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ON THE STRUCTURE OF ROSENTHAL'S SPACE X_φ IN ORLICZ FUNCTION SPACES

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ABSTRACT. Several kinds of complemented subspaces of Orlicz function spaces $L^\varphi[0,1]$ are studied. In particular X_φ, a natural generalization of Rosenthal's spaces X_p, $1 \le p \le \infty$, is analyzed. Several isomorphic and structural properties of these spaces X_φ are given.

0. Introduction. Given an Orlicz function space $L^\varphi[0,1]$, what do the complemented subspaces look like? In the particular case of $L^p[0,1]$ spaces, $1 < p < \infty$, Lindenstrauss and Rosenthal [**10**] have given a characterization of their complemented subspaces in terms of $\mathcal{L}_p$-spaces. But later it was shown that there exist at least uncountable many mutually nonisomorphic $\mathcal{L}_p$-spaces, $1 < p \neq 2 < \infty$ [**3**].

In view of the above, it appears improbable that a complete classification of complemented subspaces of $L^\varphi[0,1]$ spaces will be obtained. For this reason, we limit ourselves to study here of several remarkable kinds of complemented subspaces of reflexive $L^\varphi[0,1]$ spaces. Such spaces will be defined in Section 2, the spaces X_φ and $l^\varphi(w)(l_2)$. The space X_φ was introduced in [**18**] as a generalization of Rosenthal's space X_p. The space X_p, $1 < p < \infty$, was the first example of a complemented subspace of $L^p[0,1]$ nonisomorphic to the trivial subspaces $l_2, l_p, l_2 \oplus l_p$, $L^p[0,1]$ or $(l_2 \oplus l_2 \oplus \cdots)_p$. The space X_p has interesting properties which have been studied in [**17, 9, 1**].

In [**18**], the space X_φ has been studied in relation with the structure of $L^\varphi[0,1]$, proving that every sequence of independent symmetric random variables in $L^\varphi[0,1]$ spans a subspace of $L^\varphi[0,1]$ isomorphically embedded in X_φ. Nevertheless, here, the structure of these kinds of complemented subspaces of $L^\varphi[0,1]$, their isomorphic properties, and

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the relationship between them can be studied using the weight Orlicz sequence spaces $l^\varphi(w)$ (they have already been used in [**5, 7, 16**]).

Section 3 is dedicated to investigating the spaces $l^p(w)$, in particular when the weight sequence w verifies that $w_n \underset{n\to\infty}{\to} 0$ and $\sum_{n=1}^\infty w_n = \infty$. These spaces $l^\varphi(w)$ are all mutually isomorphic, Theorem 3.5; and they have an unconditional basis. But, in general, they do not have any symmetric basis as we prove in Proposition 3.4. Afterward, isomorphic properties of these spaces $l^\varphi(w)$ are also given. It is remarkable that some of these properties are always verified for every Banach space with a symmetric basis.

Finally, in Section 4 we show that the complemented subspaces $l^\varphi(w)(l_2)$ of $L^\varphi[0,1]$, $\sum_{n=1}^\infty w_n < \infty$, are not isomorphic to any subspace of X_φ when $\alpha_\varphi^\infty > 2$, being α_φ^∞ the lower index of φ at ∞.

1. Preliminaries. An Orlicz function φ is a convex nondecreasing continuous function defined for $x \geq 0$ so that $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(x) \underset{x\to\infty}{\to} \infty$. We always assume that $\varphi \in \Delta_2$, by which we mean φ verifies the Δ_2-condition, i.e., there exists a $K > 0$ such that $\varphi(2x) \leq K\varphi(x)$ for each $x > 0$. Let (Ω, Σ, μ) be a positive measure space. The Orlicz space $L^\varphi(\mu)$ is defined as the set of equivalence classes of μ-measurable scalar functions f on Ω such that

$$I_\varphi(f) = \int \varphi(|f(t)|)\, d\mu(t) < \infty.$$

The space $L^\varphi(\mu)$, endowed with Luxemburg norm

$$\|f\|_\varphi = \inf\{r > 0 : I_\varphi(f/r) \leq 1\},$$

is a separable Banach space. For $\Omega = [0,1]$ or $(0,\infty)$ and μ the Lebesgue measure, we denote $L^\varphi(\mu)$ by $L^\varphi[0,1]$ or $L^\varphi(0,\infty)$, respectively.

For $\Omega = \mathbf{N}$ and $w = (w_n = \mu(n))_{n=1}^\infty$ we get the weighted Orlicz sequence spaces $l^\varphi(w)$. The unit vector sequence $(e_n)_{n=1}^\infty$ is an unconditional basis in $l^\varphi(w)$. When $w_n = 1$ for each $n \in \mathbf{N}$, then we denote by l^φ the usual Orlicz sequence space. Moreover, in this case the basis (e_n) is symmetric. Two Orlicz functions φ and ψ are said to be equivalent, $\varphi \sim \psi$ if there exists a constant $K > 0$ such that $K^{-1}\varphi(x) \leq \psi(x) \leq K\varphi(x)$ for every $x \geq 0$; in the same way, it defines the equivalence at 0 or at ∞.

