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INVERSE SCATTERING AND INFINITE MATRICES

M. KOVALYOV

1. IST for the KdV. The inverse scattering transform and its application to the KdV equation have been known for quite awhile. There has been a lot of research done as well as some attempts to explain why the IST works. In this paper we will show that the whole theory can be obtained using a very simple linear algebra type of argument shown in Example 1. We illustrate our ideas in detail on the KdV as the simplest equation to which the IST can be applied and then sketch them for a more general case of first order systems.

We start with the fundamental observation that the KdV

$$u_t - 6uu_x + u_{xxx} = 0, \qquad \lim_{x\to\pm\infty} u(x,t) = 0 \tag{1}$$

can be written as

$$\dot{L} = [L, A] \tag{2a}$$

where

$$\begin{aligned} A &= 4\frac{\partial^3}{\partial x^3} - 3\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right) \\ &= 4\frac{\partial^3}{\partial x^3} - 6u\frac{\partial}{\partial x} - \frac{\partial u}{\partial x} \end{aligned} \tag{2b}$$

and

$$L = -\frac{\partial^2}{\partial x^2} + u, \qquad \lim_{x\to\pm\infty} u(x) = 0. \tag{2c}$$

We will proceed in the following way. First we will study some properties of the operator (2c) without additional assumptions (1) and (2a). We will obtain a relationship (called Gelfand-Levitan-Marchenco equation) between the spectrum and eigenfunctions of L and $u(x)$.

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Thereafter we will show how the condition (2a) can affect the spectrum and eigenfunctions of L and how it will change the Gelfand-Levitan-Marchenco equation.

It is well known from the theory of ODEs that L possesses both discrete and continuous spectra. The discrete spectrum of L is simple and consists of an at most countable set (can be finite or even empty) of negative numbers, whereas the continuous spectrum of L is double and fills up the positive half of the real axis. The eigenfunctions of L corresponding to both continuous and discrete spectra satisfy

$$L\psi = k^2\psi \tag{3a}$$
$$\psi(x,k) \sim e^{-ikx}, \quad x \to +\infty. \tag{3b}$$

In addition, as $x \to -\infty$, they satisfy the asymptotic

$$\begin{gathered} \psi(x,k) \sim a(-k)e^{-ikx} - b(k)e^{ikx}, \\ a(k)a(-k) - b(k)b(-k) = 1 \end{gathered} \tag{3c}$$

where $a(-k)$, $b(k)$ are some complex functions.[1] $\psi(x,k)$ is a solution of (3b) and so is $\psi(x,-k)$. Since the Wronskian $W(\psi(x,k),\psi(x,-k))$ of these two functions is independent of x, we have the following string of equalities

$$\begin{aligned} 2ik &= \lim_{x\to+\infty} W(\psi(x,k),\psi(x,-k)) \\ &= \lim_{x\to-\infty} W(\psi(x,k),\psi(x,-k)) \\ &= 2ik(a(k)a(-k) - b(k)b(-k)) \end{aligned}$$

and therefore $a(k)a(-k) - b(k)b(-k) = 1$. If k is real the coefficients also satisfy $\overline{a(k)} = a(-k)$, $\overline{b(k)} = b(-k)$, and as a consequence $|a(k)|^2 - |b(k)|^2 = 1$.

The solution of (3) can be written as

$$\psi(x,k) = e^{-ikx} + \int_x^{+\infty} \frac{\sin k(x-y)}{k} u(y)\psi(y,k)\,dy. \tag{4}$$

We can think of (4) as a linear equation for a generalized matrix $\psi(y,k)$

$$\int_{-\infty}^{+\infty} \theta(y-x)\left[\delta(x-y) - \frac{\sin k(x-y)}{k}u(y)\right]\psi(y,k)\,dy = e^{-ikx} \tag{5}$$

with a triangular matrix[2]

$$\theta(y-x)\left[\delta(x-y)-\sin\frac{k(x-y)}{k}u(y)\right]dy \tag{6}$$

in front of ψ and the nonhomogeneous term e^{-ikx}. Here $\theta(y-x)=1$ if $x\le y$ and 0 otherwise.

To solve (5) for $\psi(y,k)$, we need to multiply (5) by the "inverse" of (6). Although we do not know what the inverse of (6) is, we can suppose that (analogously to the ordinary matrices) the "inverse" of a triangular matrix is also triangular (this can actually be proved [**3**]), i.e., it must have the form

$$\theta(y-x)[\delta(x-y)+K(x,y)]\,dy \tag{7}$$

with some function $K(x,y)$, i.e.,

$$\begin{aligned}\psi(x,k)&=\int_{-\infty}^{+\infty}\theta(y-x)[\delta(x-y)+K(x,y)]e^{-iyk}\,dy\\&=e^{-ikx}+\int_x^{+\infty}K(x,y)e^{-iky}\,dy.\end{aligned} \tag{8}$$

Substituting (8) into (3) we obtain[3]

$$\begin{aligned}&-\left(2\frac{dK(x,x)}{dx}+u\right)e^{-ikx}+\lim_{s\to+\infty}\left[K(x,s)ik+\frac{\partial K(x,s)}{\partial s}\right]e^{-iks}\\&\quad+\int_x^{+\infty}\left[\frac{\partial^2K(x,s)}{\partial x^2}-\frac{\partial^2K(x,s)}{\partial s^2}-uK(x,s)\right]e^{-iks}\,ds=0.\end{aligned} \tag{9}$$

Since the functions e^{-ikx} are linearly independent (9) holds if and only if the coefficient of each one of them is zero, i.e.,

$$\begin{aligned}2\frac{dK(x,x)}{dx}+u&=0\\\lim_{s\to+\infty}\left[K(x,s)ik+\frac{\partial K(x,s)}{\partial s}\right]e^{-iks}&=0\\\frac{\partial^2K(x,s)}{\partial x^2}-\frac{\partial^2K(x,s)}{\partial s^2}-uK(x,s)&=0.\end{aligned} \tag{9a}$$

The set of all three equations comprises a nonlinear Goursat problem for $K(x,s)$; it has a solution and thus the triple of equations is not contradictory. The first of these equations gives us $u(x)$ in terms of the function $K(x,x)$. Thus, in order to find $u(x)$ we need to find all $\psi(x,k)$s or (which is the same) $K(x,y)$ which can be viewed as coefficients of the $\psi(x,k)$ in the basis e^{-iky}.

