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AN EXAMPLE OF DUAL CONTROL

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The C^*-algebras introduced by the author [**8**] to study index theorems on open manifolds are now known to be analytic counterparts of certain module categories appearing in controlled topology [**2**]. An apparently significant distinction is that the modules used in controlled topology are locally finite-dimensional, whereas the Hilbert spaces used to construct the C^*-algebras are locally infinite-dimensional. In the author's opinion, this distinction arises because the analysis of elliptic operators (which are the basic 'cycles' in analytic representations of K-homology) itself constitutes a form of 'control,' but in a 'spectral' rather than a 'spatial' direction. The purpose of this note is to reinforce this point of view by an example.

Let M be a compact odd-dimensional manifold, and let D be a generalized Dirac operator on M acting on a Hilbert space H of L^2 sections of the appropriate bundle. Then it is well known [**1, 5**] that D gives a cycle in Kasparov's analytic K-homology for M, and therefore gives a map

$$K^1(M) \longrightarrow \mathbf{Z}.$$

We will show how this map may be obtained using controlled C^*-algebra theory.

Recall [**8, 6**] that the basic object needed to define the C^*-algebra of a coarse space X is an X-module, that is, a Hilbert space equipped with an action of $C_0(X)$. Now we observe

Lemma 1. *The operator D endows H with the structure of an $|\mathbf{R}|$-module.*

(The notation $|\mathbf{R}|$ refers to the underlying coarse space of $\mathbf{R}$.) To see this we just use the spectral theorem, defining the action of $f \in C_0(\mathbf{R})$ on H by the operator $f(D)$. Observe that elliptic theory

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for D (specifically, the fact that D has compact resolvent) implies that this $|\mathbf{R}|$-module is *locally finite-dimensional* in the sense that the projection operator corresponding to any compact subset of $|\mathbf{R}|$ has finite-dimensional range.

For a continuous function φ on M, let A_φ denote the corresponding multiplication operator on H.

Definition 2. An operator A on H is a *smooth operator* if the iterated commutators

$$\mathrm{Ad}\,(D)^n(A) = [D, \ldots , [D, A] \ldots]$$

are all bounded on H.

Multiplication by a smooth function φ is a smooth operator. In fact, any pseudodifferential operator of order zero, with scalar principal symbol, is smooth, although we will not need this fact.

Let $\mathcal{S}(\mathbf{R})$ denote the Schwarz class of rapidly-decaying functions on $\mathbf{R}$. For $f \in \mathcal{S}(\mathbf{R})$, let

$$K(f) = \int |t\hat{f}(t)|\, dt.$$

One proves easily that the 'norm' $K(f)$ is translation invariant and is homogeneous under dilations: $K(x \mapsto f(\lambda x)) = \lambda K(f)$.

Lemma 3. *Let A be a smooth operator on $L^2(M)$, and let f be a function in $\mathcal{S}(\mathbf{R})$. Then the commutator $B = [f(D), A]$ is smooth also. Moreover, there is a constant C such that*

$$\|\mathrm{Ad}\,(D)^n(B)\| \le C\|\mathrm{Ad}\,(D)^{n+1}(A)\|K(f).$$

In particular, this estimate holds when $A = A_\varphi$ for a smooth function φ.

Compare [**4**, Section 3] for a similar estimate.

Proof. Use Fourier analysis to write

$$f(D) = \frac{1}{2\pi} \int e^{itD} \hat{f}(t)\, dt$$

so that

$$[f(D), A] = \frac{1}{2\pi} \int X_t \hat{f}(t)\, dt$$

where $X_t = [e^{itD}, A]$. It is easy to see that

$$\frac{d}{dt} X_t = iDX_t + i[D, A]e^{itD} = iDX_t + Y_t,$$

where the term Y_t is uniformly bounded in norm because $[D, A]$ is bounded. Using Duhamel's principle to solve this inhomogeneous hyperbolic equation, we find that $\|X_t\| \leq \|[D, A]\||t|$. Hence $\|[f(D), A]\| \leq K(f)/2\pi$ as required. This gives the estimate for $\|B\|$. To get the estimates for the commutators we need only remark that

$$\mathrm{Ad}\,(D)^n B = [f(D), \mathrm{Ad}\,(D)^n (A)]$$

since D commutes with $f(D)$. $\square$

We now need an elementary observation.

Lemma 4. *Let $H = \oplus_{n \in \mathbf{Z}} H_n$ be a Hilbert sum of Hilbert spaces, and suppose that a matrix A_{mn} of operators $H_m \to H_n$ is given. Suppose that there is a function $\rho \in l^1(\mathbf{Z})$ such that*

$$\|A_{mn}\| \leq \rho(m - n)$$

for all m and n. Then the matrix (A_{mn}) represents a bounded operator A on H, with

$$\|A\| \leq \|\rho\|_{l^1}.$$

For the reader's convenience we repeat the proof, see [**9**, Lemma 13.0.4].

