ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 4, Fall 1997

POINT OF CONTINUITY PROPERTY AND SCHAUDER BASES

GINÉS LÓPEZ AND JUAN F. MENA

ABSTRACT. We get a characterization of point of continuity property in Banach spaces with a shrinking Schauder finite-dimensional decomposition. We also prove that a Banach space with a shrinking Schauder finite-dimensional decomposition has the point of continuity property if every subspace with a shrinking Schauder basis has it.

1. Introduction. We begin by recalling some geometrical properties in Banach spaces: (see [**2, 4** and **6**]).

Let X be a Banach space, C a closed, bounded, convex and nonempty subset of X and τ a topology in X.

C is said to have the point of τ-continuity property (τ-PCP) if for every closed subset, F, of C the identity map from (F,τ) into $(F,\|\ \|)$ has some point of continuity.

If C satisfies the above definition with τ the weak topology in X, then C is said to have the point of continuity property (PCP).

C is said to have the Radon-Nikodym property (RNP) if for every measure space (Ω,Σ,μ) and for every $F:\Sigma\to X$, μ-continuous vector measure, such that

$$\frac{F(A)}{\mu(A)}\in C \quad \forall\, A\in\Sigma, \quad \mu(A)>0$$

there is $f:\Omega\to X$ Bochner integrable with

$$F(A)=\int_A f\,d\mu \quad \forall\, A\in\Sigma.$$

C is said to have the Krein-Milman property (KMP) if each closed, convex and nonempty subset of C is the closed convex hull of its extreme points.

Received by the editors on December 10, 1994.
1991 AMS *Mathematics Subject Classification.* Primary 46B22, secondary 46B20.
Partially supported by DGICYT PB 93-1142.

Finally, we will say that X has some of the above properties if B_X, the closed unit ball of X, has it.

It is known that RNP implies PCP and that the converse is false (see [**5**]). RNP implies KMP [**9**] but whether the converse is true is an open problem; however, if one supposes PCP, then RNP and KMP are equivalent [**10**].

Bourgain [**3**] showed that RNP is determined by subspaces with a Schauder finite-dimensional decomposition (FDD). The same is true for PCP, but it is unknown if RNP (PCP) is determined by subspaces with a Schauder basis. In this note we will prove that a Banach space has PCP if every subspace with a Schauder basis has τ-PCP, where τ is the weak topology of the basis (Corollary 4).

If X is a Banach space with a Schauder basis $\{e_n\}$ and associated functionals $\{f_n\}$, we call weak topology of the basis in X, $w_{\{e_n\}}$, to the weak topology $\sigma(X, \overline{\text{lin}}\{f_n : n \in \mathbf{N}\})$.

Now we introduce some notation: (see [**11**]).

A sequence $\{G_n\}$ of finite-dimensional subspaces of a Banach space X with $G_n \neq \{0\}$ for all $n \in \mathbf{N}$ is said to be a Schauder finite-dimensional decomposition (FDD) if for every $x \in X$ there is a unique sequence $\{y_n\} \subset X$, with $y_n \in G_n$ for all $n \in \mathbf{N}$ such that

$$x = \sum_{i=1}^{+\infty} y_i = \lim_{n \to +\infty} \sum_{i=1}^{n} y_i.$$

Given $\{G_n\}$ a Schauder FDD of X, a sequence of subspaces $\{F_n\}$ of X is said to be a block Schauder decomposition of X (with respect to $\{G_n\}$), or shortly, a blocking of $\{G_n\}$, if it is of the form

$$F_n = \text{lin}\{G_i : t_{n-1} < i \leq t_n\}$$

where $\{t_n\}$ is an increasing sequence of positive integers with $t_0 = 0$.

It is clear that $\{F_n\}$ is a Schauder FDD of X.

Let $(\{x_n\}, \{f_n\})$ be a biorthogonal system in X; $\{x_n\}$ is said to be a basis with parentheses of X if it is complete minimal and there is an increasing sequence of natural numbers $\{m_n\}$ such that

$$x = \lim_{n \to +\infty} \sum_{i=1}^{m_n} f_i(x) x_i \quad \forall\, x \in X.$$

The basis with parentheses is said to be shrinking if $\overline{\mathrm{lin}}\{f_n : n \in \mathbf{N}\} = X^*$.