To see how it is being done, we look at the following:

Example 1. Let E be an $n \times n$ matrix whose columns form an orthonormal basis, and let Q be an upper-triangular matrix with ones on the main diagonal and small elements above it. Consider $G = QE$. Then we have

$$QEFE^* = Q^{*-1}$$

where $F = (G^*G)^{-1}$. If we know EFE^*, we can recover all the elements of Q. We already know that the diagonal elements of Q are ones, and the ones below the main diagonal are zeros. Since Q^{*-1} is lower triangular with ones on the main diagonal we obtain $(n-1)n/2$ linearly independent linear equations for $(n-1)n/2$ unknown elements of Q (above the main diagonal) by setting the diagonal elements of $QEFE^*$ to 1 and the ones above the main diagonal to 0. If the columns of E comprise the standard basis, i.e., if E = identity, the matrix EFE^* simply becomes F.

According to (8),

$$\psi(x,k) = \int_{-\infty}^{+\infty} \theta(y-x)(\delta(x-y) + K(x,y))e^{-iky}\, dy.$$

We can view this as an analogue of $G = QE$ in Example 1, with $\psi(x,k), e^{-iky}$ and $\theta(y-x)[\delta(x-y) + K(x,y)]\, dy$ corresponding respectively to QE, E and Q.

The analogue of the matrix G^*G in Example 1 would be the scalar product matrix

$$\mathcal{S}(k,l) = \langle \psi(x,k), \psi(x,l) \rangle = \int_{-\infty}^{+\infty} \psi(x,k)\overline{\psi(x,l)}\, dx,$$

whose entries are given by the formulas

$$\text{if } k, l \in \mathcal{C},$$

(10a)
$$\mathcal{S}(k,l) = \frac{2\pi}{1-|r(k)|^2}\delta(k-l) - \frac{2\pi\bar{r}(k)}{1-|r(k)|^2}\delta(k+l),$$
$$r(k) = \frac{b(k)}{a(k)}$$

(10b)
$$\text{if } k \in \mathcal{C}, \quad l \in \mathcal{D}, \qquad S(k,l) = 0.$$

(10c)
$$\text{if } k, l \in \mathcal{D}, \qquad \mathcal{S}(k,l) = -i\frac{a'(-k)}{b(-k)}\delta_{lk}, \qquad a'(-k) = \frac{\partial a(-k)}{\partial k}$$

where $\mathcal{D} = \{k \mid k \text{ is imaginary, } \operatorname{Im} k < 0,\ k^2 \in \text{discrete spectrum}\}$ and $\mathcal{C} = \{k \mid k \text{ is real, } k^2 \in \text{continuous spectrum}\}$.

Proof of (10a). $L\psi(x,k) = k^2\psi(x,k)$ implies $\langle L\psi(x,k), \psi(x,l)\rangle = k^2\langle\psi(x,k), \psi(x,l)\rangle$. Using $(L\psi(x,k))\overline{\psi(x,l)} = \psi(x,k)\overline{L\psi(x,l)} + (\partial/\partial x)(\psi(x,k)\partial\bar{\psi}(x,l)/\partial x - (\partial\psi(x,k)/\partial x)\psi(x,l))$ and $L\psi(x,l) = l^2\psi(x,l)$ we can rewrite this as

$$(k^2-l^2)\langle\psi(x,k),\psi(x,l)\rangle = \left(\psi(x,k)\frac{\partial\bar{\psi}(x,l)}{\partial x} - \frac{\partial\psi(x,k)}{\partial x}\bar{\psi}(x,l)\right)\Big|_{-\infty}^{+\infty}.$$

Behavior of $\psi(x,k)$ and $\psi(x,l)$ at $\pm\infty$ is given by the asymptotics in (3). Substituting those asymptotics into the righthand side of our expression, we arrive at the formula:

$$\begin{aligned}
&(k^2-l^2)\langle\psi(x,k),\psi(x,l)\rangle \\
&= i(k^2-l^2)\bigg\{\frac{e^{i(l-k)x}}{k-l} - a(-k)a(l)\frac{e^{-i(l-k)x}}{k-l} + b(k)b(-l)\frac{e^{i(l-k)x}}{k-l} \\
&\qquad + b(k)a(l)\frac{e^{i(l+k)x}}{k+l} - a(-k)b(-l)\frac{e^{-i(l+k)x}}{k+l}\bigg\}_{x\to+\infty} \\
&= (k^2-l^2)\bigg\{|a(k)|2\frac{e^{i(l-k)x} - e^{-i(l-k)x}}{i(l-k)} \\
&\qquad - a(-l)b(l)\frac{e^{i(l+k)} - e^{-i(l+k)x}}{i(l+k)}\bigg\}_{x\to+\infty} \\
&\quad + \text{terms weakly going to zero.}
\end{aligned}$$

Combining this with $(1/i(k\pm l))e^{i(k\pm l)x}|_{x\to+\infty} - (1/(i(k\pm l)))e^{i(k\pm l)x}|_{x\to-\infty} = \int_{-\infty}^{+\infty} e^{i(k\pm l)x}\,dx = 2\pi(\delta(k\pm l)$ we obtain (10a). $\square$

Proof of (10b). If $l \in \mathcal{D}$ and $k \in \mathcal{C}$ or $\mathcal{D}$, then repeating the argument at the beginning of the previous proof we get $(k^2 - l^2)\langle\psi(x,k),\psi(x,l)\rangle = \psi(x,k)\partial\bar{\psi}(x,l)/\partial x - (\partial\psi(x,k)/\partial x)\bar{\psi}(x,l)|_{-\infty}^{+\infty}$. Since $\lim_{x\to\pm\infty}\psi(x,l) = 0$, the righthand side vanishes giving us $\langle\psi(x,k),\psi(x,l)\rangle = 0$, provided $|k| \neq |l|$. $\square$

Proof of (10c). We have from the Proof of (10a) for all k and l: $\langle\psi(x,k),\psi(x,l)\rangle = \{e^{i(l-k)x}/(i(l-k)) - a(-k)a(l)e^{-i(l-k)x}/(i(l-k)) + b(k)b(-l)e^{i(l-k)x}/(i(l-k)) + b(k)a(l)e^{i(l+k)x}/(i(k+l)) - a(-k)b(-l)$ $e^{-i(l+k)x/(i(k+l))}\}_{x\to\infty}$. If $a(-k) = 0$ and $l \to -k$, the second and the last terms vanish, the first and third cancel out and $\langle\psi(x,k),\psi(x,l)\rangle = \lim_{l\to -k} b(k)a(l)e^{i(l+k)x}/(i(k+l)) = ia'(-k)b(k) = -ia'(-k)/(b(-k))$ because of $b(k) = -1/(b(-k))$ which in turn follows from (3c) and $a(-k) = 0$. Here $a'(-k) = \partial a(-k)/\partial k$.

