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CONSTRUCTION OF THE SOLUTIONS OF DIFFERENCE EQUATIONS IN THE FIELD OF MIKUSIŃSKI OPERATORS

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ABSTRACT. We construct the solutions of certain difference equations with variable coefficients in the field of Mikusiński operators $\mathcal{F}$. The method we are using is very similar to the method used for the difference equations with variable numerical coefficients. We analyze the character of the solutions of the difference equation obtained by using this method.

The considered difference equations can be treated as the discrete analogues for the differential equations whose coefficients are operator functions in the field $\mathcal{F}$. Therefore the obtained solutions can be treated as the approximate solutions for the corresponding differential equations.

1. Introduction. The set of continuous functions $\mathcal{C}_+$ with supports in $[0,\infty)$, with the usual addition and the multiplication given by the convolution

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau, \quad t > 0,$$

is a ring. By the Titchmarsh theorem, $\mathcal{C}_+$ has no divisors of zero, hence its quotient field can be defined (see [**2**], and, for more advanced topics, [**3**]). The elements of this field, the Mikusiński operator field $\mathcal{F}$, are called *operators*. They are quotients of the form

$$\frac{f}{g}, \quad f \in \mathcal{C}_+, \quad 0 \not\equiv g \in \mathcal{C}_+,$$

where the last division is observed in the sense of convolution. Every continuous function $a = a(t)$, $t \geq 0$, defines a unique operator which we denote simply by a. In that case, we write

$$a = \{a(t)\}.$$

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We shall denote by $\mathcal{F}_c$ the subset of $\mathcal{F}$ consisting of the operators representing continuous functions.

Among the most important operators are the integral operator $l = \{1\}$, its inverse operator, the differential operator s and the identity operator I. It holds

$$ls = I, \qquad l^\alpha = \left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}, \quad \alpha > 0.$$

Clearly, s and I are not defined by any continuous function.

If x is a function with a continuous nth derivative, then

$$\{x^{(n)}(t)\} = s^n x - s^{n-1}x(0) - \cdots - x^{(n-1)}(0)I.$$

An *operational function* $u(x)$ is a function that maps a set of real numbers into the set of operators (see [**2**, Part 3, Chapter I]). It is continuous on an interval (A, B) if there exist an operator q and a continuous function $f(x,t)$ on the domain $\Omega = \{(x,t) \mid A < x < B,\, t \geq 0\}$, such that

$$u(x) = q \cdot \{f(x,t)\}, \quad x \in (A, B).$$

If, additionally, for a fixed $x_0 \in (A, B)$, the quotient

$$\frac{f(x,t) - f(x_0,t)}{x - x_0}$$

uniformly tends to the limit when $x \to x_0$ in every closed interval $[0, T]$, then the operational function $u(x)$ is *differentiable* at the point $x_0 \in (A, B)$. In that case the product

$$u'(x_0) = q\left\{\frac{\partial f(x_0,t)}{\partial x}\right\}$$

is the (first) derivative of the operational function $u(x)$ at the point x_0 (see [**2**, Part III, Chapter I, Section 7]). A $k-$times differentiable function $(k = 2, 3, \dots)$ is defined analogously.

In this paper, we consider the differential equation with variable coefficients in the field $\mathcal{F}$

$$(1) \quad \sum_{i=0}^{p} A_i(x)s^i u''(x) + \sum_{i=0}^{q} B_i(x)s^i u'(x) + \sum_{i=0}^{r} C_i(x)s^i u(x) = f(x),$$

with the conditions

$$u(0) = E, \qquad u(1) = F \tag{2}$$

where E and F are operators from $\mathcal{F}$, and $f(x), A_i(x)$, $i = 0, 1, \dots, p$, $B_i(x)$, $i = 0, 1, \dots, q$, $C_i(x)$, $i = 0, 1, \dots, r$, are operator functions.

The differential equation (1) can be written in the form

$$P(x)u''(x) + Q(x)u'(x) + R(x)u(x) = f(x),$$

where

$$P(x) = \sum_{i=0}^{p} s^i A_i(x), \qquad Q(x) = \sum_{i=0}^{q} s^i B_i(x), \qquad R(x) = \sum_{i=0}^{r} s^i C_i(x).$$

From now on we take $N \in \mathbf{N}$ and put $h := 1/N$. As is usual in numerical analysis, instead of $u'(x)$ we shall put

$$\frac{u(x+h) - u(x-h)}{2h}$$

while instead of $u''(x)$ we shall put

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2}.$$

So we obtain the difference equation in the field $\mathcal{F}$ corresponding to (1):

$$P(x)\,\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + Q(x)\,\frac{u(x+h) - u(x-h)}{2h} + R(x)u(x) = f(x). \tag{3}$$

If we denote by $x_n = x_{n-1} + h$, where $x_0 = 0$ and $h > 0$, $n = 0, 1, 2, ..., N$, and define the operators P_n, Q_n, R_n and f_n by $P_n = P(x_n)$, $Q_n = Q(x_n)$, $R_n = R(x_n)$ and $f_n = f(x_n)$, respectively, then the equation (3) becomes

$$a_n u_{n-1} + b_n u_n + c_n u_{n+1} = f_n, \quad n = 1, \dots, N-1. \tag{4}$$

The conditions

$$u_0 = E, \qquad u_N = F, \tag{5}$$

are in fact the conditions given by (2) of the differential equation (1). The coefficients of equation (4) are of the form

$$a_n = \frac{I}{h^2}\left(P_n - \frac{Q_n h}{2}\right) = \alpha_{1,n} I + s^{r_1}(\alpha_{2,n} I + \phi_{a,n}), \tag{6}$$

$$b_n = -\frac{I}{h^2}\left(2P_n - R_n h^2\right) = \beta_{1,n} I + s^{r_2}(\beta_{2,n} I + \phi_{b,n}), \tag{7}$$

$$c_n = \frac{I}{h^2}\left(P_n + \frac{Q_n h}{2}\right) = \gamma_{1,n} I + s^{r_1}(\gamma_{2,n} I + \phi_{c,n}), \tag{8}$$

where in (6), (7) and (8) $r_1 = \max\{p, q\}$ and $r_2 = \max\{p, r\}$ are natural numbers, $\alpha_{1,n}$, $\beta_{1,n}$, $\gamma_{1,n}$, $\alpha_{2,n}$, $\beta_{2,n}$, $\gamma_{2,n}$, $n = 1, \dots, N-1$, are numerical constants, assumed to be nonzero, and $\phi_{a,n}, \phi_{b,n}, \phi_{c,n}$, $n = 1, \dots, N-1$, are operators from $\mathcal{F}_c$. Moreover, for the operators f_n from the righthand side of (4) and the operators E and F from (5) we shall assume that

$$f_n = s^{\kappa}(f_{1,n} I + f_{c,n}), \quad n = 1, ..., N-1, \tag{9}$$

$$E = s^{\sigma}(E_1 I + E_c), \tag{10}$$

$$F = s^{\sigma}(F_1 I + F_c), \tag{11}$$

where κ and σ are integers, $f_{1,n}$, E_1 and F_1 are nonzero numerical constants and $f_{c,n}$, E_c and F_c are operators from $\mathcal{F}_c$.

