ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 4, Fall 1997

THE DENTABILITY IN THE SPACES WITH TWO TOPOLOGIES

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ABSTRACT. Following the purely geometric way, we study here the notion of r-dentability in the locally convex spaces (lcs) with two comparative topologies, as a generalization of the dentability and w^*-dentability. For this purpose, the notion of r-dentable points is introduced as well as the notions of r- and $E'_r(M)$-strongly exposed points. The study is based on the method resulting from two lemmas of R.R. Phelps [**6**]. S. Gjinushi has given a local version of the first lemma in a normed vector space [**3**] and E. Saab [**7**] has proved that this lemma is true in a barreled locally convex Hausdorff space. Here it will be proved that the first lemma of Phelps is true in every topological vector space supplied with two comparative topologies; the other lemma as well will be proved in locally convex (topological) spaces with two comparative topologies. Two theorems are obtained for r-dentability in a local vector space with two topologies in which the stronger topology is that of the type B.M. [**7**]. Then these two theorems imply six known theorems for dentability and w^*-dentability as corollaries.

1. Introduction. Let E be a vector space and r and r_0 two topologies in it, in connection with which E is a topological vector space (tvs). We consider that $r < r_0$. The tvs (E, r_0) will be denoted by E and (E, r) by E_r, while their topological duals are denoted each by E' or E'_r, respectively. Since $r < r_0$ we have that $E'_r \subset E'$. The points of E will be denoted by x, while those of E'_r or E' by x'. The bounded (closed) sets in tvs E_r will be called r-bounded (r-closed), while those in tvs E, bounded (closed). Then $\gamma(0)$ denotes the system of neighborhoods of the origin in the tvs E. For every r-bounded $B \subset E$, and for every $x' \in E'_r$ and $a > 0$, the set

$$\{x \in B \mid x'(x) \geq \sup x'(B) - a\}$$

will be called a slice of B and will be denoted by $T(x', a, B)$.

Received by the editors on May 27, 1992, and in revised form on March 26, 1995.

Definition 1. An r-bounded set $B \subset E$ is called an r-*dentable set* in the tvs E, if for every neighborhood $V \in \gamma(0)$, there exists a slice $T(x', a, B)$ of B which is "V-small," i.e.,

$$\forall V \in \gamma(0) \ni x' \in E'_r \ni a > 0, \quad T - T \subset V.$$

As in proposition 2.3.2 [**1**], it can be proved that the set B is r-dentable if and only if, when

$$\forall V \in \gamma(0) \ni x \in B, \quad x \notin \overline{C}^r(B \backslash x + V),$$

where $\overline{C}^r(B \backslash x + V)$ is the r-closed convex hull of the set $(B \backslash x + V)$.

Definition 2. Let σ be a system of r-bounded sets in E. The tvs E is called $\sigma - r$-*dentable* if every set $B \in \sigma$ is r-dentable. The tvs E is called r-*dentable* if every r-bounded set in it is r-dentable.

Remark 1. For $r = r_0$ in Definition 1 the notion of the dentability for $B \subset E$ is obtained. Taking the dual space E', $r = w^* = \sigma(E', E)$ and $r_0 = \beta(E, E')$ (the strong topology in E'), implies the notion of w^*-dentability of $B \subset E'$. Note that if in the first case the σ-system is the system of bounded sets in the tvs E, and in the second case the σ-system is the system of equicontinuous sets in the tvs E', then Definition 2 leads to the notions of the dentability of E [**3**] and the w^*-dentability of dual E' [**3**], respectively.

The definitions of the notions of the r-dentable points, as well as of the r and $E'_r(M)$-strongly exposed points, will be given in Section 3, as the generalization of the analogous notions, given in [**3**]. It is easy to see that the proposition of Rieffel ([**1**], Corollary 2.2.3) is true for r-dentability as well:

Proposition 1. *If the vector space E is a locally convex space for each one of the two topologies r and r_0, where $r < r_0$, then the r-dentability of the r-closed convex hull of an r-bounded set $D \subset E$ implies the r-dentability of D.*

2. The two lemmas of Phelps for r-dentability.

Lemma 1. *Let A be a closed bounded absolutely convex set in E, which is a* tvs *for the two topologies r and r_0, where $r < r_0$. If every subset of A is r-dentable and for the closed set $B \subset A$ there exists $x_0' \in E'$ such that*

$$D = \{x \in B \mid x_0'(x) = 0\} \neq \varnothing, \qquad \{x \in B \mid x_0'(x) > 0\} \neq \varnothing,$$

then for every neighborhood $V \in \gamma(0)$ there exists a slice $T_1 = T_1(x_1', a_1, B)$ of B, which is V-small and for which $T_1 \cap D = \varnothing$ ($\varnothing$ is the empty set).

