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SOME GENERALIZATIONS OF UNIVERSAL MAPPINGS

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ABSTRACT. A mapping $f : X \to Y$ is *universal* if, for each mapping $g : X \to Y$, there is a point x in X such that $f(x) = g(x)$. We define several classes of mappings which properly contain the universal mappings and we establish relationships between these mappings, the fixed point property, the span of continua and the Borsuk-Ulam theorem.

1. Introduction, definitions and observations. In 1967, W. Holsztyński [**8**] defined universal mappings between topological spaces. A mapping $f : X \to Y$ is *universal* if, for each mapping $g : X \to Y$, there is a point x in X such that $f(x) = g(x)$. Holsztyński used this property of mappings to obtain several fixed point theorems. Others have also used this property to obtain fixed point results, e.g., see [**19, 18, 4, 15** and **16**].

In this paper we offer several generalizations of the universal property. We show that each of these properties has some utility in obtaining fixed point theorems. Furthermore, several of these properties have nice relationships to the notion of span of continua which was introduced by A. Lelek [**13**] in 1964. In Sections 2 through 4 we study these classes of mappings. We consider spaces which admit only certain of these mappings onto themselves, relationships to span and relationships to the fixed point property.

By a *continuum* we will mean a compact connected metric space. A continuous function will be referred to as a *map* or *mapping*. The following definitions are made for mappings between topological spaces although our interest will be primarily in mappings between continua. The definition of universal is given above.

A mapping $f : X \to Y$ is *weakly universal* if for each mapping $g : X \to X$ there is a point x in X such that $f(x) = fg(x)$. A

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mapping $f : X \to Y$ is *weakly universal with respect to a class* $\mathcal{F}$ of self mappings on X if, for each $g \in \mathcal{F}$, there is a point x in X such that $f(x) = fg(x)$. From an historical point of view, the notion of weakly universal mappings is not an entirely new idea. H. Hopf, see [**6, 7**], defined free mappings in 1937; a mapping $f : X \to Y$ is *free* if and only if it is not weakly universal. Most of his results involving free mappings were concerned with mappings on spheres or manifolds. In [**7**], he obtained a generalization of the Borsuk-Ulam theorem. The following restatement of the Borsuk-Ulam theorem shows its connection to weakly universal maps.

Theorem (Borsuk-Ulam). *Each mapping f from the n-sphere S^n into Euclidean k-space E^k, where $k \leq n$, is weakly universal with respect to the antipodal mapping on S^n.*

There have been many generalizations of the Borsuk-Ulam theorem. Some can be found in [**1, 2, 12, 22** and **25**].

A mapping $f : X \to Y$ is *semi-universal* if whenever K is a subcontinuum of X such that $f(K) = f(X)$ and $g : K \to X$ is a mapping, then there is a point x in K such that $f(x) = fg(x)$. A mapping $f : X \to Y$ is *pseudo-universal* if f is weakly universal with respect to each mapping $g : X \to X$ that has a periodic point of period two.

The following diagram is easy to establish.

$$\begin{array}{ccccc} & & \mathbf{u} & & \\ & & \Downarrow & & \\ \mathbf{s-u} & \Longrightarrow & \mathbf{w.u.} & \Longrightarrow & \mathbf{p-u} \end{array}$$

We list below a few observations, each of which is easy to verify.

(1) [**8**]. If $f : X \to Y$ is universal, then f is surjective and Y has the fixed point property.

(2) If X has the fixed point property, then each mapping from X is weakly universal.

(3) If $\mathrm{id} : X \to X$ is weakly universal, then X has the fixed point property.

(4) If $f : X \to Y$ is a weakly universal homeomorphism, then f is universal.

We will say that a continuum Y is in class (U), respectively class (SU), class (WU) and class (PU), provided that whenever X is a continuum and f is a mapping from X onto Y, f must be universal, respectively, semi-universal, weakly universal and pseudo-universal. In [**21**], Nadler has defined a continuum Y to be in class $(\hat{U})$ if for every mapping f of a continuum X onto Y, $\hat{f} : C(X) \to C(Y)$ is universal.