The solutions of (4) are also in the field of Mikusiński operators and, therefore, it is of interest to analyze their character. Actually, we use a procedure from [**1**] and the well-known fact that the field of Mikusiński operators has very good algebraic properties. Thus the usual addition and multiplication with operators can be treated in the same way as with complex numbers.

As will be pointed out in Section 3, the differential equation (1) may represent a partial differential equation with certain initial and boundary conditions.

Finally, in Section 4, we estimate the difference between the exact and the approximate solution of the differential equation (1) with the conditions (2) in $\mathcal{F}$, the last being the solution of the difference equation (4) with (5).

Let us remark that in papers [**4, 5**] and [**6**], the difference equations with constant coefficients (first and/or second order) in the field of Mikusiński operators were analyzed. In paper [**7**], we constructed the exact and the approximate solution of a difference equation with variable coefficients, such that $\alpha_{2,n} = \beta_{2,n} = \gamma_{2,n} = 0$, $n \in \mathbf{N}$, in (2), (3) and (4), respectively.

2. Solutions of difference equations. Let us start with the theorem that gives a recurrence relation for the solutions u_n, $n = 0, 1, \dots, N$, of the difference equation (4). As remarked in the Introduction, we use the procedure (and also the notations) from [**1**, Chapter 2, Section 5].

Theorem 1. *Solutions of the difference equation* (4) *with the conditions* (5) *can be written as*

$$\begin{aligned} &u_N = F, \\ &u_n = L_{n+1/2}u_{n+1} + K_{n+1/2}, \end{aligned} \qquad n = 0, 1, 2, \dots, N-2, N-1,$$

where $L_{1/2} = 0$ and $K_{1/2} = E$, and the operators $L_{n+1/2}$ and $K_{n+1/2}$, $1 \le n \le N-1$, have the forms

$$L_{n+1/2} = \frac{-c_n}{b_n + a_n L_{n-1/2}}, \tag{13}$$

$$K_{n+1/2} = \frac{f_n - a_n K_{n-1/2}}{b_n + a_n L_{n-1/2}}. \tag{14}$$

Proof. Starting from the condition (5), where we have $u_0 = E$, and taking

$$L_{1/2} = 0, \qquad K_{1/2} = E,$$

we have

$$u_0 = L_{1/2}u_1 + K_{1/2}.$$

Using the equation (4), we can write

$$a_1u_0 + b_1u_1 + c_1u_2 = f_1, \quad \text{or} \quad u_1 = \frac{-c_1}{b_1}u_2 + \frac{a_1E - f_1}{-b_1}$$

(note that $b_1 \neq 0$, since we assumed $\beta_{1,1} \neq 0$). Denoting

$$L_{3/2} := \frac{-c_1}{b_1}, \qquad K_{3/2} := \frac{a_1 E - f_1}{-b_1}, \tag{15}$$

we get

$$u_1 = L_{3/2} u_2 + K_{3/2}.$$

Further on, using the equation (4), we can write

$$a_2 u_1 + b_2 u_2 + c_2 u_3 = f_2, \quad \text{or} \quad u_2 = L_{5/2} u_3 + K_{5/2}.$$

If we assume that for some $n \in \{3, \dots, N-2\}$, it holds

$$u_{n-1} = L_{n-1/2} u_n + K_{n-1/2}, \tag{16}$$

where $L_{n-1/2}$ and $K_{n-1/2}$, $n = 2, \dots, N-2$, are of the form (13) and (14), respectively, then putting (16) in equation (4), we obtain the relation (12). □

Note that, starting from the condition $u_N = F$ and using Theorem 1, we get

$$u_{N-1} = L_{N-1/2} F + K_{N-1/2},$$

wherefrom

$$u_{N-2} = L_{N-1-1/2} u_{N-1} + K_{n-1-1/2}.$$

Continuing this procedure, by decreasing n, we get u_n, for $n = N-3, N-4, \dots, 1$.

In order to see the character of the solution of the problem (4), (5), we have to analyze the characters of the operators $L_{n+1/2}$ and $K_{n+1/2}$ for $n = 1, 2, \dots, N-1$. As we shall see, we have three cases to analyze:

(i) $r_2 > r_1$;

(ii) $r_2 = r_1$;

(iii) $r_2 < r_1$.

Let us start with the first case, $r_2 > r_1$.

Proposition 1. *Assume in equation* (4) *the coefficients* a_n, b_n *and* c_n, $1 \leq n \leq N-1$, *are of the form* (6), (7) *and* (8), *respectively, and* *the operators f_n and E are of the form* (9) *and* (10), *respectively. If* $r_2 > r_1$, *then*

(a) *the operators $L_{n+1/2}$ are from $\mathcal{F}_c$, $n = 1, 2, \ldots, N-1$, and can be written as*

$$L_{n+1/2} =: l^{r_2-r_1}(\mathcal{L}_{1,n+1/2}I + \mathcal{L}_{c,1,n+1/2}), \quad n = 2, \ldots, N-1, \tag{17}$$

where $\mathcal{L}_{1,n+1/2}$ are numerical constants and $\mathcal{L}_{c,1,n+1/2}$ are operators from $\mathcal{F}_c$;

(b) *the operators $K_{n+1/2}$, $n = 1, 2, \ldots, N-1$, can be written as*

$$K_{n+1/2} = l^{\lambda_n}(\mathcal{K}_{1,n+1/2}I + \mathcal{K}_{c,1,n+1/2}), \tag{18}$$

where $\lambda_1 = \min\{r_2-\kappa, r_2-r_1-\sigma\}, \ldots, \lambda_n = \min\{r_2-\kappa, r_2-r_1+\lambda_{n-1}\}$, $n = 2, \ldots, N-1$, $\mathcal{K}_{1,n+1/2}$ are numerical constants and $\mathcal{K}_{c,1,n+1/2}$ are operators from $\mathcal{F}_c$.