Recall that an Orlicz function φ is p-convex, respectively q-concave, if $\varphi(x)/x^p$ is nondecreasing on $\mathbf{R}^+$, respectively $\varphi(x)/x^q$ is nonincreasing.

For other properties on Orlicz functions, as the associated indices, as well as our terminology, we refer to the books of Lindenstrauss and Tzafriri [**12, 13**], see also [**11, 14, 15, 20**].

2. Complemented subspaces of $L^\varphi[0,1]$**.** Let I be the interval $[0,1]$ or $(0,\infty)$, and $\{A_n\}_{n=1}^\infty$ be a sequence of mutually disjoint measurable subsets of I. The averaging projection P, defined by

$$P(f) = \sum_{n=1}^{\infty} \frac{\int_{A_n} f(t)\,d\mu(t)}{\mu(A_n)} \chi_{A_n} \tag{1}$$

from $L^\varphi(I)$ onto $[\chi_{A_n}]$, shows that $[\chi_{A_n}]$ is complemented in $L^\varphi(I)$ [**13**, Theorem 2.a.4]. Moreover, it is obvious that $[\chi_{A_n}]$ is isometric to the space $l^\varphi(w)$ where $w = (w_n = \mu(A_n))_{n=1}^\infty$.

For our purposes we need to consider two representations of a reflexive and separable Orlicz function space $L^\varphi[0,1]$:

A) $L^\varphi[0,1] \approx L^{\bar\varphi}(0,\infty)$ [**8**, Theorem 8.6], where $\bar\varphi(x) = x^2$ if $x \in [0,1]$ and $\bar\varphi(x) = \varphi(x)$ if $x > 1$.

B) $L^\varphi[0,1] \approx L^\varphi(l_2)$, where $L^\varphi(l_2)$ is the completion of the space of all sequences $(f_1, f_2, \dots)$ of functions of $L^\varphi[0,1]$ which are eventually zero, with respect to the norm

$$\|(f_1, f_2, \dots)\|_{L^\varphi(l_2)} = \left\| \left(\sum_{n=1}^{\infty} |f_n|^2 \right)^{1/2} \right\|_\varphi$$

(cf. [**13**, Theorem 2.d.4]). Of course, $L^\varphi(l_2)$ is isometric to the Bochner space $L^\varphi(([0,1],\Sigma,\mu), l_2)$.

The above representation A) together with (1) yields that the weighted Orlicz sequence spaces $l^{\bar\varphi}(w)$ are isomorphic to complemented subspaces of $L^\varphi[0,1]$, for every weight sequence w. It is not hard to recognize the following four cases:

i) If $\inf_n w_n > 0$, then $l^{\bar\varphi}(w) = l_2$.

ii) If $\sum_{n=1}^\infty w_n < \infty$, then $l^{\bar\varphi}(w) = l^\varphi(w)$.

iii) If $\inf_n w_n = 0$, $\sum_{n=1}^{\infty} w_n = \infty$ and for every subsequence (w_{n_k}) with $w_{n_k} \underset{k\to\infty}{\to} 0$ it holds that $\sum_{k=1}^{\infty} w_{n_k} < \infty$, then a subsequence (w_{n_k}) with $\sum_{k=1}^{\infty} w_{n_k} < \infty$ can be found so that $l^{\bar{\varphi}}(w) = l_2 \oplus l^{\varphi}(w_{n_k})$.

iv) If $w \in \Lambda$, we mean that there exists a subsequence $(w_{n_k})_{k=1}^{\infty}$ such that

$$(*) \qquad w_{n_k} \underset{k\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \sum_{k=1}^{\infty} w_{n_k} = \infty,$$

we obtain the class of spaces $l^{\bar{\varphi}}(w)$ which are mutually isomorphic as we will prove in the next section, Theorem 3.5.

In this paper, if $w \in \Lambda$, we may and will assume, without loss of generality, that the whole sequence verifies $(*)$.

Definition 2.1. Let X_φ be the class of spaces $l^{\bar{\varphi}}(w)$, where $w \in \Lambda$.

This definition can be found in [**18**], where it is proved that $X_\varphi = X_p$ when $\varphi(x) = x^p$, since X_p is Rosenthal's space. Also in [**18, 16**] there can be found a proof of the following theorem.

Theorem 2.2. *Each subspace of a separable Orlicz function space* $L^\varphi[0,1]$ *spanned either by a sequence of mutually disjoint functions or by a sequence of independent symmetric random variables is isomorphic to a subspace of* X_φ.