For a continuum X, let d denote a metric on X, and let π_1 and π_2 denote the first and second projections of $X \times X$ onto X. For Z a subcontinuum of $X \times X$, let $Z^{-1} = \{(x, y) \mid (y, x) \in Z\}$. The span of X is *zero*, respectively the *semispan* of X is *zero*, the *symmetric span of* X *is zero*, denoted by $\sigma(X) = 0$, respectively, $\sigma_0(X) = 0$, $s(X) = 0$, if whenever Z is a continuum in $X \times X$ such that $\pi_1(Z) = \pi_2(Z)$, respectively $\pi_2(Z) \subseteq \pi_1(Z)$, $Z = Z^{-1}$, then Z intersects the diagonal in $X \times X$. The three types of span zero become *surjective span zero* if we require that $\pi_1(Z) = X$ in the definitions above. The notations are, respectively, $\sigma^*(X) = 0$, $\sigma_0^*(X) = 0$ and $s^*(X) = 0$. For more information regarding the span of continua, see [**13**] and [**3**].

There are many results in the literature concerning universal mappings. Three that are of particular interest in this paper are listed below.

Let B^n denote the closed unit ball in Euclidean n-space E^n. A mapping $f : X \to B^n$ from a continuum onto B^n is *AH-essential* if $f|_{f^{-1}(S^{n-1})} : f^{-1}(S^{n-1}) \to S^{n-1}$ cannot be extended to a mapping $F : X \to S^{n-1}$.

Theorem 1 [**14**, Lemma and **10**, Proposition 1.1]. *A mapping* $f : X \to B^n$ *is universal if and only if it is AH-essential.*

Theorem 2 [**8**, Theorem 3]. *Each mapping of a continuum onto an arc-like continuum is universal.*

Theorem 3 [**8**, Corollary 1.]. *If* X *is an inverse limit of* ANR*s and all bonding mappings are universal, then* X *has the fixed point property.*

With regard to the fixed point property, universal mappings have been useful in a number of ways. In light of Theorem 3, one might ask what properties of mappings between specific ANRs would imply the mappings are universal. Indeed, the fixed point theorems for inverse limits in [**4, 15, 16, 18** and **19**] are of this nature.

In [**20**], S. Nadler uses results about the universality of induced mappings in hyperspaces to show that certain hyperspaces have the fixed point property.

2. Semi-universal mappings.

Theorem 4. *If $f_1 : X \to Y$ is semi-universal and $f_2 : Y \to Z$ is a mapping, then $f_2 \circ f_1$ is semi-universal.*

Theorem 5. *Let X be a continuum. The following statements are equivalent.*

(1) $X \in \text{class}\,(U)$.

(2) $X \in \text{class}\,(SU)$.

(3) $\sigma_0^*(X) = 0$.

Proof. That (1) and (3) are equivalent is well-known. So, we show the equivalence of (1) and (2).

Suppose $Y \in \text{class}\,(U)$. Let $f : X \to Y$ be a surjective mapping, and let K be a subcontinuum of X such that $f(K) = Y$. Let $g : K \to X$ be any mapping. Now $f|_K : K \to Y$ is a mapping onto Y and by assumption is universal. Thus, $f|_K$ and f_g have a coincidence point, i.e., there is a point $x \in K$ such that $f(x) = fg(x)$. Hence, f is semi-universal.

Suppose $Y \in \text{class}\,(SU)$. Let $f : X \to Y$ be a surjective mapping and $g : X \to Y$ any mapping. By assumption, the projection $\pi_1 : Y \times Y$ is semi-universal. Let $K = \{(f(x), g(x)) \mid x \in X\}$. Now K is a subcontinuum of $Y \times Y$ and $\pi_1(K) = Y$. Let $h : K \to Y \times Y$ be defined by $h(x,y) = (y,x)$. There must be a point $(f(x), g(x))$ in K such that $\pi_1(f(x), g(x)) = \pi_1 h(f(x), g(x))$. Hence, $f(x) = g(x)$. It follows that f is universal. $\square$

James F. Davis [**3**] has shown that $\sigma(X) = 0$ if and only if $\sigma_0(X) = 0$, and $\sigma_0(X) = 0$ implies that $\sigma_0^*(X) = 0$. Hence, we get the following corollary.