Finally it is easy to see that a Banach space has a Schauder FDD if and only if it has a basis with parentheses [**11**, Proposition 13.11].

In fact, if $\{G_n\}$ is a Schauder FDD of X and

$$G_n = \mathrm{lin}\{e_i : m_{n-1} < i \leq m_n\}$$

where $\{e_i\}$ are linearly independent vectors of norm one and $\{m_n\}$ is an increasing sequence of integers with $m_0 = 0$, then $\{e_n\}$ is a basis with parentheses with respect to $\{m_n\}$.

Now we will construct a family of closed and convex subsets in any Banach space with a Schauder FDD, that is, with a basis with parentheses, to get a characterization of PCP in Banach spaces with a shrinking Schauder FDD (Corollary 3).

In the sequel, X will denote a Banach space with a basis with parentheses $\{e_n\}$ with respect to $\{m_n\}$, and normalized, with associated functionals $\{f_n\}$.

Let $\Gamma = \mathbf{N}^{(\mathbf{N})} \cup \{\alpha_0\}$. That is, an element of Γ is a finite sequence of natural numbers and α_0 denotes the empty sequence. $|\alpha|$ will be the length of α for all $\alpha \in \Gamma$, and we put $|\alpha_0| = 0$.

We define an order in Γ by

$$\alpha \leq \beta \quad \text{if } |\alpha| \leq |\beta|$$

and

$$\alpha_i = \beta_i, \quad 1 \leq i \leq |\alpha|, \quad \forall\, \alpha, \beta \in \Gamma \backslash \{\alpha_0\}$$

and $\alpha_0 \leq \alpha$ for all $\alpha \in \Gamma$.

Γ is a countable set with a partial order and minimum element, α_0, and there is a bijective order-preserving map, ϕ, from Γ into $\mathbf{N}$.

For every $\alpha \in \Gamma$ we define $x_\alpha = \sum_{\gamma \leq \alpha} e_{\phi(\gamma)}$ and $f_\alpha = f_{\phi(\alpha)}$. Then

$$f_\gamma(x_\alpha) = 1, \quad \forall\, \gamma \leq \alpha$$

and

$$f_\gamma(x_\alpha) = 0 \quad \text{in other cases.}$$

Doing $\Lambda = \overline{\mathrm{co}}\,\{x_\alpha : \alpha \in \Gamma\}$ we obtain a closed and convex subset of X. Furthermore, $f_{\alpha_0}(x) = 1$ and $f_\alpha(x) \geq 0$ for all $x \in \Lambda$, $\alpha \in \Gamma$.

If $\{v_n\}$ is a basic sequence in X, with the same construction we get a new closed and convex subset of X which we will denote by $\Lambda_{\{v_n\}}$. Then we have a family of convex, closed and nonempty subsets of X.

2. Main results. The following result is a generalization of [**1**, Proposition 2.3].

Theorem 1. *Let X be a Banach space with a Schauder* FDD $\{G_n\}$, *and let K be a closed, convex, bounded and nonempty subset of X failing* PCP. *Then there are $\{F_n\}$ blocking of $\{G_n\}$, $\{v_n\}$ basic sequence of X with $v_n \in F_n$ for all $n \in \mathbf{N}$, Y a closed subspace of X with basis, F a subset of K with $F \subset Y$ and an isomorphism onto its image, $T : Y \to X$, such that $T(F) = \Lambda_{\{v_n\}}$.*

Proof. As we have said in the introduction, let $\{e_n\}$ be the basis with parentheses with respect to $\{m_n\}$ obtained from $\{G_n\}$ and with associated functionals $\{f_n\}$.

Without loss of generality we can suppose that $\{G_n\}$ is monotone. By [**5**] we can find a nonempty subset A of K and $\delta > 0$ such that every w-neighborhood of A has diameter at least δ.