Proof. We apply the method of the proof given by Phelps for this lemma in the normed spaces [**6**], with the respective technical modifications to extend it in topological vector spaces. Let us fix a point $z \in B$, for which $x'(z) > 0$ and let us put $r = 1/x_0'(z)$. Let $T_x : E \to E$ be the linear map given by

$$\forall\, y \in E, \quad T_x(y) = y - 2rx_0'(y)(z - x).$$

Then $T_x \circ T_x = \mathrm{id}_E$ (identity map in E), $T_x(s) = s$ for every $s \in D$, and $(1/2)[T_x(z)+z] = x$ for $y = z$. The family $(T_x)_{x\in D}$ is an equicontinuous family of linear operators in each one of the two topologies r and r_0. We prove this for the topology r_0; the proof for the other case is the same. Let $V \in \gamma(0)$, and let $u \in \gamma(0)$ denote an equilibrated neighborhood of the origin in E, such that $u+u \subset V$. Then there exists $\lambda > 0$ such that $\lambda(z - D) \subset u$. For this $\lambda > 0$ there exists the neighborhood $u_1 \in \gamma(0)$, for which $|x_0'(y)| \leq 1/(2r)$ for every $y \in u_1$, because $x_0' \in E_r'$. If we take $u_0 = u \cap u_1$, we have

$$\begin{aligned} y \in u_0 \Longrightarrow T_x(y) &= y - 2rx_0'(y)(z - x) \\ &= y + \frac{-2rx_0'(y)}{\lambda}\lambda(z - x) \in u + u \subset V, \end{aligned}$$

because u is an equilibrated neighborhood of the origin in the E and

$$\lambda(z - x) \in \lambda(z - D) \subset u, \qquad \left|\frac{-2rx_0'(y)}{\lambda}\right| \leq 1.$$

This shows that the family $(T_x)_{x\in D}$ is an equicontinuous family of linear operators. For every $x \in D$ and $y \in B$, and for $K = B \cup (\cup_{x\in D} T_x B)$ and $M = \sup |x_0'(B)|$, we have

$$\begin{aligned} T_x(y) &= y - 2rx_0'(y)(z-y) \subset A + 2rM \cdot 2A \\ &= (1+4rM)A = M_1 A, \end{aligned}$$

where $M_1 = 1 + 4rM \geq 1$. Then $K \subset M_1 A$ or $K/M_1 \subset A$, because $B \subset A \subset M_1 A$; this proves that the sets K/M_1 and K are r-dentable sets in the tvs E.

Now let $V \in \gamma(0)$. The set D is closed in the tvs E because $D = B \cap x_0'^{-1}(\{0\})$ and B is a closed set, while $x_0' \in E'$. From the fact that $z \notin D$, it results that there exists an equilibrated neighborhood $u_1 \in \gamma(0)$ such that $(z+u_1) \cap D = \varnothing$. We assume that $u_1 \subset V$. From the fact that $(T_x)_{x\in D}$ is an equicontinuous family, it results that there exists an equilibrated neighborhood $u_2 \in \gamma(0)$ for which

$$(y_1, y_2) \in E^2 \wedge y_1 - y_2 \in u_2 \wedge x \in D \quad \Longleftrightarrow \quad T_x(y_1 - y_2) \in u_1.$$