The structure of the matrix $\mathcal{S}$ is "block-diagonal" with first 1×1 blocks $-i(a'(-k)/b(-k))\delta_{lk}$ along the main diagonal followed by "2×2 blocks"

$$\frac{2\pi}{1-|r(k)|^2}\delta(k-l) - \frac{2\pi\bar{r}(k)}{1-|r(k)|^2}\delta(k+l).$$

To understand why we say this we need to look at the discrete analogue of (10a) which is $(2\pi/(1-|r_k|^2))\delta_{kl} - (2\pi\bar{r}_k/(1-|r_k|^2))\delta_{k-l}$, $k,l \in \{1,-1,2,-2,\ldots,n,-n\}$. This expression gives us a block-diagonal matrix whose rows and columns are labeled with the numbers $1,-1,2,-2,\ldots,n,-n$ rather than $1,2,3,\ldots,2n$ as it is usually done. Analogue of the whole matrix $\mathcal{S}$ would look something like this:

$$S = \begin{pmatrix}
\alpha_1 & & & & & & & \\
& \ddots & & & & & 0 & \text{elsewhere} \\
& & \alpha_m & & & & & \\
& & & \frac{2\pi}{1-|r_1|^2} & \frac{-2\pi\bar{r}_1}{1-|r_1|^2} & & & \\
& & & \frac{-2\pi r_1}{1-|r_1|^2} & \frac{2\pi}{1-|r_1|^2} & & & \\
& & & & & \ddots & & \\
& & & & & & \frac{2\pi}{1-|r_n|^2} & \frac{-2\pi\bar{r}_n}{1-|r_n|^2} \\
0 & \text{elsewhere} & & & & & & \\
& & & & & & \frac{-2\pi r_n}{1-|r_n|^2} & \frac{2\pi}{1-|r_n|^2}
\end{pmatrix}$$

with rows and columns labeled appropriately.

The analogue of F in Example 1 would be the spectral matrix $\mathcal{F}(l,m)$ defined as the inverse of $\mathcal{S}$, i.e., the solution of

$$\underset{l\in D\cup\mathcal{C}}{\mathcal{E}} \mathcal{S}(k,l)\mathcal{F}(l,m) = \tilde{\delta}(k,m) \tag{11}$$

where[4]

$$\begin{aligned}\underset{l\in\mathcal{D}\cup\mathcal{C}}{\mathcal{E}} \mathcal{S}(k,l)\mathcal{F}(l,m) = \sum_{l\in\mathcal{D}} &\mathcal{S}(k,l)\mathcal{F}(l,m)\\ &+\int_{l\in\mathcal{C}} \mathcal{S}(k,l)\mathcal{F}(l,m)\,dl\end{aligned}$$

and

$$\tilde{\delta}(k,m) = \begin{cases} \delta_{km} & k,m\in\mathcal{D},\\ \delta(k-m) & k,m\in\mathcal{C},\\ 0 & k\in\mathcal{D},\ m\in\mathcal{C} \text{ or } k\in\mathcal{C},\ m\in\mathcal{D}.\end{cases}$$

The analogue of EFE^* in Example 1 would be the matrix $\Phi(y,z)$ defined to be[5]

$$\Phi(y,z) = \underset{l\in\mathcal{D}\cup\mathcal{C}}{\mathcal{E}}\ \underset{m\in\mathcal{D}\cup\mathcal{C}}{\mathcal{E}} \overline{e^{-ily}}\mathcal{F}(l,m)e^{-imz}.$$

One can easily verify that

$$\mathcal{F}(l,k) = \begin{cases} (ib(-k)/a'(-k))\delta_{lk} & l,k\in\mathcal{D},\\ (1/2\pi)\delta(l-k) + (1/2\pi)\bar{r}(k)\delta(k+l) & l,k\in\mathcal{C},\\ 0 & l\in\mathcal{D},\\ & k\in\mathcal{C} \text{ or vice versa.}\end{cases} \tag{12}$$

To better understand how we arrive at (12) let us look at a finite dimensional analogue. If S is as before,

$$S^{-1} = \begin{pmatrix} \frac{1}{\alpha_1} & & & & & & & \\ & \ddots & & & & & 0 & \text{elsewhere} \\ & & \frac{1}{\alpha_m} & & & & & \\ & & & \frac{1}{2\pi} & \frac{\bar{r}_1}{2\pi} & & & \\ & & & \frac{r_1}{2\pi} & \frac{1}{2\pi} & & & \\ & & & & & \ddots & & \\ & & & & & & \frac{1}{2\pi} & \frac{\bar{r}_n}{2\pi} \\ 0 & \text{elsewhere} & & & & & \frac{r_n}{2\pi} & \frac{1}{2\pi} \end{pmatrix}.$$

Replacing blocks

$$\begin{pmatrix} \dfrac{1}{2\pi} & \dfrac{\bar{r}_k}{2\pi} \\ \dfrac{r_k}{2\pi} & \dfrac{1}{2\pi} \end{pmatrix} \quad \text{with} \quad \frac{1}{2\pi}\delta(k-l) + \frac{1}{2\pi}\bar{r}(k)\delta(k+l)$$

we obtain $\mathcal{F}$.