Corollary 1. *If $\sigma(X) = 0$, then $X \in \text{class}\,(U)$.*

Theorem 6. *If $f : X \to S^1$ is unessential, then f is semi-universal.*

Proof. Since f is unessential, there is a mapping $\psi : X \to E^1$ such that $f(x) = e^{i\psi(x)}$ for all $x \in X$. Now $\psi(X)$ is an arc or a point, and it follows from Theorems 2 and 5 that $\psi : X \to \psi(X)$ is semi-universal. By Theorem 4, f is semi-universal. $\square$

Let X be a connected topological space with closed subsets H, A and B; and suppose that A and B are disjoint. We say that H *weakly cuts* A from B in X if each closed connected set in X that intersects both A and B must also intersect H. The connected topological space X is *s-connected between the closed disjoint subsets* A and B provided that whenever H is a closed set in X that weakly cuts A from B, then some component of H weakly cuts A from B. The connected space X is *s-connected* provided that whenever A and B are disjoint closed connected subsets of X, then X is s-connected between A and B, see [**17**].

The mapping $f : X \to Y$ is *semi-universal with respect to the class* $\mathcal{H}$ of subcontinua of X if whenever K is in $\mathcal{H}$ with $f(K) = f(X)$ and $g : K \to X$ is a mapping, then there is a point $x \in K$ such that $f(x) = fg(x)$.

Theorem 7. *Let X be a continuum, and let Y be the cone over X with vertex v. If Y is s-connected and the projection mapping π_1 of $Y - \{v\}$ onto X is semi-universal with respect to subcontinua of Y that weakly cut $\{v\}$ from $X \times \{0\}$ in Y, then Y has the fixed point property.*

Proof. Let $\pi_2 : Y \to [0, 1]$ be the projection of Y onto $[0, 1]$. Suppose $f : Y \to Y$ is a fixed point free mapping. Let $H = \{y \in Y \mid \pi_2 f(y) = \pi_2(y)\}$. Since π_2 is universal, it follows that H is not empty. It is easy to see that H weakly cuts $X \times \{0\}$ from $\{v\}$ in Y. Since Y is s-connected, some subcontinuum K of H weakly cuts $X \times \{0\}$ from $\{v\}$. Since f is fixed point free, $v \notin H \cup f(H)$. It follows that $\pi_1(K) = X$.

Since π_1 is semi-universal with respect to K, there is a point $x \in K$ such that $\pi_1(x) = \pi_1 f(x)$. Furthermore, since $x \in K \subseteq H$, $\pi_2(x) = \pi_2 f(x)$. So, $f(x) = x$, a contradiction. □

Corollary 2. *Suppose X is an inverse limit of absolute neighborhood retracts and Y is the cone over X. If the projection mapping π_1 of $Y - \{v\}$ onto X is semi-universal with respect to subcontinua of Y that weakly cut $\{v\}$ from $X \times \{0\}$ in Y, then Y has the fixed point property.*

Proof. Since the cone over X is homeomorphic to the inverse limit of cones over absolute neighborhood retracts, which are absolute retracts, and inverse limits of absolute retracts are s-connected [**18**, Theorem 3], this result follows immediately from Theorem 7. □

Theorem 8. *Let X be a continuum, and let $Y = X \times [0,1]$. If Y is s-connected and the projection mapping from Y onto X is semi-universal with respect to continua that weakly cut $X \times \{0\}$ from $X \times \{1\}$ in Y, then Y has the fixed point property.*

Proof. The proof is similar to the proof of Theorem 7. □

In light of Corollary 2 and Theorem 8, an answer to the question below would be of interest.

Question 1. For what continua X is the projection mapping of $X \times [0,1]$ onto X semi-universal with respect to continua that weakly cut $X \times \{0\}$ from $X \times \{1\}$ in $X \times [0,1]$?