Proof. (a) From relation (15) we have

$$\begin{aligned} L_{3/2} = \frac{-c_1}{b_1} &= -\frac{\gamma_{1,1}I + s^{r_1}(\gamma_{2,1}I + \phi_{c,1})}{\beta_{1,1}I + s^{r_2}(\beta_{2,1}I + \phi_{b,1})} \\ &= -\frac{l^{r_2}\gamma_{1,1}/\beta_{2,1} + l^{r_2-r_1}(\gamma_{2,1}I + \phi_{c,1})/\beta_{2,1}}{I + (\beta_{1,1}/\beta_{2,1})l^{r_2} + (\phi_{b,1}/\beta_{2,1})} \\ &= -\left(l^{r_2}\frac{\gamma_{1,1}}{\beta_{2,1}} + l^{r_2-r_1}\frac{\gamma_{2,1}I + \phi_{c,1}}{\beta_{2,1}}\right) \\ &\quad \cdot \sum_{j=0}^{\infty}(-1)^j\left(\frac{\beta_{1,1}}{\beta_{2,1}}l^{r_2} + \frac{\phi_{b,1}}{\beta_{2,1}}\right)^j. \end{aligned}$$

It is well known in the Mikusiński operator theory that for any ϕ from $\mathcal{F}_c$ the geometric sum $\sum_{i=1}^{\infty}\phi^i$ converges in the field $\mathcal{F}$ to an operator representing a continuous function. Therefore we can write

$$\begin{aligned} L_{3/2} &= -\left(l^{r_2}\frac{\gamma_{1,1}}{\beta_{2,1}} + l^{r_2-r_1}\frac{\gamma_{2,1}I + \phi_{c,1}}{\beta_{2,1}}\right)(I + \rho_1) \\ &=: l^{r_2-r_1}(\mathcal{L}_{1,3/2}I + \mathcal{L}_{c,1,3/2}), \end{aligned} \tag{19}$$

where $\mathcal{L}_{1,3/2} = -\gamma_{2,1}/\beta_{2,1}$ is a numerical constant, while ρ_1 and $\mathcal{L}_{c,1,3/2}$ are operators from $\mathcal{F}_c$; since $r_2 - r_1 > 0$, the operator $L_{3/2}$ is from $\mathcal{F}_c$, also.

From relation (13), using (19), we have

$$
\begin{aligned}
&L_{5/2}\\
&= \frac{-c_2}{b_2 + a_2 L_{3/2}}\\
&= \frac{-(\gamma_{1,2} I + s^{r_1}(\gamma_{2,2} I + \phi_{c,2}))}{\beta_{1,2} I + s^{r_2}(\beta_{2,2} I + \phi_{b,2}) - (\alpha_{1,2} I + s^{r_1}(\alpha_{2,2} I + \phi_{a,2})) l^{r_2}\left(\frac{\gamma_{1,1}}{\beta_{2,1}} I + l^{-r_1}\frac{\gamma_{2,1} I + \phi_{c,1}}{\beta_{2,1}}\right)(I + \rho_1)}\\
&= \frac{-\left(l^{r_2}\frac{\gamma_{1,2}}{\beta_{2,2}} + l^{r_2 - r_1}\frac{\gamma_{2,2} I + \phi_{c,2}}{\beta_{2,2}}\right)}{I + \frac{\beta_{1,2}}{\beta_{2,2}} l^{r_2} + \frac{\phi_{b,2}}{\beta_{2,2}} - \left(\frac{\alpha_{1,2}}{\beta_{2,2}} l^{r_2} + l^{r_2 - r_1}\frac{\alpha_{2,2} I + \phi_{a,2}}{\beta_{2,2}}\right) l^{r_2}\left(\frac{\gamma_{1,1}}{\beta_{2,1}} I + l^{-r_1}\frac{\gamma_{2,1} I + \phi_{c,1}}{\beta_{2,1}}\right)(I + \rho_1)}\\
&= -\left(l^{r_2}\frac{\gamma_{1,2}}{\beta_{2,2}} + l^{r_2 - r_1}\frac{\gamma_{2,2} I + \phi_{c,2}}{\beta_{2,2}}\right)\\
&\quad \cdot \sum_{j=0}^{\infty}\Bigg(\left(\frac{\alpha_{1,2}}{\beta_{2,2}} I + s^{r_1}\frac{\alpha_{2,2} I + \phi_{a,2}}{\beta_{2,2}}\right)\left(\frac{\gamma_{1,1}}{\beta_{2,1}} I + s^{r_1}\frac{\gamma_{2,1} I + \phi_{c,1}}{\beta_{2,1}}\right)\\
&\qquad\qquad \cdot l^{2r_2}(I + \rho_1) - \frac{\beta_{1,2}}{\beta_{2,2}} l^{r_2} - \frac{\phi_{b,2}}{\beta_{2,2}}\Bigg)^j
\end{aligned}
$$

Hence, we can write

$$
\begin{aligned}
L_{5/2} &= -\left(l^{r_2}\frac{\gamma_{1,2}}{\beta_{2,2}} + l^{r_2 - r_1}\frac{\gamma_{2,2} I + \phi_{c,2}}{\beta_{2,2}}\right)(I + \rho_2)\\
&=: l^{r_2 - r_1}(\mathcal{L}_{1,5/2} I + \mathcal{L}_{c,1,5/2}),
\end{aligned}
$$

where $\mathcal{L}_{1,5/2}$ is a numerical constant, while ρ_2 and $\mathcal{L}_{c,1,5/2}$ are operators from $\mathcal{F}_c$; therefore the operator $L_{5/2}$ is also from $\mathcal{F}_c$.