We now introduce other complemented subspaces, with unconditional basis, of $L^\varphi[0,1]$, which have the property that they are not isomorphic to a subspace of X_φ, as we will see in Section 4.

Using the representation B) and considering the spaces

$$l^\varphi(w)(l_2) = \left\{ ((x_{n,k})_k)_n \in l_2^{\mathbf{N}} : \sum_{n=1}^{\infty} \varphi\left(\left(\sum_{k=1}^{\infty} |x_{n,k}|^2 \right)^{1/2} \right) w_n < \infty \right\}$$

equipped with the norm

$$\|(x_{n,k})\|_{l^\varphi(w)(l_2)} = \|(\|(x_{n,k})_k\|_2)_n\|_\varphi$$

we have the following result:

Proposition 2.3. *Let $L^\varphi[0,1]$ be a reflexive Orlicz function space. For every weight sequence w with finite sum, the space $l^\varphi(w)(l_2)$ is isomorphic to a subspace of $L^\varphi[0,1]$. Furthermore, when $2 < \alpha_\varphi^\infty$ or $\beta_\varphi^\infty < 2$, $l^\varphi(w)(l_2)$ is isomorphic to a complemented subspace of $L^\varphi[0,1]$.*

Proof. Without loss of generality, we may assume that $\sum_{n=1}^\infty w_n \le 1$. Let $(A_n)_n$ be a sequence of mutually disjoint intervals of $[0,1]$ such that $\mu(A_n) = w_n$ for each n. Consider the elements $(r_{n,k})$ of $L^\varphi(l_2)$ defined by

$$r_{n,k} = (0, 0, \dots, \overset{k}{\smile}{\chi_{A_n}}, 0, \dots).$$

Then $[r_{n,k}]$ is isometric to $l^\varphi(w)(l_2)$.

In addition, if $\alpha_\varphi^\infty > 2$, we can assume that $\varphi(x^{1/2})$ is a convex function, then $[r_{n,k}]$ is complemented in $L^\varphi(l_2)$. To see this we define the projection

$$P((f_k)_k) = \sum_{k=1}^\infty \sum_{n=1}^\infty \Big(\int_{A_n} f_k(s)\, ds \Big/ \mu(A_n) \Big) r_{n,k}.$$

Set $g(x) = x^{1/2}$, so we have

$$\begin{aligned} &\int_0^1 \varphi \circ g \bigg(\sum_{k=1}^\infty \bigg| \sum_{n=1}^\infty \Big(\int_{A_n} f_k(s)\, ds \Big/ \mu(A_n) \Big) \chi_{A_n}(t) \bigg|^2 \bigg)\, dt \\ &\qquad = \sum_{n=1}^\infty \int_{A_n} \varphi \circ g \bigg(\sum_{k=1}^\infty \bigg| \int_{A_n} f_k(s)\, ds \Big/ \mu(A_n) \bigg|^2 \bigg)\, dt, \end{aligned}$$

applying the Jensen inequality twice,

$$\begin{aligned} &\le \sum_{n=1}^\infty \int_{A_n} \bigg(\int_{A_n} \varphi \circ g \bigg(\sum_{k=1}^\infty |f_k(s)|^2 \bigg)\, ds \Big/ \mu(A_n) \bigg)\, dt \\ &= \sum_{n=1}^\infty \int_{A_n} \varphi \circ g \bigg(\sum_{k=1}^\infty |f_k(s)|^2 \bigg)\, ds \\ &= \int_0^1 \varphi \bigg(\bigg(\sum_{k=1}^\infty |f_k(s)|^2 \bigg)^{1/2} \bigg)\, ds. \end{aligned}$$

Hence $\|P((f_k))\|_{L^\varphi(l_2)} \leq \|(f_k)\|_{L^\varphi(l_2)}$.

Now the case $\beta_\varphi^\infty < 2$ follows by duality. □

3. Weighted Orlicz sequence spaces. In this section weighted Orlicz sequence spaces $l^\varphi(w)$, whose weight w belongs to the class Λ, are considered. First, some structural results on $l^\varphi(w)$ spaces are given. Afterwards, we will state some isomorphic properties of these spaces.

Drewnowski proved in [**4**] that, given $w \in \Lambda$, then for every weight sequence $v = (v_n)$ the unit vector basis of $l^\varphi(v)$ is equivalent to a block basic sequence with constant coefficients of the unit vector basis of $l^\varphi(w)$. Using this, the next result can be deduced from Theorems 1.1, 1.5 and 1.6 of Nielsen [**15**] (as was done in [**16**]). Consider $\alpha_\varphi^0, \alpha_\varphi^\infty$, β_φ^0 and β_φ^∞, the Matuszewka indices and

$$C_\varphi(0,\infty) = \overline{\mathrm{co}}\left\{\frac{\varphi(sx)}{\varphi(x)} : s > 0\right\}.$$

