most important inequality, and certainly a keystone of the theory of inequalities," [**1**, p. 3], can be found in the literature. We refer to the monographs [**1, 2, 5, 6**] and the references therein.

In 1978, D.I. Cartwright and M.J. Field [**3**] proved the following interesting sharpening of the arithmetic mean–geometric mean inequality.

$$\frac{1}{2}\frac{1}{\max_{1\le i\le n} x_i}\sum_{i=1}^{n} p_i (x_i - A_n)^2 \le A_n - G_n, \tag{1}$$

with equality holding if and only if the x_i's corresponding to positive p_i's are all equal. Moreover, the authors pointed out that the constant $1/(2\max_{1\le i\le n} x_i)$ is best possible.

The aim of this paper is to show that inequality (1) remains valid if we replace on the lefthand side of (1) A_n by G_n. Since

$$0 \le (A_n - G_n)^2 = \sum_{i=1}^{n} p_i (x_i - G_n)^2 - \sum_{i=1}^{n} p_i (x_i - A_n)^2,$$

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we obtain a refinement of inequality (1).

Theorem. *If* $x_i > 0$, $i = 1, \dots, n$, *and* $p_i \ge 0$, $i = 1, \dots, n$, *with* $\sum_{i=1}^{n} p_i = 1$, *then*

$$\frac{1}{2}\frac{1}{\max_{1\le i\le n} x_i}\sum_{i=1}^{n} p_i(x_i - G_n)^2 \le A_n - G_n. \tag{2}$$

Proof. We follow the method of proof given in [**3**]. It suffices to establish (2) for $x_1 \le x_2 \le \cdots \le x_n$. We use induction on n. If $n = 1$, then (2) is obviously true. Let $n = 2$; we set

$$t = x_1/x_2 \in (0,1] \quad \text{and} \quad p = p_1,$$

then (2) is equivalent to

$$2pt + 2(1-p) - 2t^p - p(t - t^p)^2 - (1-p)(1-t^p)^2 \ge 0. \tag{3}$$

Inequality (3) is valid for $p = 0$ and $p = 1$. Let $p \in (0,1)$; we denote the lefthand side of (3) by $f(t)$ and obtain

$$f'''(t)\frac{t^{3-p}}{2p(p-1)} = p(2-p) + p(p+1)t - 2(2p-1)t^p = g(t),$$

say. A simple calculation yields $g(t) \ge 0$ for $t \in (0,1]$, which implies $f'''(t) \le 0$. Since $f'(1) = f''(1) = 0$, we get $f(t) \ge f(1) = 0$ for all $t \in (0,1]$.

Let $n \ge 3$; we assume that inequality (2) with $n-1$ instead of n is true. Moreover, we may suppose that $x_1 < x_2 < \cdots < x_n$; otherwise, if two of the x_i's are equal, then the validity of (2) follows from the induction hypothesis.

We define

$$\varphi : M = \Big\{ p = (p_1, \dots, p_n) \mid p_i \geq 0, \quad i = 1, \dots, n, \\ \text{with } \sum_{i=1}^n p_i = 1 \Big\} \longrightarrow \mathbf{R},$$

$$\varphi(p_1, \dots, p_n) = \sum_{i=1}^n p_i x_i - \prod_{i=1}^n {x_i}^{p_i} \\ - \frac{1}{2x_n} \sum_{i=1}^n p_i \Big(x_i - \prod_{j=1}^n {x_j}^{p_j} \Big)^2 .$$

Let φ attain its absolute minimum at $\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_n)$. We assume (for a contradiction) that $\tilde{p}$ is an interior point of M. Then there exists a real number α such that we obtain for $i = 1, \dots, n$,

$$\frac{\partial}{\partial p_i} (\varphi(p_1, \dots, p_n) - \alpha \psi(p_1, \dots, p_n)) \Big|_{p = \tilde{p}} = 0,$$

with $\psi(p_1, \dots, p_n) = \sum_{j=1}^n p_j - 1$. This implies that the function

$$h(x) = x - \tilde{G}_n \log(x) - \frac{1}{2x_n} (x - \tilde{G}_n)^2 \\ + \frac{\tilde{G}_n}{x_n} (\tilde{A}_n - \tilde{G}_n) \log(x) - \alpha,$$