Let's see that there is a subset $\{a_n : n \in \mathbf{N}\}$ of A such that $\{u_j : j \in \mathbf{N}\}$ is a basic sequence of X, where

$$u_1 = a_1, \qquad u_j = a_j - a_{\phi(\phi^{-1}(j)-)}, \quad \forall\, j > 1,$$

$\alpha- = (\alpha_1, \dots, \alpha_{n-1})$ if $\alpha = (\alpha_1, \dots, \alpha_n) \in \Gamma \backslash \{\alpha_0\}$, $n > 1$, $\alpha- = \alpha_0$ in other cases.

For this, let $\varepsilon_j = \delta 2^{-(j+1)}$ for all $j \in \mathbf{N}$, and we construct, by induction, $k_0 = 0 < k_1 < \cdots < k_n < \cdots \in \mathbf{N}$, $k_j \in \{m_n : n \in \mathbf{N}\}$ for all $j \in \mathbf{N}$ and $v_1, \dots, v_n, \dots \in X$ such that

$$\|u_j\| > \frac{\delta}{2}, \qquad \|v_j - u_j\| < \varepsilon_j,$$
$$v_j \in \mathrm{lin}\,\{e_i : k_{j-1} < i \leq k_j\} \quad \forall\, j \in \mathbf{N}.$$

We know that $\mathrm{diam}\,(A) \geq \delta$ and so there is $a_1 \in A$ such that $\|a_1\| > \delta/2$.

Let $m_p \in \mathbf{N}$ with $\|a_{1|(m_p,+\infty)}\| < \varepsilon_1$, where

$$a_{1|(m_p,+\infty)} = a_1 - \sum_{j=1}^{m_p} f_j(a_1)e_j.$$

(This is because $a_1 = \lim_{n\to+\infty} \sum_{j=1}^{m_N} f_j(a_1)e_j$.)

We define $k_1 = m_p$ and $v_1 = a_{1|[1,k_1]}$, that is,

$$v_1 = \sum_{j=1}^{k_1} f_j(a_1)e_j.$$

Now, we suppose $n \geq 1$ and $a_1, \dots, a_n$ and k_n have been already constructed.

We do $i = \phi(\phi^{-1}(n+1)-)$, $\alpha = \phi^{-1}(n+1)-$, $\beta = \phi^{-1}(n+1)$. Then $\alpha < \beta$ and so $i < n+1$ because ϕ is an order-preserving map. So a_i has been already constructed.

Let $\varepsilon = \varepsilon_{n+1}/2$ and

$$V = \{a \in A : |f_j(a_i - a)| < \varepsilon/k_n, \quad 1 \leq j \leq k_n\}.$$

Then V is a w-neighborhood of a_i in A and $\operatorname{diam}(V) \geq \delta$. Then, there is $a_{n+1} \in V : \|a_{n+1} - a_i\| > \delta/2$ and $u_{n+1} = a_{n+1} - a_i$.

If now $m_j > k_n$ and $\|u_{n+1|(m_j,+\infty)}\| < \varepsilon$, we put

$$k_{n+1} = m_j, \qquad v_{n+1} = u_{n+1|(k_n,k_{n+1}]}.$$

Then $\|u_{n+1}\| > \delta/2$ and

$$\begin{aligned}\Big\|u_{n+1} - \sum_{j=1}^{k_{n+1}} f_j(u_{n+1})e_j\Big\| &= \Big\|u_{n+1} - \sum_{j=1}^{k_n} f_j(u_{n+1})e_j - \sum_{j=k_n+1}^{k_{n+1}} f_j(u_{n+1})e_j\Big\| \\ &= \Big\|u_{n+1} - v_{n+1} - \sum_{j=1}^{k_n} f_j(u_{n+1})e_j\Big\| < \varepsilon.\end{aligned}$$

But, by definition of V and u_{n+1}, $\| \sum_{j=1}^{k_n} f_j(u_{n+1})e_j \| < \varepsilon$.

So $\|u_{n+1} - v_{n+1}\| < \varepsilon_{n+1}$ and the inductive construction is complete.

Now it is clear that $F_n = \operatorname{lin}\{e_i : k_{n-1} < i \le k_n\}$ is a blocking of $\{G_n\}$.