3. Weakly universal mappings.

Theorem 9. *If $f_1 : X \to Y$ is weakly universal and $f_2 : Y \to Z$ is a mapping, then $f_2 \circ f_1$ is weakly universal.*

Note. It follows from observation (3) in Section 1 that each continuum in class (WU) must have the fixed point property.

Question 2. Is class $(U) =$ class (WU)?

Question 3. Is each mapping $f : X \to T$ from an acyclic continuum onto a simple triod weakly universal?

A mapping $f : X \to Y$ is *weakly confluent* if whenever K is a subcontinuum of Y, there is a component H of $f^{-1}(K)$ such that $f(H) = K$.

Question 4. Is each weakly confluent mapping $f : X \to T$ from a continuum onto a tree weakly universal?

Question 5. Is each weakly confluent mapping $f : X \to B^n$ from an acyclic continuum onto the closed n-ball weakly universal?

The assumption that X is acyclic in Question 5 is necessary. The mapping from an annulus to a disk which shrinks the inner circle to a point is monotone but not weakly universal.

Question 6. If $Y \in$ class (WU) and K is a retract of Y, is $K \in$ class (WU)?

Theorem 10. *Suppose $f : X \to X$ is a self map on a continuum X. Then f has a fixed point if and only if, for each $\varepsilon > 0$, there is an ε-map of X onto a continuum Y_ε that is weakly universal with respect to f.*

Proof. Suppose f has a fixed point. Then, for $\varepsilon > 0$, the identity map on X is an ε-map that is weakly universal with respect to f.

For the opposite implication, suppose $f : X \to X$ is fixed point free, and let ε be a positive number such that $d(x, f(x)) \geq \varepsilon$ for each $x \in X$. By assumption there is a continuum Y_ε and an ε-map $g : X \to Y_\varepsilon$ that is weakly universal with respect to f. Hence, there is a point $x \in X$ such that $g(x) = gf(x)$. But since g is an ε-map, $d(x, f(x)) < \varepsilon$, which is a contradiction. □

Corollary 3. *A continuum X has the fixed point property if and only if for each $\varepsilon > 0$ there is a weakly universal ε-map from X onto a continuum Y_ε.*

Corollary 4. *Suppose X is $\mathcal{H}$-like for some class of continua $\mathcal{H}$. Then the mapping $f : X \to X$ has a fixed point if and only if each of the ε-maps is weakly universal with respect to f.*

Corollary 5. *Suppose X is $\mathcal{H}$-like. Then X has the fixed point property if and only if each of the ε-maps is weakly universal.*

If the answer to Question 3 is yes, it follows from Corollary 3 that inverse limits on simple triods have the fixed point property.

In the next theorem we apply a generalization of the Borsuk-Ulam theorem to obtain a partial solution to a question of J.B. Fugate (Univ. of Houston Prob. Book, #112).

Theorem 11. *Suppose X is a contractible continuum and $j : X \to E^n$ is an embedding of X into Euclidean n-space. If $h : X \to X$ is a periodic homeomorphism of period p, then there is a divisor k of p with $1 \leq k < p$ such that $h^k : X \to X$ has a fixed point.*

Proof. If h has a fixed point, then we are done. So assume that h is fixed point free. Then h generates a free Z_p-action on X. Since X is contractible, X is m-connected for all $m \geq 0$, see [**24**, page 51]. In particular, X is $(n-1)(p-1)$-connected. By Theorem 1 in [**1**], there is a point x in X and a $1 \leq k < p$ such that $j(x) = j(h^k(x))$. Hence, $x = h^k(x)$. Suppose, without loss of generality, that k is the least such integer for x. Now there are integers m and $0 \leq r < k$ such that $p = km + r$. So

$$x = h^p(x) = h^{km+r}(x) = h^r((h^k)^m(x)) = h^r(x).$$

By choice of k, this implies that $r = 0$. So k is a divisor of p and x is a fixed point of h^k. □

Corollary 6. *If X is contractible, $j : X \to E^n$ is an embedding, and $h : X \to X$ is a periodic homeomorphism of period p, where p is a prime number, then h has a fixed point.*

Question 7. If $f : X \to B^2$ is a surjective mapping and $H^1(X) \approx 0$, is f weakly universal with respect to periodic homeomorphisms on X?