with $\tilde{A}_n = \sum_{i=1}^n \tilde{p}_i x_i$ and $\tilde{G}_n = \prod_{i=1}^n {x_i}^{\tilde{p}_i}$, has n distinct zeros, namely, $x_1, \dots, x_n$. Hence,

$$H(x) = -x_n x h'(x) = x^2 - (\tilde{G}_n + x_n)x + \tilde{G}_n (x_n + \tilde{G}_n - \tilde{A}_n)$$

has $n - 1$ distinct zeros in (x_1, x_n). The value

$$r = \frac{1}{2} (\tilde{G}_n + x_n) + \sqrt{((\tilde{G}_n + x_n)/2)^2 - \tilde{G}_n (x_n + \tilde{G}_n - \tilde{A}_n)}$$

is a zero of H. Since

$$r - x_n = \sqrt{((x_n - \tilde{G}_n)/2)^2 + \tilde{G}_n (\tilde{A}_n - \tilde{G}_n)} - (x_n - \tilde{G}_n)/2 \geq 0,$$

we conclude that H has at most one zero in (x_1, x_n). This implies $n-1 \leq 1$, which contradicts the assumption $n \geq 3$. Thus, $\tilde{p}$ is a boundary point of M. Hence, one component of $\tilde{p}$ is equal to 0. Let $\tilde{p}_k = 0$, $k \in \{1, \ldots, n\}$; using $\max_{1 \leq i \leq n,\ i \neq k} x_i \leq x_n$ and the induction hypothesis, we obtain for all $(p_1, \ldots, p_n) \in M$:

$$\begin{aligned} \varphi(p_1, \ldots, p_n) &\geq \varphi(\tilde{p}_1, \ldots, \tilde{p}_n) \\ &\geq \sum_{\substack{i=1 \\ i \neq k}}^{n} \tilde{p}_i x_i - \prod_{\substack{i=1 \\ i \neq k}}^{n} x_i^{\ \tilde{p}_i} \\ &\quad - \frac{1}{2 \max_{1 \leq i \leq n,\ i \neq k} x_i} \sum_{\substack{i=1 \\ i \neq k}}^{n} \tilde{p}_i \left(x_i - \prod_{\substack{j=1 \\ j \neq k}}^{n} x_j^{\ \tilde{p}_j} \right)^2 \\ &\geq 0. \end{aligned}$$

This completes the proof of the theorem. □

Remarks. 1) An analysis of the proof reveals that the sign of equality holds in (2) if and only if the x_i's corresponding to positive p_i's are all equal.

2) In a recently published note, L. Grafakos [**4**] has given an elegant elementary proof of the following inequality due to Hilbert.

If a_n, $n \in \mathbf{Z}$, are real numbers which are square summable, then

$$\sum_{j \in \mathbf{Z}} \left(\sum_{\substack{n \in \mathbf{Z} \\ n \neq j}} \frac{a_n}{j-n} \right)^2 \leq \pi^2 \sum_{n \in \mathbf{Z}} a_n^2, \tag{4}$$

where the constant π^2 is best possible.

Grafakos emphasized that his proof of (4) uses only one inequality, namely, $2ab \leq a^2 + b^2$. If we use inequality (2) with $n = 2$ or any other suitable refinement of $2ab \leq a^2 + b^2$, then we obtain a sharpening of (4). For instance, applying $2ab \leq a^2 + b^2 - \delta(b-a)^2$, $\delta < 1$, we get

$$\sum_{j \in \mathbf{Z}} \left(\sum_{\substack{n \in \mathbf{Z} \\ n \neq j}} \frac{a_n}{n-j} \right)^2 + \delta \sum_{\substack{n \in \mathbf{Z} \\ n \neq m}} \sum_{m \in \mathbf{Z}} \left(\frac{a_m - a_n}{m-n} \right)^2 < \pi^2 \sum_{n \in \mathbf{Z}} a_n^2,$$

valid for all real numbers a_n, $n \in \mathbf{Z}$, which are square summable and not all equal to 0.

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