Correspondingly,

$$\begin{aligned} \Phi(y,z) = \delta(y-z) - \sum_{k\in\mathcal{D}} \frac{b(-k)}{i(\partial a(-k)/\partial(-k))} e^{-ik(y+z)} \\ + \frac{1}{2\pi}\int_{-\infty}^{+\infty} r(k)e^{ik(y+z)}\,dz. \end{aligned} \tag{13}$$

Since the elements $\Phi(y,z)$ are analogous to the entries of EFE^* in Example 1, the expression $QEFE^*$ corresponds to

$$\int_{-\infty}^{+\infty} \theta(y-x)[\delta(x-y) + K(x,y)]\Phi(y,z)\,dz,$$

and in the spirit of Example 1 it must satisfy

$$\int_x^{\infty} [\delta(x-z) + K(x,z)]\Phi(y,z)\,dz = 0 \quad \text{for } x < y.$$

This identity can be rewritten as

$$K(x,y) + \tilde{F}(x+y) + \int_x^{\infty} K(x,z)\tilde{F}(z+y)\,dz = 0, \tag{14}$$

where $\tilde{F}(x+y) = \Phi(y,z) - \delta(x-y)$.

Equation (14) is the celebrated Gelfand-Levitan-Marchenco equation.

Usually $K(x,y)$ is obtained by solving the integral equation (14). How to do it is described in a variety of books on integral equations and the inverse scattering theory.

Now let L be the operator defined by (3a) with u dependent on time. Then the eigenvalues of L, i.e., points of the discrete spectrum, are time-independent and are constants of motion. Indeed, $L\psi = \lambda\psi$. Multiplying this by ψ we obtain $\langle\psi, L\psi\rangle = \lambda(\psi,\psi)$. Differentiating in time and using $\dot{L} = [L, A]$, we obtain $\langle\dot{\psi}, L\psi\rangle + \langle\psi, LA\psi\rangle - \langle\psi, AL\psi\rangle + \langle\psi, L\dot{\psi}\rangle = \dot{\lambda}\langle\psi,\psi\rangle + \lambda\langle\dot{\psi},\psi\rangle + \lambda\langle\psi,\dot{\psi}\rangle$. Because L is self-adjoint on $L^2(\mathbf{R})$ and $\langle\dot{\psi}, L\psi\rangle = \lambda\langle\dot{\psi},\psi\rangle$, we have that $\langle\psi, L\dot{\psi}\rangle = \langle L\psi,\dot{\psi}\rangle = \lambda\langle\psi,\dot{\psi}\rangle$ and $\langle\psi, LA\psi\rangle - \langle\psi, AL\psi\rangle = \langle L\psi, A\psi\rangle - \langle\psi, AL\psi\rangle = \lambda\langle\psi, A\psi\rangle - \lambda\langle\psi, A\psi\rangle = 0$. All terms but one cancel out leaving us with $0 = \dot{\lambda}\langle\psi,\psi\rangle$ and since $\langle\psi,\psi\rangle \neq 0$ we must have $\dot{\lambda} = 0$ and therefore eigenvalues of L must be time-independent. The continuous spectrum of L continuous fills up the positive semi-axis; it can also be considered time independent.

The second consequence of (2) is that if ψ is an eigenfunction of L corresponding to λ, then so is $\dot{\psi} + A\psi$, i.e.,

$$(L-\lambda)(\dot{\psi} + A\psi) = 0$$

which implies that $\dot{\psi} + A\psi$ is proportional to ψ. To see it, we differentiate in time $L\psi = \lambda\psi$ and use $\dot{L} = [L, A]$ to get $\dot{L}\psi + L\dot{\psi} = \lambda\dot{\psi}$. Replacing $\dot{L}$ with $[L, A]$ gives use $LA\psi - AL\psi - (L-\lambda)\dot{\psi} = 0$ or $(L-\lambda)(\dot{\psi} + A\psi) = 0$.

Since $\dot{\psi} + A\psi$ is in the eigenspace of k^2, it must be a linear combination of $\psi(x,k)$ and $\psi(x,-k)$

$$\dot{\psi} + A\psi = \alpha\psi(x,k) + \beta\psi(x,-k).$$

For $x \to +\infty$, $\dot{\psi} + A\psi \sim 4ik^3e^{-ikx}$ and $\alpha\psi(x,k) + \psi(x,-k) \sim \alpha e^{-ikx} + \beta e^{ikx}$, so we must have $\alpha = 4ik^3$, $\beta = 0$, which gives us

$$\dot{\psi} + A\psi = 4ik^3\psi, \quad k^2 = \lambda. \tag{15}$$

As x approaches $-\infty$, this equation gives us

$$\dot{a}(-k)e^{-ikx} - \dot{b}(k)e^{ikx} = \left(-4\frac{\partial^3}{\partial x^3} + 4ik^3\right)(a(-k)e^{-ikx} - b(k)e^{ikx})$$

which in turn yields

$$\dot{a}(k,t) = 0, \qquad \dot{b}(k,t) = 8ik^3b(k,t)$$

or

$$a(k,t) = a(k,0); \qquad b(k,t) = b(k,0)e^{8ik^3t}. \tag{16}$$

Remark. If L and A were $n \times n$ matrices we would have been able to solve (2) for L in the form $L(t) = (\psi\beta)L(0)(\psi\beta)^{-1}$, with $(\psi\beta) = -A(\psi\beta)$. Indeed, assuming for simplicity that all eigenvalues of L are simple, we would have the jth eigenvalue of L, ψ_j and corresponding eigenvector ψ_j satisfy $(L - \lambda_j)(\dot{\psi}_j + A\psi_j) = 0$ and therefore $\dot{\psi}_j + A\psi_j = \beta_j\psi_j$ for some β_j because each λ_j is simple. The last identity is equivalent to $(d/dt)(\psi\beta) = -A\psi\beta$ where ψ is the matrix whose columns are the ψ_js and $\beta = \operatorname{diag}(e^{-\beta_1 t}, \dots, e^{-\beta_n t})$. The solution of $\dot{L} = [L, A]$ then would be given as $L(t) = (\psi\beta)L(0)(\psi\beta)^{-1}$.