If the answer to Question 7 is yes, then one can apply Corollary 4 to show that periodic homeomorphisms on disk-like continua have fixed points, which would answer a question of Fugate and McLean (Univ. of Houston Prob. Book, #110).

4. Pseudo universal mappings.

Theorem 12. *If $f_1 : X \to Y$ is pseudo universal and $f_2 : Y \to Z$ is a mapping, then $f_2 \circ f_1$ is pseudo universal.*

Theorem 13. *Let X be a continuum. The following statements are equivalent.*

(1) $X \in \text{class}\,(PU)$.

(2) *Each mapping of a continuum onto X is weakly universal with respect to involutions.*

(3) $s^*(X) = 0$.

Proof. (1) ⇒ (2). This implication is obvious.

(2) ⇒ (3). Let Z be a subcontinuum of $X \times X$ such that $\pi_1(Z) = X$ and $Z \cap Z^{-1}$ is not empty. Then $\hat{Z} = Z \cup Z^{-1}$ is also a subcontinuum of $X \times X$ and $\pi_1(\hat{Z}) = X$. Let $\alpha : \hat{Z} \to \hat{Z}$ be defined by $\alpha(x, y) = (y, x)$. We point out that α^2 is the identity map on $\hat{Z}$ and $\pi_1 \circ \alpha = \pi_2|_{\hat{Z}}$. By assumption, $\pi_1 : \hat{Z} \to X$ is pseudo universal with respect to α. So there is a $z \in \hat{Z}$ such that $\pi_1(z) = \pi_1\alpha(z) = \pi_2(z)$. Thus $\hat{Z}$ intersects the diagonal in $X \times X$, as does Z. Therefore, it follows from Theorem 2 in [**3**] that $s^*(X) = 0$.

(3) $\Rightarrow$ (1). Suppose $s^*(X) = 0$. Let $f : M \to X$ be a mapping from a continuum M onto X and let $g : M \to M$ be a mapping with a point of period two. Let $Z = \{(f(x), fg(x)) \mid x \in M\}$. Now Z is a subcontinuum of $X \times X$ and $\pi_1(Z) = X$. Let p be a point of M such that $g^2(p) = p$. Then $(fg(p), f(p)) \in Z^{-1}$ and $(fg(p), f(p)) = (f(g(p)), fg(g(p))) \in Z$. Hence $Z \cap Z^{-1}$ is nonempty. It follows that Z meets the diagonal in $X \times X$. Thus there is a point $x \in M$ such that $f(x) = fg(x)$. Therefore, $X \in \text{class}\,(PU)$. $\square$

Theorem 14. class (WU) *is a proper subclass of* class (PU).

Proof. It is clear that class $(WU) \subseteq$ class (PU).

Let Σ_2 be the dyadic solenoid. Now $s^*(\Sigma_2) = 0$, so $\Sigma_2 \in$ class (PU). However, since Σ_2 admits a fixed point free mapping, it follows from the note in Section 3 that Σ_2 is not in class (WU). $\square$

Note 3. W.T. Ingram's example M in [**11**] is not in class (PU) since $s^*(M) > 0$.

A mapping $f : X \to Y$ is *pseudo confluent* if, whenever K is an irreducible continuum in Y, there is a component H of $f^{-1}(K)$ such that $f(H) = K$. A continuum Y is in class (P) if each mapping from a continuum onto Y is pseudo confluent.

Theorem 15. *If* $M \in$ class (PU), *then* M *is unicoherent,* M *is not a triod and* M *is irreducible.*

Proof. Since $s^*(M) = 0$ we may use the proof of Theorem 3 in [**3**] to get that M is unicoherent and not a triod. It follows from Theorem 3.2 in [**23**] that M is irreducible. $\square$

Question 8. If $M \in$ class (PU), is M hereditarily unicoherent? Is M atriodic?

Theorem 16. *If* $M \in$ class (PU), *then* $M \in$ class (P).