For $u = u_1 \cap u_2$, there exists a u-small slice T of $K : T - T \subset u$. If $x_0 \in K$ and $x_1'(x_0) > \sup x_1'(K) - a$, then $x_0 \in B$, or there exists $x \in D$ for which $x_0 \in T_x B$. In the first case we take the slice $T_1(x_1', a_1, B)$ where $a_1 = a - \sup x_1'(K) + \sup x_1'(B)$ while in the second one we take the slice $T_1(x_1', a_1, T_x B)$, where $a_1 = a - \sup x_1'(K) + \sup x_1'(T_x B)$ (it is easy to show that $a_1 > 0$). Then $T_1 \subset T$. If $T_1 \cap D \neq \varnothing$, there exists $t \in T_1 \cap D$. Then $t \in T$ and $(1/2)(T_t(z) + z) = t$, from which $x_1'(t) = 2^{-1}(x_1'(z) + x_1'(T_t(z))) \geq \sup x_1'(K) - a$. From the fact that z, $T_t(z) \in K$, $z \in T \vee T_t(z) \in T$ is obtained. If $t, z \in T$, we have that $t - z \in T - T \subset u \subset u_1$ in contradiction with the fact that $(z + u_1) \cap D = \varnothing$. If $t, T_t(z) \in T$, the same argument shows that $t \in (z + u_1) \cap D$, too, which is a contradiction as well. This shows that $T_1 \cap D = \varnothing$. If $x_0 \in B$, then the required slice is $T_1 = T_1(x_1, a_1, B)$, because $T_1 \cap D = \varnothing$ and $T_1 - T_1 \subset T - T \subset u \subset u_1 \subset V$. If $x_0 \in T_x B$, (for any x!), then we have that

$$T_x^{-1}(T_1(x_1', a_1, T_x B) \subset B, T_x^{-1}(T_1(x_1', a_1, T_x B) \cap D = \varnothing,$$

because $T_x(y) = y$ for every $y \in D$ and $T_1 \cap D = \varnothing$. From the fact that $x_1' \in E_r'$ and $(T_x)_{x\in D}$ is an equicontinuous family in E_r,

$x'_1 \circ T_x \in E'_r$ is obtained. In this case it is easy to prove that the set $T_2 = T_x^{-1}(T_1(x'_1, a_1, T_x B)) = T_x(T_1(x'_1, a_1, T_x B))$ is a slice of B, and that $T_2 \cap D = \varnothing$. The set T_2 is a V-small slice of B too; indeed, if $y_1, y_2 \in T_2$, then $T_x(y_1), T_x(y_2) \in T_1 \subset T$ or $T_x(y_1) - T_x(y_2) \in T - T \subset u \subset u_2$. Accordingly,

$$y_1 - y_2 = T_x(T_x(y_1) - T_x(y_2)) \in u_1 \subset V.$$

This last relation shows that $T_2 - T_2 \subset V$. $\square$

Remark 2. It is easy to show that this lemma holds true, when $A = E$ and the tvs is r-dentable, while B is a closed bounded set in E.

From Lemma 1, the following is obtained.

Lemma 2. *Let A be a closed bounded absolutely convex set in the space E, which is a Hausdorff locally convex* (*topological*) *space* (Hlcs) *for the two topologies r and r_0, and let $r < r_0$.*

If every subset of the set A is r-dentable and C is a closed convex subset of A, while B is a subset of A, then for every $V \in \gamma(0)$ and for every slice $T_0 = T_0(x'_0, a, C)$ of C and $\varepsilon > 0$, there exists the slice $T_1 = T_1(x'_1, a_1, C)$ of C such that

$$T_1 \subset T_0, \quad T_1 - T_1 \subset V, \qquad |x'_0 - x'_1|_{B \cup C} \leq \varepsilon.$$