To determine $u(t,x)$ all we have to do is to replace a, b and r in (10) and the part of Section 1 thereafter with the $a(k,t)$, $b(k,t)$ and $r(k,t) = b(k,t)/a(k,t)$ given by (16). To solve the KdV for some given initial data $u(x,0)$ we have to compute $a(k,0)$, $b(k,0)$ and $r(k,0)$ corresponding to $u(x,0)$ and then use the Gelfand-Levitan-Marchenco equation with $a(k,t)$, $b(k,t)$ and $r(k,t)$ given by (16) to determine the function $K(t,x,s)$. The solution of the KdV then is obtained via $u(x,t) = -2dK(t,x,x)/dx$.

2. IST for the first order systems. In this section we apply the ideas of Section 1 to the first order systems. For simplicity's sake, we take a pair of 2×2 equations

$$\psi_x = U\psi, \qquad U = \begin{pmatrix} 0 & q(x) \\ r(x) & 0 \end{pmatrix}, \qquad \lim_{x\to\pm\infty} U(x) = 0$$
$$\psi_t = -T\psi, \qquad \lim_{x\to\pm\infty} T(x) = \begin{pmatrix} -A & 0 \\ 0 & A \end{pmatrix}.$$

The compatibility condition for them is an equation of motion $U_t = [U + \partial/\partial x, T]$ that can be solved with the IST. Let

$$\lambda_l = \lambda \begin{pmatrix} \delta_{l1} \\ \delta_{l2} \end{pmatrix}, \quad \lambda \in \mathbf{C},\ l = 1 \text{ or } 2,$$

and $\mu_m = (\delta_{m1}\delta_{m2})\mu$, $\mu \in \mathbf{C}$, $m = 1$ or 2 and let $\psi(x, \lambda_l)$ and $\varphi(\mu_m, x)$ be the solutions of the equations:

$$-\frac{\partial \psi(x,\lambda_l)}{\partial x} = U\psi(x,\lambda_l) + i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi(x,\lambda_l) \tag{1a}$$

$$\psi(x,\lambda_l) \sim \begin{pmatrix} e^{-i\lambda x} & 0 \\ 0 & e^{i\lambda x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \delta_{l1} \\ \delta_{l2} \end{pmatrix}, \quad x \to +\infty \tag{1b}$$

$$\begin{aligned} \frac{\partial \varphi(\mu_m,x)}{\partial x} &= \varphi(\mu_m,x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\quad + i\mu\varphi(\mu_m,x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{2a}$$

$$\varphi(\mu_m,x) \sim (\delta_{m1},\delta_{m2}) \begin{pmatrix} e^{i\mu x} & 0 \\ 0 & e^{-i\mu x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x \to +\infty. \tag{2b}$$

In addition these functions satisfy

$$\begin{aligned} \psi(x,\lambda_l) \sim &\begin{pmatrix} e^{-i\lambda x} & 0 \\ 0 & e^{i\lambda x} \end{pmatrix} \begin{pmatrix} \tilde{a}(\lambda) & \tilde{b}(\lambda) \\ -b(\lambda) & a(\lambda) \end{pmatrix} \\ &\cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \delta_{l1} \\ \delta_{l2} \end{pmatrix}, \quad x \to -\infty \end{aligned} \tag{1c}$$

$$\begin{aligned} \varphi(\mu_m,x) \sim &(\delta_{m1}\delta_{m2}) \begin{pmatrix} a(\mu) & -\tilde{b}(\mu) \\ b(\mu) & \tilde{a}(\mu) \end{pmatrix} \\ &\cdot \begin{pmatrix} e^{i\mu x} & 0 \\ 0 & e^{-i\mu x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x \to -\infty \end{aligned} \tag{2c}$$

with $a(\lambda)\tilde{a}(\lambda) + b(\lambda)\tilde{b}(\lambda) = 1$.

Reasoning for (1c) is the same as for (1)–(3c). To prove (2c), we let $\psi(x,\lambda)$ be the matrix whose lth column is $\psi(x,\lambda_l)$ and $\varphi(\mu,x)$ the matrix whose mth row is $\varphi(m_\mu,x)$. then

$$\begin{aligned} &\frac{\partial}{\partial x}\left[\varphi(\mu,x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi(x,\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\ &\qquad = i(\mu-\lambda)\varphi(\mu,x)\psi(x,\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned}$$

because of (1a) and (2a). Using this formula with $\lambda = \mu$ and the conditions (1b) and (2b), we obtain

$$\varphi(\lambda, x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi^{-1}(x, \lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and (2c) follows.

The functions $\psi(x, \lambda_l)$ and $\varphi(\mu_m, x)$ can be written as

$$\begin{aligned} \psi(x, \lambda_l) &= \left\{ \begin{pmatrix} e^{-i\lambda x} & 0 \\ 0 & e^{i\lambda x} \end{pmatrix} + \int_x^{\infty} K(x, s) \begin{pmatrix} e^{-i\lambda s} & 0 \\ 0 & e^{i\lambda s} \end{pmatrix} ds \right\} \\ &\quad \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \delta_{l1} \\ \delta_{l2} \end{pmatrix} \\ &= \int_{-\infty}^{+\infty} \theta(s - x) \left[\begin{pmatrix} \delta(x-s) & 0 \\ 0 & \delta(x-s) \end{pmatrix} + K(x, s) \right] \\ &\qquad \cdot \begin{pmatrix} e^{-i\lambda s} & 0 \\ 0 & e^{i\lambda s} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \delta_{l1} \\ \delta_{l2} \end{pmatrix} ds \end{aligned} \tag{1d}$$

and

(2d)

$$\begin{aligned} \varphi(\mu_m, x) = (\delta_{m1} \delta_{m2}) \int_{-\infty}^{+\infty} \theta(x - s) &\left[\begin{pmatrix} \delta(s-x) & 0 \\ 0 & \delta(x-s) \end{pmatrix} + \tilde{K}(x, s) \right] \\ &\cdot \begin{pmatrix} e^{i\mu x} & 0 \\ 0 & e^{-i\mu x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ds \end{aligned}$$

with 2×2 matrices $K(x, s)$ and $\tilde{K}(x, s)$. The proof is similar to that of (1)–(5).

Once we know $K(x, s)$ we can determine U similarly to how we did for (1)–(9a), so all we need to do is to determine the $K(x, s)$.