If we suppose that for some n, $4 \le n \le N-1$, the operator $L_{n-1/2}$ is from $\mathcal{F}_c$ and can be written as

$$
\begin{aligned}
L_{n-1/2} &= -\left(l^{r_2}\frac{\gamma_{1,n-1}}{\beta_{2,n-1}} + l^{r_2 - r_1}\frac{\gamma_{2,n-1} I + \phi_{c,n-1}}{\beta_{2,n-1}}\right)(I + \rho_{n-1})\\
&=: l^{r_2 - r_1}(\mathcal{L}_{1,n-1/2} I + \mathcal{L}_{c,1,n-1/2}),
\end{aligned}
$$

(for some ρ_{n-1} from $\mathcal{F}_c$), then we have

$$
\begin{aligned}
&L_{n+1/2}\\
&= \frac{-c_n}{b_n + a_n L_{n-1/2}}\\
&= -\frac{\gamma_{1,n} I + s^{r_1}(\gamma_{2,n} I + \phi_{c,n})}{\beta_{1,n} I + s^{r_2}(\beta_{2,n} I + \phi_{b,n}) + (\alpha_{1,n} I + s^{r_1}(\alpha_{2,n} I + \phi_{a,n}))L_{n-1/2}}\\
&= -\frac{l^{r_2}\frac{\gamma_{1,n}}{\beta_{2,n}} + l^{r_2-r_1}\frac{\gamma_{2,n} I + \phi_{c,n}}{\beta_{2,n}}}{I + \frac{\beta_{1,n}}{\beta_{2,n}} l^{r_2} + \frac{\phi_{b,n}}{\beta_{2,n}} + \big(\frac{\alpha_{1,n}}{\beta_{2,n}} l^{r_2} + l^{r_2-r_1}\frac{\alpha_{2,n} I + \phi_{a,n}}{\beta_{2,n}}\big)L_{n-1/2}}\\
&= -\left(l^{r_2}\frac{\gamma_{1,n}}{\beta_{2,n}} + l^{r_2-r_1}\frac{\gamma_{2,n} I + \phi_{c,n}}{\beta_{2,n}}\right)\\
&\quad\cdot \sum_{j=0}^{\infty}(-1)^j\left(\frac{\beta_{1,n}}{\beta_{2,n}} l^{r_2} + \frac{\phi_{b,n}}{\beta_{2,2}} + \left(\frac{\alpha_{1,n}}{\beta_{2,n}} l^{r_2} + l^{r_2-r_1}\frac{\alpha_{2,n} I + \phi_{a,n}}{\beta_{2,n}}\right)\right.\\
&\qquad\qquad \left.\cdot\, l^{r_2-r_1}(\mathcal{L}_{1,n-1/2} I + \mathcal{L}_{c,1,n-1/2})\right)^j.
\end{aligned}
$$

So we can write

$$
\begin{aligned}
L_{n+1/2} &= -\left(l^{r_2}\frac{\gamma_{1,n}}{\beta_{2,n}} + l^{r_2-r_1}\frac{\gamma_{2,n} I + \phi_{c,n}}{\beta_{2,n}}\right)(I + \rho_n)\\
&=: l^{r_2-r_1}(\mathcal{L}_{1,n+1/2} I + \mathcal{L}_{c,1,n+1/2}),
\end{aligned}
$$

where $\mathcal{L}_{1,n+1/2}$ is a numerical constant, while $\mathcal{L}_{c,1,n+1/2}$ and ρ_n are operators from $\mathcal{F}_c$. Thus the operator $L_{n+1/2}$, $1 \le n \le N-1$, belongs also to $\mathcal{F}_c$.

(b) The operator $K_{3/2}$ can be transformed as follows:

$$
\begin{aligned}
K_{3/2} = \frac{f_1 - a_1 E}{b_1} &= \frac{f_1 - (\alpha_{1,1} I + s^{r_1}(\alpha_{2,1} I + \phi_{a,1}))E}{\beta_{1,1} I + s^{r_2}(\beta_{2,1} I + \phi_{b,1})}\\
&= \frac{l^{r_2} f_1 - (l^{r_2}\alpha_{1,1} + l^{r_2-r_1}(\alpha_{2,1} I + \phi_{a,1}))E}{\beta_{2,1}}\\
&\quad\cdot \frac{I}{I + (\beta_{1,1}/\beta_{2,1}) l^{r_2} + (\phi_{b,1}/\beta_{2,1})}\\
&= \frac{l^{r_2} f_1 - (l^{r_2}\alpha_{1,1} + l^{r_2-r_1}(\alpha_{2,1} I + \phi_{a,1}))E}{\beta_{2,1}}(I + \rho_1).
\end{aligned}
$$

Since $f_1 = s^{\kappa}(f_{1,1}I + f_{c,1})$ and $E = s^{\sigma}(E_1 I + E_c)$, we have

(20)

$$\begin{aligned}
&K_{3/2}\\
&= \frac{l^{r_2-\kappa}(f_{1,1}I+f_{c,1})-(l^{r_2-\sigma}\alpha_{1,1}+l^{r_2-r_1-\sigma}(\alpha_{2,1}I+\phi_{a,1}))(E_1I+E_c)}{\beta_{2,1}}\\
&\quad \cdot (I+\rho_1)\\
&=: l^{\lambda_1}(\mathcal{K}_{1,3/2}I + \mathcal{K}_{c,1,3/2}),
\end{aligned}$$

where $\lambda_1 = \min\{r_2-\kappa, r_2-r_1-\sigma\}$ (note that $r_2-\sigma > r_2-r_1-\sigma$), $\mathcal{K}_{1,3/2}$ is a numerical constant and $\mathcal{K}_{c,1,3/2}$ represents a continuous function.