By [**11**, Theorem 15.21], $\{v_n\}$ is a basic sequence and by [**8**, Proposition 1.a.9], $\{u_n\}$ is a basic sequence equivalent to $\{v_n\}$.

Let's define $F = \overline{\operatorname{co}}\{a_n : n \in \mathbf{N}\}$, $Y = \overline{\operatorname{lin}}\{u_n : n \in \mathbf{N}\}$ and

$$\bar{u}_\alpha = u_{\phi(\alpha)}, \qquad \bar{a}_\alpha = a_{\phi(\alpha)}, \qquad \bar{v}_\alpha = v_{\phi(\alpha)}, \quad \forall\, \alpha \in \Gamma.$$

By the above construction there is an isomorphism onto its image $T : Y \to X$ such that $T(\bar{u}_\alpha) = \bar{v}_\alpha$ for all $\alpha \in \Gamma$.

By definition, $\bar{u}_{\alpha_0} = \bar{a}_{\alpha_0}$ and $\bar{u}_\alpha = \bar{a}_\alpha - \bar{a}_{\alpha-}$ for all $\alpha \neq \alpha_0$.

Then $\bar{a}_\alpha = \sum_{\gamma < \alpha} \bar{u}_\gamma$ for all $\alpha \in \Gamma$ and $F \subset Y$.

Furthermore, $T(\bar{a}_\alpha) = \sum_{\gamma \le \alpha} T(\bar{u}_\gamma) = \sum_{\gamma \le \alpha} \bar{v}_\gamma$.

But we have constructed $\Lambda_{\{v_n\}}$ and, by definition, $T(F) = \Lambda_{\{v_n\}}$ and $F \subset \overline{\operatorname{co}}(A) \subset K$. So the proof is complete. $\square$

Corollary 2. *With the same hypotheses and notations of Theorem* 1, *if we suppose that the Schauder* FDD *is shrinking, then* $\Lambda_{\{v_n\}}$ *fails* PCP.

Proof. In Theorem 1 we obtain

$$|f_j(u_{n+1})| < \varepsilon_{n+1}/(2k_n), \quad 1 \le j \le k_n,$$

and $\{u_n\}$ is bounded, so $\{u_n\} \to 0$ weakly because the Schauder FDD is shrinking.

But $x_{(\alpha,i)} = x_\alpha + v_{(\alpha,i)}$ for all $\alpha \in \Gamma$ for all $i \in \mathbf{N}$, then

$$\bar{a}_{(\alpha,i)} = \bar{a}_\alpha + \bar{u}_{(\alpha,i)} \quad \forall\, \alpha \in \Gamma, \quad \forall\, i \in \mathbf{N},$$

and $\{\bar{a}_{(\alpha,i)}\}$ converges weakly to $\bar{a}_\alpha$ when $i \to +\infty$ for all $\alpha \in \Gamma$.

Furthermore, $\|\bar{a}_{(\alpha,i)} - \bar{a}_\alpha\| = \|\bar{u}_{(\alpha,i)}\| \ge \delta/2$ for all $\alpha \in \Gamma$, $i \in \mathbf{N}$, and we obtain that $\{\bar{a}_\alpha : \alpha \in \Gamma\}$ is a nonempty, closed and bounded subset without points of $(w - \| \|)$-continuity. Then $\{x_\alpha : \alpha \in \Gamma\}$ is a nonempty, closed and bounded subset of $\Lambda_{\{v_n\}}$ without points of $(w - \| \|)$-continuity and $\Lambda_{\{v_n\}}$ fails PCP. $\square$

The following consequence is a characterization of PCP in Banach spaces with a shrinking Schauder FDD.