We get bounded solutions if $\lambda_l \in \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ and $\mu_m \in M_0 \cup M_1 \cup M_2$, where $\lambda_0 = \{\lambda_l \mid \lambda \in \mathbf{R}, l = 1 \text{ or } 2\}$, $\Lambda_1 = \{\lambda_l \mid l = 1, \tilde{a}(\lambda) = 0, \operatorname{Re} i\lambda > 0\}$, $\Lambda_2 = \{\lambda_l \mid l = 2, a(\lambda) = 0, \operatorname{Re} i\lambda < 0\}$, $M_0 = \{\mu_m \mid \mu \in \mathbf{R}, m = 1 \text{ or } 2\}$, $M_1 = \{\mu_m \mid m = 1, a(\mu) = 0, \operatorname{Re} i\mu < 0\}$, $M_2 = \{\mu_m \mid m = 2, \tilde{a}(\mu) = 0, \operatorname{Re} i\mu > 0\}$.

For $\varphi(\mu_m, x)$, $\psi(x, \lambda_l)$ bounded at $\pm\infty$, we define $S(\mu_m, \lambda_l) = \int_{-\infty}^{+\infty} \varphi(\mu_m, x) \psi(x, \lambda_l)\, dx$.

The nonzero entries of $\mathcal{S}$ are:

if $\lambda_l \in \Lambda_0$, $\mu_m \in M_0$, $r(\lambda) = b(\lambda)/a(\lambda)$, $\tilde{r}(\lambda) = \tilde{b}(\lambda)/\tilde{a}(\lambda)$,

$$\mathcal{S}(\mu_m, \lambda_l) = \frac{2\pi\delta(\lambda-\mu)}{1+\tilde{r}(\lambda)r(\lambda)}(\delta_{m1}\delta_{m2})\begin{pmatrix} -1 & -\tilde{r}(\lambda) \\ r(\lambda) & -1 \end{pmatrix}\begin{pmatrix} \delta_{l1} \\ \delta_{l2} \end{pmatrix}; \tag{3a}$$

if $\kappa \in \Lambda_1$, $\kappa_2 \in M_2$,

$$\mathcal{S}(\kappa_2, \kappa_1) = i\frac{\tilde{a}'(\kappa)}{\tilde{b}(k)}; \tag{3b}$$

if $\kappa_2 \in \Lambda_2$, $\kappa_1 \in M_1$,

$$\mathcal{S}(\kappa_1, \kappa_2) = -\frac{ia'(\kappa)}{b(\kappa)}. \tag{3c}$$

Proof of (3a). Let $\psi(x,\lambda)$, $\varphi(\mu,x)$ be as defined after (2c). Then

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \varphi(\mu,x)\psi(x,\lambda)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dx \qquad (4)\\
&= \frac{1}{i(\mu-\lambda)}\varphi(\mu,x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\psi(x,\lambda)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Bigg|_{-\infty}^{+\infty} \\
&= \frac{1}{i(\mu-\lambda)}\Bigg\{\begin{pmatrix} e^{-i(\lambda-\mu)x} & 0 \\ 0 & e^{i(\lambda-\mu)x} \end{pmatrix}_{x\to+\infty} \\
&\qquad - \begin{pmatrix} a(\mu)\tilde{a}(\lambda)e^{i(\mu-\lambda)x} + \tilde{b}(\mu)b(\lambda)e^{-i(\mu-\lambda)x} & -a(\mu)\tilde{b}(\lambda)e^{i(\mu-\lambda)x} + \tilde{b}(\mu)a(\lambda)e^{-i(\mu-\lambda)x} \\ b(\mu)\tilde{a}(\lambda)e^{i(\mu-\lambda)x} - \tilde{a}(\mu)b(\lambda)e^{-i(\mu-\lambda)x} & -b(\mu)\tilde{b}(\lambda)e^{i(\mu-\lambda)x} - \tilde{a}(\mu)a(\lambda)e^{-i(\mu-\lambda)x} \end{pmatrix}\Bigg\}_{x\to-\infty} \\
&= \frac{1}{i(\mu-\lambda)}\left\{\begin{matrix} [1-\tilde{b}(\lambda)b(\mu)]e^{-i(\lambda-\mu)x} - \tilde{a}(\mu)a(\lambda)e^{i(\lambda-\mu)x} & -\tilde{b}(\mu)a(\lambda)e^{-i(\lambda-\mu)x} + a(\mu)\tilde{b}(\lambda)e^{i(\lambda-\mu)x} \\ \tilde{a}(\lambda)b(\mu)e^{i(\lambda-\mu)x} - b(\lambda)\tilde{a}(\mu)e^{-i(\lambda-\mu)x} & -[1-b(\lambda)\tilde{b}(\mu)]e^{-i(\lambda-\mu)x} + \tilde{a}(\lambda)a(\mu)e^{i(\lambda-\mu)x} \end{matrix}\right\}_{x\to+\infty}
\end{aligned}$$

If λ and μ are real, we can write it as

$$\frac{1}{i(\mu-\lambda)}\left\{\begin{matrix} e^{i(\lambda-\mu)x} - b(\lambda)\tilde{b}(\lambda)e^{i(\lambda-\mu)x} - a(\lambda)\tilde{a}(\lambda)e^{-i(\lambda-\mu)x} & a(\lambda)\tilde{b}(\lambda)(e^{i(\lambda-\mu)x} - e^{-i(\lambda-\mu)x}) \\ -b(\lambda)\tilde{a}(\lambda)e^{-i(\lambda-\mu)x} - e^{i(\lambda-\mu)x}) & -e^{-i(\lambda-\mu)x} + b(\lambda)\tilde{b}(\lambda)e^{-i(\lambda-\mu)x} + a(\lambda)\tilde{a}e^{i(\lambda-\mu)x} \end{matrix}\right\}_{x\to+\infty},$$