Also, we have

$$\begin{aligned}
&K_{5/2}\\
&= \frac{f_2 - a_2 K_{3/2}}{b_2 + a_2 L_{3/2}}\\
&= \frac{l^{r_2} f_2 - (l^{r_2}\alpha_{1,2} + l^{r_2-r_1}(\alpha_{2,2}I + \phi_{a,2}))K_{3/2}}{\beta_{2,2}}\\
&\quad \cdot \frac{I}{I + \frac{\beta_{1,2}}{\beta_{2,2}} l^{r_2} + \frac{\phi_{b,2}}{\beta_{2,2}} - \left(\frac{\alpha_{1,2}}{\beta_{2,2}} l^{r_2} + l^{r_2-r_1}\frac{\alpha_{2,2}I+\phi_{a,2}}{\beta_{2,2}}\right)L_{3/2}}\\
&=: \frac{l^{r_2} f_2 - (\alpha_{1,2} l^{r_2} + s^{r_1-r_2}(\alpha_{2,2}I + \phi_{a,2}))K_{3/2}}{\beta_{2,2}} \cdot (I+\rho_2)\\
&= \frac{l^{r_2-\kappa}(f_{1,2}+f_{c,2})-(l^{r_2}\alpha_{1,2}+l^{r_2-r_1}(\alpha_{2,2}I+\phi_{a,2}))l^{\lambda_1}(\mathcal{K}_{1,3/2}I+\mathcal{K}_{c,1,3/2})}{\beta_{2,2}}\\
&\quad \cdot (I+\rho_2)\\
&=: l^{\lambda_2}(\mathcal{K}_{1,5/2}I + \mathcal{K}_{c,1,5/2}),
\end{aligned}$$

where $\lambda_2 = \min\{r_2-\kappa, r_2-r_1+\lambda_1\}$, $\mathcal{K}_{1,5/2}$ is a numerical constant and $\mathcal{K}_{c,1,5/2}$ represents a continuous function. Using the mathematical induction, we obtain

$$
\begin{aligned}
&(21)\\
&K_{n+1/2}\\
&= \left(\frac{l^{r_2} f_n - (\alpha_{1,n} l^{r_2} + s^{r_1-r_2}(\alpha_{2,n} I + \phi_{a,n}))K_{n-1/2}}{\beta_{2,n}} \right) \cdot (I+\rho_n)\\
&= \left(\frac{l^{r_2} f_n - (\alpha_{1,n} l^{r_2} + s^{r_1-r_2}(\alpha_{2,n} I + \phi_{a,n}))l^{\lambda_{n-1}}(\mathcal{K}_{1,n-1/2} I + \mathcal{K}_{c,1,n-1/2})}{\beta_{2,n}} \right)\\
&\quad \cdot (I+\rho_n)\\
&= l^{\lambda_n}(\mathcal{K}_{1,n+1/2} I + \mathcal{K}_{c,1,n+1/2}),
\end{aligned}
$$

where $\lambda_n = \min\{r_2-\kappa, r_2-r_1+\lambda_{n-1}\}$, $\mathcal{K}_{1,n+1/2}$ is a numerical constant and $\mathcal{K}_{c,1,n+1/2}$, for $n = 1, \dots, N-1$, represents a continuous function. $\square$

Thus we have the following results.

Corollary 1. *If the conditions of Proposition* 1 *are satisfied such that $\kappa < r_2$ and $\sigma < r_2 - r_1$, then the operators $K_{n+1/2}$ belong to $\mathcal{F}_c$, for $n = 1, 2, \dots, N-1$.*

Note that in this case it might happen that the operators f_n and E represent continuous functions (for $\kappa < 0$ and $\sigma < 0$) or that they do not.

Corollary 2. *If the conditions of Proposition* 2 *are satisfied such that $\kappa < r_2$ and $\sigma > r_2 - r_1 > 0$ (implying that the operators f_n, $n = 1, \dots, N-1$, and E do not represent continuous functions), then $K_{3/2}$ does not represent an operator from $\mathcal{F}_c$. However, for sufficiently large N there exists $1 \le n_0 \le N-1$, such that $K_{n+1/2}$, $n = 1, \dots, n_0$, do not represent operators from $\mathcal{F}_c$ and $K_{n+1/2}$, $n = n_0+1, \dots, N-1$, do represent operators from $\mathcal{F}_c$.*

Proof. Using relation (20) we have

$$K_{3/2} = l^{\lambda_1}(\mathcal{K}_{1,3/2} I + \mathcal{K}_{c,1,3/2}),$$

where $\lambda_1 = \min\{r_2 - \kappa, r_2 - r_1 - \sigma\}$. Thus $\lambda_1 = r_2 - r_1 - \sigma$, because $r_2 - \kappa > 0$. Since $r_2 - r_1 - \sigma < 0$, the operator $K_{3/2}$ is not from $\mathcal{F}_c$.

The character of the operator

$$K_{5/2} = l^{\lambda_2}(\mathcal{K}_{1,5/2}I + \mathcal{K}_{c,1,5/2})$$

depends also on λ_2. Let us remark that from the relations

$$r_2 - r_1 + (r_2 - r_1 - \sigma) = 2(r_2 - r_1) - \sigma > \lambda_1$$

it follows that $\lambda_2 = \min\{r_2 - r_1,\, 2(r_2 - r_1) - \sigma\}$. If $\lambda_2 = 2(r_2 - r_1) - \sigma < 0$, then $K_{5/2}$ does not represent an operator from $\mathcal{F}_c$. Of course, it may happen that $\lambda_2 > 0$ and in that case $K_{5/2}$ belongs to $\mathcal{F}_c$. Continuing this procedure, we see that by increasing n the expression $r_2 - r_1 + \lambda_{n-1}$ also increases; hence, for some n_0, λ_{n_0+1} becomes greater than 0. But then $K_{n_0+1/2}$ does belong to $\mathcal{F}_c$. Clearly, this conclusion holds if N is sufficiently big. $\square$

The last proof gives us also the following

Corollary 3. *If the conditions of Proposition* 1 *are satisfied such that either*

(i) $0 < \kappa < r_2$ *and* $\sigma > (N-1)(r_2 - r_1)$, *or*

(ii) $\kappa > r_2$,

then $K_{n+1/2}$, $n = 1, ..., N-1$, *do not represent operators from* $\mathcal{F}_c$.

Let us remark that in the second part of last corollary the continuity of the operator E does not "improve" the continuity of the operators $K_{n+1/2}$.

We can give now the form of the solution u_n of the problem (4), (5).

Theorem 2. *Assume in equation* (4) *the coefficients* a_n, b_n *and* c_n, $1 \le n \le N-1$, *are of the form* (6), (7) *and* (8), *respectively,* f_n, $n = 1, \dots, N-1$, E *and* F *are of the form* (9), (10) *and* (11), *respectively. If* $r_2 > r_1$, *then the solutions of the equation* (4) *with conditions* (5) *are of the form*

$$u_n = l^{\omega_n}(\mathcal{U}_n I + \mathcal{U}_{c,n}), \quad n = 1, \dots, N-1, \tag{22}$$

where $\omega_N = -\sigma$, $\omega_n = \min\{(r_2 - r_1) + \omega_{n+1}, \lambda_n\}$, $\mathcal{U}_n$ *are numerical constants and* $\mathcal{U}_{c,n}$ *are operators from* $\mathcal{F}_c$, $n = 1, \dots, N-1$.