Proposition 3.1. *Let $l^\varphi(w)$ be a weighted Orlicz sequence space with $w \in \Lambda$ and X be a Banach space with a symmetric basis $(e_n)_n$.*

a) *$X \underset{\sim}{\subset} l^\varphi(w)$ if and only if $X \approx l^\psi$ for some Orlicz function $\psi \in C_\varphi(0,\infty)$ and (e_n) is equivalent to the unit vector basis of l^ψ.*

b) *If $X \approx l^p$, then the statement* a) *is equivalent to either $p \in [\alpha_\varphi^0, \beta_\varphi^0] \cup [\alpha_\varphi^\infty, \beta_\varphi^\infty]$ or $p \in [\alpha_\varphi^\infty, \beta_\varphi^0]$; the last case holds when $\beta_\varphi^\infty \leq \alpha_\varphi^0$.*

Under some restricted conditions we can give further information on the subspaces of $l^\varphi(w)$, which will be useful in Section 4.

Proposition 3.2. *Let φ be an Orlicz function such that $\varphi(x) = x^2$ for every $x \in [0,1]$ and $1 \leq \alpha_\varphi^\infty \leq \beta_\varphi^\infty \leq 2$. Then for every q-convex and 2-concave Orlicz function ψ, with $\beta_\varphi^\infty < q$, there exists another Orlicz function $\phi \in C_\varphi(0,\infty)$ so that $\psi \underset{\sim}{\circ} \phi$. Therefore, $l^\psi \underset{\sim}{\subset} l^\varphi(w)$ for every $w \in \Lambda$.*

Proof. We may assume that

$$\begin{aligned} \varphi(x) &\le Cx^2 \quad \text{for every} \quad x \in [0,1] \\ \varphi(x) &\le Cx^r \quad \text{for every} \quad x \ge 1 \end{aligned}$$

for some constant $C > 0$ and for some $\beta_\varphi^\infty \le r < q < 2$. According to Corollary 2.2 in [**20**] we may also assume that ψ has continuous second derivative. So, we have

$$r < q \le x\psi'(x)/\psi(x) \quad \text{for every} \quad x > 0 \tag{2}$$

and ψ is not equivalent to either of the functions x^r or x^2. Of course, $x^2 \in C_\varphi(0,\infty)$.

Consider $N(x) = \psi(x^{1/2-r})/x^{r/2-r}$ for every $x \in [0,1]$. The increasing function N satisfies

i) $N(x)/x$ is a decreasing function, $N(x)/x \underset{x\to 0}{\to} \infty$, and $N(x) \ge N'(x)x$ for every $x \in [0,1]$,

ii) $N(x) \underset{x\to 0}{\to} 0$,

iii) there exists $s > 0$ such that $xN'(x)/N(x) \ge s$ for every $x \in (0,1]$ (this follows from (2)).

Let

$$K = \int_1^\infty (r-2)t^{r-5}N''(t^{r-2})\varphi(t)\,dt$$

and

$$\phi(x) = \frac{1}{K}\int_1^\infty (r-2)t^{r-5}N''(t^{r-2})\varphi(xt)\,dt$$

for every $x \in [0,1]$. ϕ is an Orlicz function such that $\phi \underset{\sim}{\circ} \psi$ and $\phi \in C_\varphi(0,\infty)$. Namely,

$$\begin{aligned} \phi(x) &\le \frac{Cx^r}{K}\int_{1/x}^\infty (r-2)t^{2r-4-1}N''(t^{r-2})\,dt \\ &\quad + \frac{Cx^2}{K}\int_1^{1/x} (r-2)t^{r-3}N''(t^{r-2})\,dt \\ &= \frac{Cx^r}{K}\int_{(1/x)^{r-2}}^0 uN''(u)\,du \\ &\quad + \frac{Cx^2}{K}\int_1^{(1/x)^{r-2}} N''(u)\,du \end{aligned}$$

integrating, and using i) and ii) we have

$$\begin{aligned}&= \frac{Cx^r}{K} N\bigg(\bigg(\frac{1}{x}\bigg)^{r-2}\bigg) - \frac{Cx^2}{K} N'(1)\\&\le \frac{Cx^r}{K} N\bigg(\bigg(\frac{1}{x}\bigg)^{r-2}\bigg) = \frac{C}{K}\psi(x)\end{aligned}$$

for every $x \in [0,1]$.