with positive terms weakly going to zero as

$$x \to +\infty = \frac{e^{i(\lambda-\mu)x} - e^{-i(\lambda-\mu)x}}{i(\mu-\lambda)}\Big|_{x\to+\infty} \begin{pmatrix} a(\lambda)\tilde{a}(\lambda) & a(\lambda)\tilde{b}(\lambda) \\ -b(\lambda)\tilde{a}(\lambda) & a(\lambda)\tilde{a}(\lambda) \end{pmatrix}$$
$$= \frac{2\pi\delta(\lambda-\mu)}{1+\tilde{r}(\lambda)r(\lambda)} \begin{pmatrix} -1 & -\tilde{r}(\lambda) \\ r(\lambda) - 1 & \end{pmatrix}$$

and this implies (3a). □

Proof of (3b). $\mathcal{S}(\mu_1, \lambda_2)$ is the element in the first column and second row of (4) in the Proof of (3a). Set $\lambda = \kappa$ and take

$$\lim_{x\to\infty} \lim_{\mu\to\kappa} \frac{\tilde{a}(\kappa)b(\mu)e^{i(\kappa-\mu)x} - b(\kappa)\tilde{a}(\mu)e^{-i(\kappa-\mu)x}}{i(\mu-\kappa)} = -\frac{b(\kappa)\tilde{a}(\kappa)}{i}.$$

Since $\tilde{a}(\kappa) = 0$, $\tilde{b}(\kappa)b(\kappa) = 1$ and we obtain (3b). □

Proof of (3c). Similar to that of (3b). □

Define $\mathcal{F}(\lambda_l, \mu_m)$ to be the inverse of $S(\mu_m, \lambda_l)$, i.e., the solution of $\mathcal{E}_{\mu_m} \mathcal{F}(\lambda_l, \mu_m) S(\mu_m, \nu_n) = \tilde{\delta}_{\lambda_l \nu_n}$, where

$$\tilde{\delta}_{\lambda_l \nu_n} = \begin{cases} \delta_{ln}\delta(\lambda-\nu) & \text{if } \lambda_l, \nu_n \in \Lambda_0 \\ 1 & \text{if } \lambda_l = \nu_n \in \Lambda_1 \\ & \text{or } \lambda_l = \nu_n \in \Lambda_2 \\ 0 & \text{otherwise,} \end{cases}$$

and by $\mathcal{E}_{\mu_m}$ we mean generalization of summation defined as

$$\mathop{\mathcal{E}}_{\mu_m} \mathcal{F}(\lambda_l, \mu_m) S(\mu_m, \nu_n) = \int_{-\infty}^{+\infty} \sum_{m=1}^{2} \mathcal{F}(\lambda_l, \mu_m) S(\mu_m, \nu_n)\, d\mu$$
$$+ \sum_{\mu_1 \in M_1} \mathcal{F}(\lambda_l, \mu_1) S(\mu_1, \nu_n)$$
$$+ \sum_{\mu_1 \in \mu_2} \mathcal{F}(\lambda_l, \mu_2) S(\mu_2, \nu_n).$$

The entries of $\mathcal{F}$ are:

if $\lambda_l \in \Lambda_0$, $\mu_m \in M_0$,

$$\mathcal{F}(\lambda_l, \mu_m) = \frac{1}{2\pi}\delta(\lambda-\mu)^{(\delta_{l1}\delta_{l2})}\begin{pmatrix} 1 & -\tilde{r}(\lambda) \\ r(\lambda) & 1 \end{pmatrix}\begin{pmatrix} \delta_{m1} \\ \delta_{m2} \end{pmatrix} \tag{5a}$$

if $\kappa_1 \in \Lambda_1$, $\kappa_2 \in M_2$,

$$\mathcal{F}(\kappa_1, \kappa_2) = \frac{\tilde{b}(\kappa)}{i\tilde{a}'(\kappa)} \tag{5b}$$

if $\kappa_2 \in \Lambda_2$, $\kappa_1 \in M_1$,

$$\mathcal{F}(\kappa_2, \kappa_1) = -\frac{b(\kappa)}{ia'(\kappa)}; \tag{5c}$$

all the other entries are zeros.

The linear algebraic analogue of what we just did is as follows.

Example 2. Let Q and Q' be upper triangular matrices with ones on the main diagonal, and let E and E' be two matrices such that columns of each one of them form an orthonormal basis. Define $G = QEN^{-1}$ and $G' = E'Q'N$ with N being an invertible diagonal matrix. Also define $S = G'G$ and $F = S^{-1} = NE^{-1}Q^{-1}N^{-1}Q'^{-1}E'^{-1}$. Then the matrix $QEFE' = QENE^{-1}Q^{-1}N^{-1}Q'^{-1} = Q'^{-1} + Q''$, where Q'' is an upper triangular matrix with zeros on the main diagonal. Thus $QEFE'$ is an upper triangular matrix with ones on the main diagonal, and we can determine Q the same way as we did in Example 1.

The analogy between Example 2 and the quantities described above is as follows:

$\psi(x,\lambda_l)$ and $\varphi(\mu_m, x)$ correspond to	G and G'
$\theta(s-x)\left[\delta(x-s)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + K(x,s)\right]$	Q
$\begin{pmatrix} e^{-i\lambda s} & 0 \\ 0 & e^{i\lambda s} \end{pmatrix}\begin{pmatrix} \delta_{l1} \\ \delta_{l2} \end{pmatrix}$	E
$(\delta_{m1}\delta_{m2})\begin{pmatrix} e^{i\mu x} & 0 \\ 0 & e^{-i\mu x} \end{pmatrix}$	E'
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\delta(x-y)$	N and N^{-1}
$\mathcal{S}, \mathcal{F}$ and Φ	S, F and EFE'

where $\Phi(x,y)$ is

$$\int_{-\infty}^{+\infty}\begin{pmatrix} e^{i\lambda(s-y)} & -\tilde{r}(\lambda)e^{i\lambda(s+y)} \\ r(\lambda(e^{i\lambda(s+y)} & e^{-i\lambda(s-y)} \end{pmatrix} d\lambda$$
$$+\begin{pmatrix} 0 & -\sum_{\lambda_1\in\Lambda_1}(\tilde{b}(\lambda_1)/(i\tilde{a}'(\lambda_1)))e^{-i\lambda_1(s+y)} \\ \sum_{\lambda_2\in\Lambda_2}(b(\lambda_2)/(ia'(\lambda_2)))e^{i\lambda_2(s+y)} & 0 \end{pmatrix}$$

The expression $\Phi(x,y)+\int_x^\infty K(x,s)\Phi(s,y)\,ds$ is analogous to $QEFE'$ and thus must be zero for $y<x$. If we introduce $\tilde{F}(x+y)=\Phi(x,y)-\delta(x-y)$ we obtain the Gelfand-Levitan-Marchenco equation $K(x,y)+\tilde{F}(x+y)+\int_x^\infty K(x,s)\tilde{F}(s+y)\,ds=0$.