Proof. From the first relation in (12) we obtain that u_N has the form (22), where $\omega_N = -\sigma$. From (12), (17) and (21) we have

$$\begin{aligned} u_{N-1} &= l^{r_2 - r_1 - \sigma}(\mathcal{L}_{1,N-1/2} I + \mathcal{L}_{c,1,N-1/2})(F_1 I + F_c) \\ &\quad + l^{\lambda_{N-1}}(\mathcal{K}_{1,N-1+1/2} I + \mathcal{K}_{c,1,N-1+1/2}) \\ &= l^{\omega_{N-1}}(\mathcal{U}_{N-1} I + \mathcal{U}_{c,N-1}), \end{aligned}$$

where $\omega_{N-1} = \min\{r_2 - r_1 - \sigma, \lambda_{N-1}\}$, $\mathcal{U}_{N-1}$ is a numerical constant and $\mathcal{U}_{c,N-1}$ is an operator from $\mathcal{F}_c$. Continuing this procedure we obtain relation (22). $\square$

Corollary 4. *Assume the conditions of Theorem* 2 *are fulfilled and the operators* f_n, $n = 1, \dots, N-1$, E *and* F *are of the form* (9), (10) *and* (11), *respectively, either represent continuous functions or* $\sigma < r_2 - r_1$, $\kappa < r_2$. *Then the solutions of the problem* (4) *and* (5), *the operators* u_n, $n = 1, \dots, N-1$, *represent continuous functions.*

Proof. If $\sigma < 0$, $\kappa < 0$, then in Proposition 1 and its Corollary 1 it was shown that the operators $L_{n+1/2}$, and $K_{n+1/2}$ for every $1 \le n \le N-1$ are from $\mathcal{F}_c$. So, $\omega_n \ge 1$, $1 \le n \le N-1$, and therefore the solutions of equation (4), given by (22) are from $\mathcal{F}_c$.

Similarly, if $0 < \sigma < r_2 - r_1$, $\kappa < 0$, we have that $\omega_n \ge 1$, $1 \le n \le N-1$. $\square$

Let us remark that in the last corollary it might happen that neither the conditions E and F in (5), nor the operators f_n are from $\mathcal{F}_c$, but still the solutions of equation (4) belong to $\mathcal{F}_c$.

Corollary 5. *Assume the conditions of Theorem* 2 *are fulfilled and* $\sigma > r_2 - r_1$, $\kappa < r_2$. *Then for sufficiently large* N *there exists* $n_0 < N/2$ *such that the solutions of the problem* (4) *and* (5), *given by* (22), u_n, $n = 1, \dots, n_0$ *and* u_n, $n = N - n_0, N - n_0 + 1, \dots, N-1$ *do not represent continuous functions, but* u_n, $n = n_0 + 1, \dots, N - n_0 - 1$, *do represent continuous functions.*

Proof. Follows from Corollary 2 of Proposition 1. □

Corollary 6. *Assume the conditions of Theorem* 2 *are fulfilled and either* $\kappa > r_2$ *or* $\sigma > (N-1)(r_2-r_1)$. *Then the solutions of the problem* (4), (5) *do not represent continuous functions.*

We turn now to (ii), i.e., to the case $r_1 = r_2$. To that end, we shall additionally suppose that the complex numbers $\mathcal{L}_{2,n+1/2}$, $n = 1, 2, \dots, N-1$, given by

$$\mathcal{L}_{2,3/2} = -\frac{\gamma_{2,1}}{\beta_{2,1}}, \qquad \mathcal{L}_{2,n+1/2} = \beta_{2,n} - \alpha_{2,n}\mathcal{L}_{2,n-1/2}, \tag{23}$$

are nonzero. Then analogously as in Proposition 1, we have

Proposition 2. *Assume in equation* (4) *the coefficients* a_n, b_n *and* c_n, $1 \le n \le N-1$, *are of the form* (6), (7) *and* (8), *respectively, and the operators* f_n *and* E *are of the form* (9) *and* (10), *respectively. If* $r_1 = r_2 = m > 0$, *then*

(a) *the operators* $L_{n+1/2}$ *can be written as*

$$L_{n+1/2} = \mathcal{L}_{2,n+1/2} I + \mathcal{L}_{c,2,n+1/2}, \quad n = 1, 2, \dots, N-1,$$

where $\mathcal{L}_{2,n+1/2}$ *are numerical constants from* (23), *and* $\mathcal{L}_{c,2,n+1/2}$ *are from* $\mathcal{F}_c$, $n = 1, \dots, N-1$;

(b) *the operators* $K_{n+1/2}$, $n = 1, 2, \dots, N-1$, *can be written as*

$$K_{n+1/2} = l^{\lambda}(\mathcal{K}_{2,n+1/2} I + \mathcal{K}_{c,2,n+1/2}),$$

where $\lambda = \min\{m-\kappa, -\sigma\}$, $\mathcal{K}_{2,n+1/2}$ *are nonzero numerical constants and* $\mathcal{K}_{c,2,n+1/2}$ *are operators representing continuous functions.*

Corollary 7. *Assume the conditions of Proposition* 2 *are satisfied such that* $\kappa < 0$ *and* $\sigma < 0$ (*implying that the operators* f_n *and* E *represent continuous functions*). *Then the operators* $K_{n+1/2}$, $n = 1, 2, \dots, N-1$, *belong to* $\mathcal{F}_c$.

Corollary 8. *Assume the conditions of Proposition* 2 *are satisfied such that* $\kappa > m > 0$ *or* $\sigma \ge 0$ (*implying that the operators* f_n,

$n = 1, \dots, N-1$, and E do not represent continuous functions). Then the operators $K_{n+1/2}$, $n = 1, \dots, N-1$, do not represent operators from $\mathcal{F}_c$.

Thus we have the following theorem.