On the other hand,

$$\begin{aligned}\phi(x) &\ge \frac{1}{K}\int_1^{1/x} (r-2)t^{r-5}N''(t^{r-2})x^2t^2\,dt\\&= \frac{x^2}{K}\int_1^{(1/x)^{r-2}} N''(u)\,du\\&= \frac{x^2}{K}\bigg(N'\bigg(\bigg(\frac{1}{x}\bigg)^{r-2}\bigg) - N'(1)\bigg)\\&\ge \frac{x^2}{K}\bigg(s\bigg(\frac{1}{x}\bigg)^{2-r} N\bigg(\bigg(\frac{1}{x}\bigg)^{r-2}\bigg) - N'(1)\bigg) \qquad (3)\end{aligned}$$

where the last inequality holds by iii). In view of i) there exists $\delta > 0$ such that $(s/2)(1/x)^{2-r}N((1/x)^{r-2}) - N'(1) > 0$ for every $x \in (0,\delta)$. Hence, we can deduce that

$$(3) \ge \frac{x^2}{K}\frac{s}{2}\bigg(\frac{1}{x}\bigg)^{2-r} N\bigg(\bigg(\frac{1}{x}\bigg)^{r-2}\bigg) = \frac{s}{2K}\psi(x)$$

for every $x \in (0,\delta)$. Therefore, it holds $\phi \underset{\sim}{\circ} \psi$.

To see that $\phi \in C_\varphi(0,\infty)$, define

$$F(t) = (1/K)(r-2)t^{r-5}N''(t^{r-2})\varphi(t) \quad \text{for each } t \ge 1.$$

So $\int_1^\infty F(t)\,dt = \phi(1) = 1$. Take the measure $\lambda(A) = \int_A F(t)\,dt$ on $\sum$ the σ-algebra of measurable sets of $[1,\infty)$. Let $\bar{\lambda}$ be the probability measure defined on $E^\infty_{\varphi,1} = \overline{\{\varphi(sx)/\varphi(x) : s \ge 1\}}$ by $\bar{\lambda}(\cap_{\lambda>0}E^\infty_{\varphi,\lambda}) = 0$, and let

$$\bar{\lambda}\bigg(\bigg\{\frac{\varphi(sx)}{\varphi(s)} \in E^\infty_{\varphi,1} : s \in A\bigg\}\bigg) = \lambda(A)$$

for every measurable set $A \subset [1,\infty)$. Since

$$\phi(x) = \int_{E^\infty_{\varphi,1}} \frac{\varphi(sx)}{\varphi(s)}\, d\bar{\lambda}(s),$$

using the Krein-Milman theorem, we conclude that

$$\phi \in C^\infty_{\varphi,1} = \overline{\mathrm{co}}\,(E^\infty_{\varphi,1}) \subset C_\varphi(0,\infty). \qquad \square$$

Corollary 3.3. *Let φ be an Orlicz function with $1 \le \alpha^\infty_\varphi \le \beta^\infty_\varphi < 2$. For every Orlicz function ψ with $\beta^\infty_\varphi < \alpha^0_\psi$ and two-concave at* 0, *it holds that $l^\psi \underset{\sim}{\subset} X_\varphi$. In particular, $l_p \underset{\sim}{\subset} X_\varphi$ for every $p \in [\alpha^\infty_\varphi, 2]$.*

Remark. The theorem above also holds for $p > 1$ instead of 2.

Now, our aim is to prove some isomorphic properties of the $l^\varphi(w)$ spaces where $w \in \Lambda$, which hold for Banach spaces with a symmetric basis. Nevertheless, $l^\varphi(w)$ spaces do not have in general such bases, as we prove in the following.

Proposition 3.4. *Let φ be an Orlicz function such that* $\min\{\alpha^0_\varphi, \alpha^\infty_\varphi\} > 1$ *and $[\alpha^0_\varphi, \beta^0_\varphi] \cap [\alpha^\infty_\varphi, \beta^\infty_\varphi] = \varnothing$, and w be a weight sequence belonging to Λ. Then $l^\varphi(w)$ does not have a symmetric basis.*

Proof. By Proposition 3.9, in [**19**], it is enough to prove that $l^\varphi(w)$ is not isomorphic to any Orlicz sequence space l^ψ.

In the case $\beta^0_\varphi < \alpha^\infty_\varphi$, by Proposition 3.1, we know that $l^p \underset{\sim}{\subset} l^\varphi(w)$ if and only if $p \in [\alpha^0_\varphi, \beta^0_\varphi] \cup [\alpha^\infty_\varphi, \beta^\infty_\varphi]$. On the other hand, by Theorem 1 in [**11**] if $l^\varphi(w) \approx l^\psi$, then $p \in [\alpha^0_\psi, \beta^0_\psi]$, which is a contradiction.

The case $\beta^\infty_\varphi < \alpha^0_\varphi$ follows by duality. $\square$

In spite of the fact that there are spaces $l^\varphi(w)$ without symmetric basis, it holds that every weighted Orlicz sequence space $l^\varphi(w)$ has the following property: *every block basic sequence with constant coefficients of the unit vector basis (e_n) spans a complemented subspace of $l^\varphi(w)$.*

Indeed, let $u_i = \sum_{n\in\sigma_i} e_n$, $i \in \mathbf{N}$ be a block basic sequence with constant coefficients of (e_n). If $s_i = \sum_{n\in\sigma_i} w_n$ for each $i \in \mathbf{N}$, then the averaging projection

$$P\bigg(\sum_{n=1}^{\infty} x_n e_n\bigg) = \sum_{i=1}^{\infty}\bigg(\sum_{n\in\sigma_i}\frac{x_n w_n}{s_i}\bigg)u_i$$

is a bounded linear projection from $l^{\varphi}(w)$ onto $[u_i]$.