So far we have dealt with the scattering problem for time-independent solutions of $\psi_x=-U\psi$ only. The functions $\psi(x,\lambda_l)$, however, cannot satisfy the second equation $\psi_t=-T\psi$. To remedy the problem the standard trick is to introduce $\psi^{(t)}(x,\lambda_1)=e^{At}\psi(x,\lambda_1)$, $\psi^{(t)}(x,\lambda_2)=e^{-At}\psi(x,\lambda_2)$, $\varphi^{(t)}(\mu_1,x)=e^{-At}\varphi(\mu_1,x)$, $\varphi^{(t)}(\mu_2,x)=e^{+At}\varphi(\mu_2,x)$.

For $x\to-\infty$,

$$\psi^{(t)}(x,\lambda_l)\sim\begin{pmatrix} \tilde{a}e^{-i\lambda x+At} & -\tilde{b}e^{-i\lambda x-At} \\ -be^{i\lambda x+At} & -ae^{i\lambda x-At} \end{pmatrix}\begin{pmatrix} \delta_{l1} \\ \delta_{l2} \end{pmatrix}.$$

Substituting this into

$$\psi_t^{(t)}(x,\lambda_l)=\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}\psi^{(t)}(x,\lambda_l),$$

which is the form of $\psi_t^{(t)}=-T\psi^{(t)}$ for large x, we obtain equations for the scattering coefficients:

$$\begin{aligned} \dot{\tilde{a}}(\lambda,t)=\dot{a}(\lambda,t)&=0 \\ 2Ab(\lambda,t)+\dot{b}(\lambda,t)&=0 \\ 2Ab(\lambda,t)-\dot{\tilde{b}}(\lambda,t)&=0, \end{aligned}$$

and therefore $a(\lambda,t)=a(\lambda,0)$; $\tilde{a}(\lambda,t)=\tilde{a}(\lambda,0)$; $b(\lambda,t)=e^{-2At}b(\lambda,0)$; $\tilde{b}(\lambda,t)=e^{2At}\tilde{b}(\lambda,0)$. All we have to do now is to rewrite all the formulas with $\psi(x,\lambda_l)$, $\varphi(\mu_m,x)$, $a(\lambda)$, $\tilde{a}(\lambda)$, $b(\lambda)$, $\tilde{b}(\lambda)$ replaced by $\psi^{(t)}(x,\lambda_l)$, $\varphi^{(t)}(\mu_m,x)$, $a(\lambda)$, $\tilde{a}(\lambda)$, $e^{-2At}b(\lambda)$, $e^{2At}\tilde{b}(\lambda)$.

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ENDNOTES

1. We write the coefficients (3c) in this somewhat strange manner to be in agreement with the standard notation used in the existing literature.

2. The word matrix here should not be understood in the regular sense, it is rather a function of two variables one of which can be understood as the column number and the other one as the row number. One can think of the expression (6) as the matrix element in the xth row and yth column. Similarly, $\psi(y,k)$ can be viewed as the element of a matrix ψ in the yth row and kth column and e^{-ikx} as the element of a matrix in the xth row and kth column.

3. Using the following identities:

$$\begin{aligned}
\frac{\partial\psi}{\partial x} &= -ike^{-ikx} - K(x,x)e^{-ikx} + \int_x^\infty \frac{\partial K(x,s)}{\partial x} e^{-iks}\,ds,\\
\frac{\partial^2\psi}{\partial x^2} &= -k^2e^{-ikx} - \frac{dK(x,x)}{dx}e^{-ikx} + ikK(x,x)e^{-ikx}\\
&\quad - \frac{\partial K(x,s)}{\partial x}e^{-ikx}\Big|_{s=x} + \int_x^\infty \frac{\partial^2 K(x,s)}{\partial x^2}e^{-iks}\,ds,\\
k^2\psi &= k^2e^{-kx} - \int_x^\infty K(x,s)\frac{\partial^2 e^{-iks}}{\partial s^2}\,ds\\
&= k^2e^{-ikx} - \int_x^\infty \frac{\partial^2 K(x,s)}{\partial s^2}e^{-iks}\\
&\quad + \int_x^\infty \frac{\partial}{\partial s}\left[K(x,s)ike^{-iks} + \frac{\partial K(x,s)}{\partial s}e^{-iks}\right]ds\\
&= k^2e^{-ikx} - \int_x^\infty \frac{\partial^2 K(x,s)}{\partial s^2}e^{-iks}\,ds\\
&\quad - K(x,x)ike^{-ikx} - \frac{\partial K(x,s)}{\partial s}e^{-ikx}\Big|_{s=x}\\
&\quad + \lim_{s\to+\infty}\left[K(x,s)ik + \frac{\partial K(x,s)}{\partial s}\right]e^{-iks}.
\end{aligned}$$

4. One can use the Riemann-Stieltjes integral to replace $\mathcal{E}$.

5. In $\Phi(y,z)$, y corresponds to the row number and z to the column number. In $\mathcal{F}(l,m)$, l corresponds to the row number and m to the column number.

REFERENCES

1. M.J. Ablowitz and P.A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, London Math. Soc. Lecture Notes Ser. **149** (1991).

2. M.J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, SIAM, Stud. Appl. Math. (1985).

3. I.M. Gelfand and B.M. Levitan, *On the determination of a differential equation from its spectral function*, Amer. Math. Soc. Trans. Ser. 2 **1** (1995), 253–304.

4. S. Novikov, S.V. Manakov, L.P. Pitaevskii and V.E. Zakharov, *Theory of solitons*: *The inverse scattering method*, English translation, 1984.

5. H. Sabatier, *Inverse scattering problem*, Springer-Verlag, New York, 1981.

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1