Proof. Suppose M is not in class (P). Let X be a continuum and $f : X \to M$ a mapping that is not pseudo confluent. Then there is an irreducible subcontinuum C of M such that no component of $f^{-1}(C)$ is mapped by f onto C. The remainder of the proof follows the proof of Theorem 4 in [**3**] with the observation that $\pi_1(Y) = X$ and thus $\pi_1(Z) = M$. $\square$

A continuum is in class (W) if each mapping from a continuum onto X is weakly confluent. Grispolakis and Tymchatyn [**5**, Theorem 5.3] have shown that, for atriodic continua, class (W) = class (P). Thus we get the following corollary to Theorem 16.

Corollary 7. *If M is atriodic and $M \in$ class (PU), then $M \in$ class (W).*

See Nadler [**21**], particularly 1.12 through 1.15, for results concerning class $(\hat{U})$ and relationships to class (W).

REFERENCES

1. Fred Cohen and J.E. Connett, *A coincidence theorem related to the Borsuk-Ulam theorem*, Proc. Amer. Math. Soc. **44** (1974), 218–220.

2. J.E. Connett, *A generalization of the Borsuk-Ulam theorem*, J. London Math. Soc. **7** (1973), 64–66.

3. James F. Davis, *The equivalence of zero span and zero semispan*, Proc. Amer. Math. Soc. **90** (1984), 133–138.

4. C.A. Eberhart and J.B. Fugate, *Weakly confluent maps on trees*, in *General topology and modern analysis*, Academic Press, New York, 1981.

5. J. Grispolakis and E.D. Tymchatyn, *Continua which are images of weakly confluent mappings only*, II, Houston J. Math. **6** (1980), 375–387.

6. Heinz Hopf, *Freie Überdeckungen and freie abbildungen*, Fund. Math. **28** (1937), 33–57.

7. ———, *Eine Verallgemeinerung bekannter abbildungs und überdeckungssätze*, Portugaliae Math. **4** (1943), 129–139.

8. W. Holsztyński, *Universal mappings and fixed point theorems*, Bull. Acad. Polon. Sci. **15** (1967), 433–438.

9. ———, *On the composition and products of universal mappings*, Fund. Math. **64** (1969), 181–188.

10. ———, *Universality of the product mappings onto products of I_n and snake-like spaces*, Fund. Math. **64** (1969), 147–155.

11. W.T. Ingram, *An atriodic tree-like continuum with positive span*, Fund. Math. **77** (1972), 99–107.

12. Steven M. Kahn, *The Conner-Miller classes of a product involution and Borsuk-Ulam type theorems*, Houston J. Math. **8** (1982), 75–84.

13. A. Lelek, *Disjoint mappings and the span of spaces*, Fund. Math. **55** (1964), 199–214.

14. O.W. Lokuciewski, *On a theorem on fixed points*, Ycπ. Mat. Hayκ **12** (1957), 171–172, (in Russian).

15. M.M. Marsh, *u-Mappings on trees*, Pacific J. Math. **127** (1987), 373–387.

16. ———, *Mappings of trees and the fixed point property*, Trans. Amer. Math. Soc. **315** (1989), 799–810.

17. ———, *s-connected spaces and the fixed point property*, Topology Proc. **8** (1983), 85–97.

18. M.M. Marsh and E.D. Tymchatyn, *Inductively weakly confluent mappings*, Houston J. Math. **18** (1992), 235–250.

19. S.B. Nadler, *Universal mappings and weakly confluent mappings*, Fund. Math. **60** (1980), 221–235.

20. ———, *Induced universal maps and some hyperspaces with the fixed point property*, Proc. Amer. Math. Soc. **100** (1987), 749–754.

21. ———, *New problems in continuum theory*, in *Continuum theory and dynamical systems*, Lecture Notes in Pure Appl. Math. **149** (1993), 201–229.

22. M. Nakaoka, *Continuous maps of manifolds with involution*, I, Osaka J. Math. **11** (1974), 129–145.

23. R.H. Sorgenfrey, *Concerning triodic continua*, Amer. J. Math. **66** (1944), 439–460.

24. Edwin Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

25. C. Yang, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson*, II, Ann. Math. **62** (1955), 271–283.

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