So, given $w, w' \in \Lambda$, it follows that $l^{\varphi}(w)_c\underset{\sim}{\subset} l^{\varphi}(w')$ and $l^{\varphi}(w')_c\underset{\sim}{\subset} l^{\varphi}(w)$. The next result answers a natural question:

Theorem 3.5. *Let w and w' be two weight sequences belonging to the class Λ, and let φ be an Orlicz function. Then $l^{\varphi}(w) \approx l^{\varphi}(w')$.*

In order to prove this theorem and another below, we need some previous results on representations of the spaces $l^{\varphi}(w)$.

Fix a weighted Orlicz sequence space $l^{\varphi}(w)$ and consider the vector space

$$l^{\varphi}(w)^{\infty} = \{x = ((x_{n,k})_k)_n \in l^{\varphi}(w)^{\mathbf{N}}\}$$

and the functional

$$\rho : l^{\varphi}(w)^{\infty} \longrightarrow [0,\infty]$$
$$\rho((x_{n,k})) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\varphi(|x_{n,k}|)w_k.$$

It is clear that ρ is a convex functional such that $\rho(0) = 0$ and $\rho(-x) = \rho(x)$ for each $x \in l^{\varphi}(w)^{\infty}$. Thus, ρ is a modular functional in the sense of [**14**] and, from Theorem 1.5 in [**14**] the vector subspace

$$l^{\varphi}(w)^{\infty}_{\rho} = \{x \in l^{\varphi}(w)^{\infty} : \lim_{\lambda\to 0}\rho(\lambda x) = 0\}$$

can be endowed with the norm

$$\|x\|_{\rho} = \inf\{r > 0 : \rho(x/r) \le 1\}.$$

Of course, the space $l^{\varphi}(w)^{\infty}_{\rho}$ is isometric to the weighted Orlicz sequence space $l^{\varphi}(v)$ with $v = \{w_1, w_1, w_2, w_1, w_2, w_3, w_1, \dots\}$.

Now it is straightforward that $l^\varphi(w)^\infty_\rho$ is a symmetric sum of $l^\varphi(w)$ in the sense given by Rosenthal in [**17**, p. 294]. Hence, in view of Proposition 11 in [**17**], we have

$$l^\varphi(w)^\infty_\rho \approx l^\varphi(w)^\infty_\rho \oplus l^\varphi(w)^\infty_\rho \approx l^\varphi(w)^\infty_\rho \oplus l^\varphi(w).$$

Proposition 3.6. *Let w be a weight sequence belonging to Λ. Then $l^\varphi(w) \approx l^\varphi(w)^\infty_\rho$. In particular, $l^\varphi(w)$ is isomorphic to its own square.*

Proof. By Proposition 11 in [**17**] it is enough to observe that $l^\varphi(w)^\infty_\rho \, {}_c\underset{\sim}{\subset} l^\varphi(w)$. □

Proof of Theorem 3.5. We have already shown that the spaces $l^\varphi(w)$ and $l^\varphi(w')$ are each isomorphic to their own square. Hence, using the well-known Pełczynski decomposition method, we deduce that $l^\varphi(w) \approx l^\varphi(w')$. □

Theorem 3.7. *Let $w \in \Lambda$ and $\varphi(x)$ be an Orlicz function nonequivalent to x at 0. If $l^\varphi(w) = H \oplus Y$, then either H or Y contains a complemented subspace isomorphic to $l^\varphi(w)$.*

Proof. It will be enough to prove the theorem for $l^\varphi(w)^\infty_\rho$. We consider the sequence $(e_{n,k})_{n,k}$ where $e_{n,k} = (0, 0, \overset{n}{\overbrace{e_k}}, 0, \dots)$. Setting $d_{n,k} = \varphi^{-1}(1/w_k)e_{n,k}$, we get an unconditional normalized basis of $l^\varphi(w)^\infty_\rho$. We point out that the sequence $(e_{n,k})_k$ is the unit vector basis of $l^\varphi(w)$ for each $n \in \mathbf{N}$.

Let P_Y be the projection onto Y with kernel H (P_H is defined similarly).