Theorem 3. *Assume in equation (4) the coefficients a_n, b_n and c_n, $1 \le n \le N-1$, are of the form (9), (10) and (11), respectively. If $r_1 = r_2 = m$, and the operators f_n, $n = 1, \dots, N-1$, E and F, given by (9), (10) and (11), respectively,*

(i) *represent continuous functions, then the solutions of the problem (4) and (5) also represent continuous functions*;

(ii) *have the forms (9), (10) and (11), respectively, such that $\sigma \ge 0$ or $\kappa \ge m$ (thus neither of them is from $\mathcal{F}_c$), then the solutions of the problem (4), (5) do not represent continuous functions.*

We have come to the case (iii), namely $r_1 > r_2$.

Proposition 3. *Assume in equation (4) the coefficients a_n, b_n and c_n, $1 \le n \le N-1$, are of the form (6), (7) and (8), respectively, and the operators f_n, $n = 1, 2, \dots, N-1$, E and F are given by (9), (10) and (11), respectively. If $r_1 > r_2$, then*

(a) *the operators $L_{n+1/2}$, $n = 1, \dots, N-1$, can be written as*

$$L_{n+1/2} = l^{(-1)^n (r_1 - r_2)} (\mathcal{L}_{3,n+1/2} I + \mathcal{L}_{c,3,n+1/2}), \tag{24}$$

where $\mathcal{L}_{3,n+1/2}$ are numerical constants and $\mathcal{L}_{c,3,n+1/2}$ are operators from $\mathcal{F}_c$. Hence the operators $L_{2k+1/2}$ are from $\mathcal{F}_c$, while the operators $L_{(2k+1)+1/2}$ are not, $k = 1, \dots, [(N-1)/2]$;

(b) *the operators $K_{n+1/2}$, $n = 1, 2, \dots, N-1$, can be written as*

$$K_{n+1/2} = l^{\lambda_n} (\mathcal{K}_{3,n+1/2} I + \mathcal{K}_{c,3,n+1/2}),$$

where $\lambda_1 = \min\{r_2 - \kappa, r_2 - r_1 - \sigma\}, \dots, \lambda_{2k} = \min\{2r_1 - r_2 - \kappa, r_1 - r_2 + \lambda_{2k-1}\}$, $\lambda_{2k+1} = \min\{r_2 - \kappa, r_2 - r_1 + \lambda_{2k}\}$, $k = 1, \dots, [(N-1)/2]$, where $\mathcal{K}_{3,n+1/2}$ are numerical constants and $\mathcal{K}_{c,3,n+1/2}$ are operators from $\mathcal{F}_c$.

Proof. (a) From relation (19) for $r_1 > r_2$ it follows that $L_{3/2}$ is not an operator from $\mathcal{F}_c$ and can be written as

$$L_{3/2} = l^{r_2 - r_1}(\mathcal{L}_{3,3/2} I + \mathcal{L}_{c,3,3/2}),$$

where $\mathcal{L}_{3,3/2}$ is a nonzero numerical constant and $\mathcal{L}_{c,3,3/2}$ is an operator representing a continuous function. From relation (13) we have

$$\begin{aligned} &L_{5/2} \\ &= \frac{-c_2}{b_2 + a_2 L_{3/2}} \\ &= \frac{-(\gamma_{1,2} I + s^{r_1}(\gamma_{2,2} I + \phi_{c,2}))}{\beta_{1,2} I + s^{r_2}(\beta_{2,2} I + \phi_{b,2}) - (\alpha_{1,2} I + s^{r_1}(\alpha_{2,2} I + \phi_{a,2}))\Big(l^{r_2}\frac{\gamma_{1,1}}{\beta_{2,1}} + l^{r_2 - r_1}\frac{\gamma_{2,1} I + \phi_{c,1}}{\beta_{2,1}}\Big)(I + \rho_1)}. \end{aligned}$$

Thus we can write

$$\begin{aligned} L_{5/2} &= -\bigg(l^{2r_1 - r_2}\frac{\gamma_{1,2}}{\delta_2} + l^{r_1 - r_2}\frac{\gamma_{2,2} I + \phi_{c,2}}{\delta_2}\bigg)(I + \psi_2) \\ &= l^{r_1 - r_2}(\mathcal{L}_{3,5/2} I + \mathcal{L}_{c,3,5/2}) \end{aligned}$$

where $\delta_2 := \alpha_{2,2}\gamma_{2,1}/\beta_{2,1}$ is a numerical constant and ψ_2 is an operator representing continuous function which is obtained analogously as the operator ρ_2. Therefore we can conclude that the operator $L_{5/2}$ is from $\mathcal{F}_c$. Continuing this procedure, we obtain the form (24) for all other operators $L_{n+1/2}$, $n = 3, \dots, N-1$.

(b) Omitted. $\square$

Note that in this case the operators $l^j L_{n+1/2}$ and $l^j K_{n+1/2}$, $n = 1, 2, \dots N$, for $j > r_1 - r_2$ are from $\mathcal{F}_c$.

From the last proposition we get the following statements.

Theorem 4. *Assume in equation* (4) *the coefficients* a_n, b_n *and* c_n, $1 \le n \le N$, *are of the form* (6), (7) *and* (8), *respectively, and the operators* f_n, $n = 1, 2, \dots, N-1$, E *and* F *are given by* (9), (10) *and* (11), *respectively. If* $r_1 > r_2$, *then the solutions of the problem* (4), (5) *are of the form*

$$u_n = l^{\omega_n}(\mathcal{U}_{3,n} I + \mathcal{U}_{c,3,n}), \quad n = 1, 2, \dots, N-1,$$

where $\omega_N = -\sigma$, $\omega_n = \min\{(-1)^n(r_1 - r_2) + \omega_{n+1}, \lambda_n\}$, $\mathcal{U}_{3,n}$ *are numerical constants and* $\mathcal{U}_{c,3,n}$ *are operators from* $\mathcal{F}_c$, $n = 1, 2, \dots, N-1$.