Proof. We use the Cauchy-Schwarz inequality. Let $u, v \in H$ be represented by series $\sum u_m$ and $\sum v_n$ with respect to the Hilbert sum decomposition. Then

$$\begin{aligned}|\langle Au, v\rangle| &\le \sum_{m,n} \|A_{mn}\|^{1/2}\|u_m\|\|A_{mn}\|^{1/2}\|v_n\| \\ &\le \Big(\sum_{m,n}\|A_{mn}\|\|u_m\|^2\Big)^{1/2}\Big(\sum_{m,n}\|A_{mn}\|\|v_n\|^2\Big)^{1/2} \\ &\le \|\rho\|_{l^1}\|u\|\|v\|,\end{aligned}$$

giving the result. □

Proposition 5. *Any smooth operator A belongs to the algebra $C^*(|\mathbf{R}|; H)$, that is, it is locally compact and a norm limit of finite propagation operators.*

Proof. All operators on H are locally compact, because H is locally finite-dimensional. Let P_n be the spectral projection onto the subspace H_n of H corresponding to the interval $[n, n+1)$ in the spectrum of D, and let $A_{mn} = P_m A P_n$. In view of the preceding lemma, it will suffice to prove that $\|A_{mn}\| = O(|m-n|^{-2})$, since then truncating the matrix A_{mn} at $|m-n| \le R$ for some large constant R will produce a finite propagation operator which can be made as close as we please to A. We will prove, in fact, that $\|A_{mn}\| = O(|m-n|^{-\infty})$.

To do this, pick a Schwarz function $g(t)$ with $g(t) = 0$ for $t \in [-1, 0]$ and $g(t) = 1$ for $t \in [1, 2]$. For each $|m-n| \ge 2$, define a Schwarz function f_{mn} on $\mathbf{R}$ by $f_{mn}(t) = g((t-n-1)/(m-n-1))$ if $m > n$ and $f_{mn}(t) = 1 - f_{nm}(t)$ if $m < n$, so that $f_{mn}(t) = 0$ for $t \in [n, n+1]$, $f_{mn}(t) = 1$ for $t \in [m, m+1]$ and $K(f_{mn}) = O(|m-n|^{-1})$. We have

$$\begin{aligned}A_{mn} = P_m A P_n &= P_m f_{mn}(D) A P_n - P_m A f_{mn}(D) P_n \\ &= P_m \mathrm{Ad}\,(f_{mn}(D))(A) P_n.\end{aligned}$$

Clearly this computation can be repeated k times to give

$$A_{mn} = P_m \mathrm{Ad}\,(f_{mn}(D))^k(A) P_n$$

for any positive integer k. But the preceding results then show that

$$\|\mathrm{Ad}\,(f_{mn}(D))^k(A)\| = O(K(f_{mn})^k) = O(|m-n|^{-k})$$

as required. □

Representing a continuous function as a uniform limit of smooth functions, we get

Corollary 6. *For any continuous function* φ *on* M*, the multiplication operator* A_φ *belongs to* $C^*(|\mathbf{R}|; H)$*.*

Thus we have shown that the representation of $C(M)$ on H gives rise to a C^*-algebra homomorphism $C(M) \to C^*(|\mathbf{R}|)$. The K-theory of the algebra $C^*(|\mathbf{R}|)$ has been calculated [**3**] and it is 0 in dimension 0, $\mathbf{Z}$ in dimension 1. Thus we have obtained a homomorphism

$$i : K^1(M) = K_1(C(M)) \longrightarrow K_1(C^*(|\mathbf{R}|)) = \mathbf{Z}.$$

We claim that this is simply the index homomorphism obtained from Kasparov's theory.

Remark. Strictly speaking, the computation of the K-theory of $C^*(|\mathbf{R}|)$ is valid only when the module H is sufficiently large, see [**3**] for the precise condition required. The module defined above (for a compact manifold) is certainly *not* sufficiently large in this sense. However, any module can be embedded in a sufficiently large one, so we still obtain the homomorphism i as claimed.

Theorem 7. *The homomorphism* i *is the Kasparov index map.*

Proof. Using the ideas of the index theorem for partitioned manifolds [**7**] we know that the isomorphism $K_1(C^*(|\mathbf{R}|)) \to \mathbf{Z}$ can be described as follows. Let P be the projection on H which corresponds (in the $|\mathbf{R}|$-module structure) to the positive real axis $\mathbf{R}^+$. For a unitary $u \in C^*(|\mathbf{R}|)$ representing an element of K_1 form the 'Toeplitz operator' $T_u = Pu$ on the range of P. One can show this operator is Fredholm, and the integer we want is its index. However, in our situation, P is just the positive spectral projection of D, and the operator T_u really is a Toeplitz operator. The result follows since it is known that the Kasparov pairing is given by the formation of a Toeplitz index. □

Final remarks. Everything could be made to work for noncompact manifolds. However, we have chosen to emphasize the compact case in order to make the following point. In bounded topology, all compact objects are trivial. Nevertheless, the index is a nontrivial phenomenon on compact manifolds. The reason that the index can nonetheless be described in a controlled fashion is that a compact manifold still has a noncompact 'phase' direction. Estimates on differential operators provide control in the phase direction.

The proof that a smooth operator has finite propagation in the dual, given above, does not depend on ellipticity (though local compactness does). In fact, if D is any essentially self-adjoint first order differential operator and A is 'smooth' with respect to D, then A is in the C^*-algebra generated by the finite propagation operators on the spectrum, by the same arguments as above. On the other hand, if an operator is 'smooth' relative to arbitrary families of operators D, it is in fact pseudodifferential; see [**9**, Lemma 8.5.2]. Following up this line of thought leads to a 'controlled' characterization of the pseudodifferential operators: an operator on M is in the C^*-algebra generated by pseudodifferential operators (of order zero) if and only if it is in the C^*-algebra of finite propagation operators for every 'dual control' structure defined by a family of commuting essentially self-adjoint first order differential operators on M.

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