Consider $(d^*_{n,k})_{n,k}$ the biorthogonal functionals of $(d_{n,k})_{n,k}$. For each $k \in \mathbf{N}$, set

$$A_k = \{n : d^*_{n,k}(P_Y(d_{n,k})) \ge 1/2\}$$

and

$$B_k = \{n : d^*_{n,k}(P_H(d_{n,k})) \ge 1/2\}.$$

Without loss of generality we may assume that there exists $I \subset \mathbf{N}$ such that $\sum_{k\in I} w_k = \infty$ and A_k is infinite for every $k \in \mathbf{N}$. In other cases the statement above can be held for the sets B_k. Thus, if $k \in I$ and $n \in A_k$, then

$$\|P_Y\| \geq \|P_Y(d_{n,k})\|_\rho \geq |d^*_{n,k}(P_Y(d_{n,k}))| \geq 1/2. \tag{4}$$

Moreover, for every $k \in I$ and $d^*_{i,j}$, $i, j \in \mathbf{N}$, we have

$$d^*_{i,j}(P_Y(d_{n,k})) \underset{n\to\infty}{\overset{n\in A_k}{\longrightarrow}} 0. \tag{5}$$

In fact, if there exists $k_0 \in I$, $d^*_{i_0,j_0}$, $\varepsilon > 0$ and a sequence $(n_m)_m$ of A_{k_0} such that $d^*_{i_0,j_0}(P_Y(d_{n_m,k_0})) \geq \varepsilon$ for each $m \in \mathbf{N}$, then we would get

$$\begin{aligned}\left(\sum_{m=1}^{\infty} |x_m|\right)\|P_Y\| &\geq \left\|\sum_{m=1}^{\infty} x_m d_{n_m,k_0}\right\|_\rho \|P_Y\| \\ &\geq \left|\sum_{m=1}^{\infty} |x_m| d^*_{i_0,j_0}(P_Y(d_{n_m,k_0}))\right| \\ &\geq \varepsilon \sum_{m=1}^{\infty} |x_m|;\end{aligned}$$

therefore, the basic sequence $(d_{n_m,k_0})_{m=1}^\infty$ of $l^\varphi(w)^\infty_\rho$ would be equivalent to the unit vector basis of l_1. So $\sum_{m=1}^\infty \varphi(|x_m|\varphi^{-1}(1/w_{k_0}))w_{k_0}$ converges if and only if $\sum_{m=1}^\infty |x_m| < \infty$, hence $\varphi(x) \underset{\sim}{\circ} x$, which is a contradiction.

Now denote by Q_N the following linear operator

$$Q_N\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} x_{n,k} d_{n,k}\right) = \sum_{n=1}^{N}\sum_{k=1}^{N} x_{n,k} d_{n,k}.$$

Since (4) and (5) hold, we may find a bijection $\sigma : \mathbf{N} \to I$, indices $n_m \in A_{\sigma(m)}$, and an increasing sequence of positive integers $N_1 < N_2 < \cdots < N_m < \cdots$ such that

$$\|P_Y(d_{n_m,\sigma(m)}) - Q_{N_m}(P_Y(d_{n_m,\sigma(m)}))\|_\rho \leq 1/2^{m+1}2$$

and

$$\|Q_{N_{m-1}}(P_Y(d_{n_m,\sigma(m)}))\|_\rho \leq 1/2^{m+1}2$$

for every $m \geq 2$.

Let $(u_m)_m$ be the sequence of $l^\varphi(w)_\rho^\infty$ defined by

$$\begin{aligned} u_1 &= Q_{N_1}(P_Y(d_{n_1,\sigma(1)})) \\ u_m &= Q_{N_m}(P_Y(d_{n_m,\sigma(m)})) - Q_{N_{m-1}}(P_Y(d_{n_m,\sigma(m)})) \end{aligned}$$

for every $m \geq 2$. $(u_m)_m$ is an unconditional block basic sequence of $(d_{n,k})$, whose unconditional constant is equal to 1. Now, since

$$\sum_{m=1}^{\infty} \|P_Y(d_{n_m,\sigma(m)}) - u_m\|_\rho < 1/2$$

by Proposition 1.a.9 in [**12**] we obtain that the sequence $(P_Y(d_{n_m,\sigma(m)}))_{m=1}^\infty$ is equivalent to the block basic sequence $(u_m)_m$.

Note that if K is the basis constant of $(P_Y(d_{n_m,\sigma(m)}))_{m=1}^\infty$, then there exists $N \in \mathbf{N}$ such that

$$\|P_Y(d_{n_{m+N},\sigma(m+N)}) - u_{m+N}\|_\rho < \frac{1}{2^m 8\|P_Y\|4K}. \tag{6}$$

Now the following two statements about $(u_m)_{m=1}^\infty$ hold

i) $(u_m)_m$ is equivalent to $(d_{n_m,\sigma(m)})_m$.

ii) There exists a projection T from $l^\varphi(w)_\rho^\infty$ onto $[u_m]$ such that $\|T\| \leq 4\|P_Y\|$.