Corollary 3. i) *Let X be a Banach space with a Schauder* FDD*. If X fails* PCP*, then there is $\{v_n\}$ basic sequence of X such that $\Lambda_{\{v_n\}}$ is bounded.*

ii) *Let X be a Banach space with a shrinking Schauder* FDD *$\{G_n\}$. If there exists $\{F_n\}$ blocking of $\{G_n\}$ and $\{v_n\}$ seminormalized basic sequence with $v_n \in F_n$ for all $n \in \mathbf{N}$ such that $\Lambda\{v_n\}$ is bounded, then X fails* PCP*.*

Proof. i) It is a very easy consequence of Theorem 1.

ii) $x_{(\alpha,i)} = x_\alpha + v_{(\alpha,i)}$ for all $\alpha \in \Gamma$, $i \in \mathbf{N}$, then $\{x_{(\alpha,i)}\}$ converges weakly to x_α when $i \to +\infty$ for all $\alpha \in \Gamma$ because the Schauder FDD is shrinking.

Now we take the basic sequence normalized and we obtain a subset without PCP isomorphic to $\Lambda_{\{v_n\}}$. $\square$

Corollary 4. *Let X be a Banach space such that every subspace of X with a Schauder basis, $\{v_n\}$, has $w_{\{v_n\}}$-*PCP*. Then X has* PCP*.*

Proof. Let's suppose that X fails PCP. By [**3**] there is a Z subspace of X with a Schauder FDD failing PCP. Now Theorem 1 says that there is a basic sequence $\{v_n\}$ of Z such that $\Lambda_{\{v_n\}}$ fails PCP with the weak topology of the basis, and the proof is complete. $\square$

It was proved in [**7**, Theorem IV.6] that PCP, for Banach spaces not containing l_1, is determined by the subspaces with a Schauder basis. As a consequence of our results we get, in a particular case, that in fact PCP is determined by the subspaces with a shrinking Schauder basis.

Corollary 5. *Let X be a Banach space with a shrinking Schauder* FDD. *Then the following are equivalent*:

i) X *has* PCP.

ii) *Every subspace of* X *with a Schauder basis has* PCP.

iii) *Every subspace of* X *with a shrinking Schauder basis has* PCP.

Proof. i) $\Rightarrow$ ii) and ii) $\Rightarrow$ iii) are evident.

iii) $\Rightarrow$ i). We assume that X fails PCP. By Theorem 1 we obtain a subspace of X with a Schauder basis. By Corollary 2, this subspace fails PCP and by [**11**, Remark 15.9] the basis is shrinking. $\square$

It is clear that the hypotheses on X in the above corollary imply that X does not contain l_1. So the equivalence between i) and ii) can be obtained from the aforementioned result by Ghoussoub and Maurey [**7**, Theorem IV.6].

REFERENCES

1. S. Argyros, E. Odell and H. Rosenthal, *On certain convex subsets of* c_0, Lecture Notes in Math. **1332** (1988), 80–111.

2. J. Bourgain, *La proprieté de Radon-Nikodym*, Publ. Math. Univ. Pierre et Marie Curie **36** (1977).

3. ———, *Dentability and finite-dimensional decompositions*, Studia Math. **67** (1980), 135–148.

4. R.D. Bourgain, *Geometric aspects of convex sets with the Radon-Nikodym property*, Lecture Notes in Math. **993** (1980).

5. J. Bourgain and H. Rosenthal, *Geometrical implications of certain finite-dimensional decompositions*, Bull. Soc. Math. Belg. Sér. A **32** (1987), 57–82.

6. J. Diestel and J.J. Uhl, Jr., *Vector measures*, Math. Surveys **15** (1977).

7. N. Ghoussoub and B. Maurey, *G_δ-embeddings in Hilbert space*, II, J. Funct. Anal. **78** (1988), 271–305.

8. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces* I, Ergeb. Math. Grenzgeb. **92** (1977).

9. R.R. Phelps, *Dentability and extreme points in Banach spaces*, J. Funct. Anal. **17** (1974), 78–90.

10. W. Schachermeyer, *The Radon-Nikodym property and the Krein-Milman property are equivalent for strongly regular sets*, Trans. Amer. Math. Soc. **303** (1987), 673–687.

11. I. Singer, *Bases in Banach spaces*, II, Springer Verlag, New York, 1981.

Universidad de Granada, Departamento de Análisis Matemático, 18071 Granada, Spain
E-mail address: glopez@goliat.ugr.es

Universidad de Granada, Departamento de Análisis Matemático, 18071 Granada, Spain
E-mail address: jfmena@goliat.ugr.es