Corollary 9. *Assume the conditions of Theorem* 4 *are fulfilled such that* $\sigma < r_2 - r_1 < 0$, $\kappa < r_2$. *Then the solutions of the problem* (4), (5) *represent continuous functions.*

3. An application. Let us consider the partial differential equation

$$\sum_{i=0}^{p} A_i(x)\frac{\partial^{2+i}u(x,t)}{\partial x^2 \partial t^i} + \sum_{i=0}^{q} B_i(x)\frac{\partial^{1+i}u(x,t)}{\partial x \partial t^i} + \sum_{i=0}^{r} C_i(x)\frac{\partial^{i}u(x,t)}{\partial t^i} = f(x,t), \tag{25}$$

for $0 < x < 1$, $t > 0$, where $p, q, r \in \mathbf{N}$, and the coefficients $A_i(x)$, $i = 0, 1, \dots, p$, $B_i(x)$, $i = 0, 1, \dots, q$, $C_i(x)$, $i = 0, 1, \dots, r$, are continuous functions depending on the variable x, while $f(x,t)$ and $u(x,t)$ are the given and the unknown function of two variables. We assume that the solution $u = u(x,t)$ of equation (25) satisfies certain appropriate conditions, namely we take

$$\left.\frac{\partial^{\mu+\nu}u(x,t)}{\partial x^\mu \partial t^\nu}\right|_{t=0} = 0, \tag{26}$$

for $\mu = 0$, $\nu = 0, 1, \dots, r-1$, $\mu = 1$, $\nu = 0, 1, \dots, q-1$, $\mu = 2$, $\nu = 0, 1, \dots, p-1$, and, moreover,

$$u(0,t) = E(t), \qquad u(1,t) = F(t). \tag{27}$$

In (27), $E(t)$ and $F(t)$ are continuous functions, depending only on the variable t.

In the field of Mikusiński operators $\mathcal{F}$, the problem (25), (26), (27) corresponds to the problem

$$\sum_{i=0}^{p} A_i(x)s^i u''(x) + \sum_{i=0}^{q} B_i(x)s^i u'(x) + \sum_{i=0}^{r} C_i(x)s^i u(x) = f(x), u(0) = E, \qquad u(1) = F, \tag{28}$$

where E and F are given operators from $\mathcal{F}_c$. Note that this problem, in view of the special righthand side and boundary conditions, is a special case of the problem (1), (2), considered at the beginning.

The difference analogue for the equation (28) is

$$a_n u_{n-1} + b_n u_n + c_n u_{n+1} = f_n \tag{29}$$

(i.e., equation (4)), with the coefficients a_n, b_n and c_n, given by (6), (7) and (8), respectively.

The equation (29) with the conditions

$$u_0 = E, \qquad u_N = F,$$

leads us to the problem (4), (5). But in this case the operators E, F and f_n are from $\mathcal{F}_c$ and from Theorems 2, 3 and 4, it follows that its solutions are from $\mathcal{F}_c$.

4. The error of approximation. In order to give the error of approximation in the field $\mathcal{F}$, we have to say few words on the comparison of operators. Namely, for two operators $a = \{a(t)\}$ and $b = \{b(t)\}$ from $\mathcal{F}_c$, we define

$$a \le b \quad \text{iff} \quad a(t) \le b(t) \quad \text{for each } t \ge 0$$

(see [**2**, p. 237]). Analogously, we shall say for two operator functions that

$$a(x) \le_T b(x), \quad x \in [c, d],$$

if $a(x)$ and $b(x)$ are representing continuous real valued functions of two variables, $a(x) = \{a(x, t)\}$, $b(x) = \{b(x, t)\}$ and

$$a(x, t) \le b(x, t) \quad \text{for } t \in [0, T],\ x \in [c, d].$$

The absolute value of an operator a from $\mathcal{F}_c$, $a = \{a(t)\}$, denoted by $|a|$, is the operator $|a| = \{|a(t)|\}$. Also, we put $|a(x)| = \{|a(x, t)|\}$.

By estimating the difference between the exact solution of the differential equation (28) and the exact solution of the difference equation (29)) (4), i.e., (4), in the field of Mikusiński operators, we conclude that the solution of the difference equation (29) can be treated as the approximate solution of the partial differential equation (25).

Let us suppose that the solution of equation (28) is from $\mathcal{F}_c$ and has a continuous fourth derivative in the field of Mikusiński operators. Let us denote by $u(x_n)$ the exact solution of equation (28) at the point x_n and by u_n, i.e., the solution of the difference equation (29), the approximate solution at x_n of the same equation, which also belongs to $\mathcal{F}_c$, $n = 1, \dots, N$. From

$$\begin{aligned}\sum_{i=0}^{p} A_i(x)s^i\biggl(u''(x_n) - \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}\biggr)&\\ + \sum_{i=0}^{q} B_i(x)s^i\biggl(u'(x_n) - \frac{u_{n+1} - u_{n-1}}{2h}\biggr)&\\ + \sum_{i=0}^{r} C_i(x)s^i(u(x_n) - u_n) = 0&,\end{aligned}$$

for $n = 1, 2, \dots, N$, we obtain

$$\begin{aligned}|u(x_n) - u_n| = \biggl|&\frac{\sum_{i=0}^{p} A_i(x)s^i}{\sum_{i=0}^{r} C_i(x)s^i}\\ &\cdot\biggl(u''(x_n) - \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}\biggr)\\ &+ \frac{\sum_{i=0}^{r} B_i(x)s^i}{\sum_{i=0}^{r} C_i(x)s^i}\biggl(u'(x_n) - \frac{u_{n+1} - u_{n-1}}{2h}\biggr)\biggr|.\end{aligned} \tag{30}$$

In this paper we give the error of approximation for $r > p \geq q$. Then we have $r_2 = r > r_1 = p$, and therefore we have the estimates

$$\begin{aligned}\biggl|\frac{\sum_{i=0}^{p} A_i(x)s^i}{\sum_{i=0}^{r} C_i(x)s^i}\biggr| &\leq_T R_1(T)l,\\ \biggl|\frac{\sum_{i=0}^{q} B_i(x)s^i}{\sum_{i=0}^{r} C_i(x)s^i}\biggr| &\leq_T R_2(T)l.\end{aligned} \tag{31}$$

From (30) and (31) it follows

$$|u(x_n) - u_n| \leq_T \frac{h^2}{6}(R_1(T) \cdot M_4(T) + R_2(T) \cdot M_3(T))l^2,$$

where

$$M_i(T) = \max_{x\in[0,1],\ t\in[0,T]} \left|\frac{\partial^i u(x,t)}{\partial x^i}\right|, \quad i = 3, 4.$$

Note that the error of approximation is $O(h^2)$, as is in the classical case.

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