According to i), $\sum_{m=1}^\infty x_m u_m$ is convergent in $l^\varphi(w)_\rho^\infty$ if and only if

$$\sum_{m=1}^{\infty} \varphi(|x_m|\varphi^{-1}(1/w_{\sigma(m)}))w_{\sigma(m)} = \sum_{k\in I} \varphi(|x_{\sigma^{-1}(k)}|\varphi^{-1}(1/w_k))w_k < \infty.$$

Consider $w' = (w_k)_{k\in I}$ belonging to the class Λ. Then, by Theorem 2.6, we have

$$l^\varphi(w)_\rho^\infty \approx l^\varphi(w) \approx l^\varphi(w') \approx [d_{n_m,\sigma(m)}] \approx [u_m] \approx [P_Y(d_{n_m,\sigma(m)})].$$

Now, from ii), Proposition 1.a.9 in [**12**] and (6) we obtain

$$l^\varphi(w)^\infty_\rho \approx [P_Y(d_{k_m,\sigma(m)})]_c \underset{\sim}{\subset} Y$$

as we wanted to show. □

This property is verified for every Orlicz sequence space l^φ, because l^φ has a symmetric basis. But, it is actually unknown whether the spaces l^φ and $l^\varphi(w)$ are primary, for $w \in \Lambda$ and φ arbitrary.

4. Relationship between complemented subspaces of $L^\varphi[0,1]$. Our goal now is to prove that the spaces X_φ and $l^\varphi(w)(l_2)$ are rather different, as subspaces of $L^\varphi[0,1]$.

Proposition 4.1. *Let $L^\varphi[0,1]$ be a reflexive Orlicz function space with either $\alpha^\infty_\varphi > 2$ or $\beta^\infty_\varphi < 2$. Then $l^\varphi(w)(l_2)$ is not isomorphic to $L^\varphi[0,1]$ for every weight sequence w with finite sum.*

Proof. Consider $1 < \alpha^\infty_\varphi \leq \beta^\infty_\varphi < 2$. It holds that $l_p \underset{\sim}{\subset} l^\varphi(w)(l_2)$ if and only if $p = 2$ or $l_p \underset{\sim}{\subset} l^\varphi(w)$ (see Proposition 3 in [**6**]). Therefore, either $p = 2$ or $p \in [\alpha^\infty_\varphi, \beta^\infty_\varphi]$. Since $X_\varphi {}_c\underset{\sim}{\subset} L^\varphi[0,1]$, from Proposition 3.2, we have that $l_p \underset{\sim}{\subset} L^\varphi[0,1]$, for each $p \in (\beta^\infty_\varphi, 2)$. Therefore $L^\varphi[0,1]$ is not isomorphic to $l^\varphi(w)(l_2)$.

The remaining case holds now by duality. □

Remark. If $\beta^\infty_\varphi < 2$, then X_φ is not isomorphic to any subspace of $l^\varphi(w)(l_2)$, for every weight sequence w with finite sum.

Next we will prove that, if $\alpha^\infty_\varphi > 2$, then X_φ is not isomorphic to any $l^\varphi(w)(l_2)$. For that, we need a lemma which is a straightforward extension of Corollary 2 in [**11**].

Lemma 4.2. *Let $l^\varphi(w)$ be a weighted Orlicz sequence space such that $\min\{\alpha^0_\varphi, \alpha^\infty_\varphi\} = s > 1$. Let ψ be an Orlicz function with $\beta^0_\psi < s$. Then every bonded linear operator from $l^\varphi(w)$ into l^ψ is compact.*

Proposition 4.3. *Let φ be an Orlicz function with $\alpha_\varphi^\infty > 2$. Then $l^\varphi(w)(l_2)$ is not isomorphic to any subspace of X_φ, for every weight sequence w with finite sum.*

Proof. We may suppose that φ is a 2-convex function. Let v be a weight sequence with $v \in \Lambda$. From Proposition 4 in [**18**], it will be sufficient to prove that $l^\varphi(w)(l_2)$ is not isomorphic to any subspace of $l^\varphi(v) \oplus l_2$.

Now the proof follows as in [**17**, p. 298], considering Lemma 4.2 and the fact that no subspace of $l^\varphi(v)$ or of $l^\varphi(w)$ is isomorphic to l_2 (see Proposition 3.1 and [**11**] Proposition 4). □

Under the same hypothesis of the proposition above, the next corollary points out the different nature of the complemented subspaces $l^\varphi(w)(l_2)$ of $L^\varphi[0,1]$.

Corollary 4.4. *The complemented subspaces $l^\varphi(w)(l_2)$ of $L^\varphi[0,1]$, where w has a finite sum, cannot be spanned either by a sequence of mutually disjoint functions of $L^\varphi[0,1]$ or by a sequence of independent symmetric random variables of $L^\varphi[0,1]$.*

Proof. This follows from Theorem 2.2 and Proposition 4.3. □

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