Proof. We follow the idea of the proof of the analogous proposition of [**3**]. Here we give only the part of the proof concerning the application of the first lemma and the case of the Hlcs supplied with two comparative topologies. Let us introduce the notation $H = \{x \in E | x'_0(x) = \sup x'_0(C) - a\}$. Due to the translation in the origin, we can suppose that the origin $0 \in H$; then $H = x_0^{-1}(\{0\})$ where $0 \in R$. Let K be the closed absolutely convex hull of the set $B \cup C$, and let $M = \sup x'_0(K)$; obviously, $M > 0$. We fix a real number $\lambda > \{2, 2M/\varepsilon\}$. If C_1 denotes the closed convex hull of the set $(\lambda K \cap H) \cup T_0$, then $C_1 \subset \lambda K \subset \lambda A$ and

$$D = \{x \in C_1 | x'_0(x) = 0\} \neq \varnothing, \qquad \{x \in C_1 | x'_0(x) > 0\} \neq \varnothing,$$

because $x_0'(0) = 0 \in \lambda K \cap H \subset C_1$ and there exists a point $x_0 \in T_0 \subset C_1$ such that $x_0'(x_0) > 0$. Due to Lemma 1, and if $V \in \gamma(0)$, then there exists a V-small slice $T_1 = T_1(x_1', a_1, C_1)$ of C_1 such that $T_1 \cap D = \varnothing$.

As in the other versions of this lemma given in [**3**], and based on the fact that $\lambda K \cap H \subset D$, $T_1 \cap \lambda K \cap H = \varnothing$, $r < r_0$ and that $x_1' \in E_r'$, we conclude that

$$T_1(x_1', a_1, C) \subset T_1(x_1', a_1, C_1),$$
$$\sup x_1'(C) = \sup x_1'(C_1) = \sup x_1'(T_0).$$

Accordingly $T_1(x_1', a_1, C) - T_1(x_1', a_1, C) \subset T_1 - T_1 \subset V$. Now we may choose a point $z \in C$ such that $x_1'(z) > \sup x_1'(C) - a_1$. After that the proof is the same as in the one of the lemma in [**3**]. □

Corollary 1. *Let E be as in the above lemma. If the family $(A_n)_{n \in N}$ is a sequence of closed bounded absolutely convex sets, in which every subset is r-dentable, and the family $(B_n)_{n \in N}$ is another sequence of bounded sets such that $B_n \subset A_n$ for all $n \in N$, then for every sequence $(V_n)_{n \in N}$ of neighborhoods of the origin of E and for every slice $T_0 = T_0(x_0', a_0, C)$ of a given closed convex C, with the property $C \subset A_n$, for all $n \in N$, there exists a sequence $(a_n)_{n \in N}$ of real positive numbers and a sequence $(T_n = T_n(x_n', a_n, C))$ of slices of C such that*

a) $T_{n+1} \subset T_n \subset T_0$,

b) $T_n - T_n \subset V_n$,

c) $|x_{n+1}' - x_n'|_{B_n \cup C} \leq a_n/2^{n+1}$, $a_{n+1} \leq a_n/2$.

Proof. We will show how to choose the slice T_{n+1} for all $n \in N \cup \{0\}$ when the slice $T_n = T_n(x_n', a_n, C)$ is known. Setting in Lemma 2 $A = A_n$, $B = B_n$, $T_0 = T_n$, $V = V_{n+1}$, $\varepsilon = a_n/2^{n+1}$, $B_0 = \varnothing$ and $A_0 = A_1$ implies that there exists a V_n-small slice $T_{n+1}' = T_{n+1}'(x_{n+1}', b_{n+1}, C) \subset T_n$ such that $|x_{n+1}' - x_n'|_{B_n \cup C} \leq a_n/2^{n+1}$. The slice T_{n+1} we are looking for is the slice $T_{n+1} = T_{n+1}(x_{n+1}', a_{n+1}, C)$ for which $a_{n+1} = \min\{a_n/2, b_{n+1}\}$. □

3. The r-dentable, r- and $E_r'(M)$-strongly exposed points.

Definition 3. Let A be an r-bounded set in E. The point $x \in A$ is called an r-*dentable point* of A if, for every $V \in \gamma(0)$, there exists a V-small slice $T(x', a, A)$ such that $x'(x) > \sup x'(A) - a$.

If, in Definition 3, the element $x' \in E'_r$ may be chosen the same for every $V \in \gamma(0)$, then the given point x is called an r-strongly exposed point of A.

It is understood that an r-strongly exposed point of A is an r-dentable one and that an r-bounded set, that has an r-dentable point, is an r-dentable set. Even in this case, an r-closed convex and r-bounded C is the r-closed convex hull of a given subset S of C, if every slice $T(x', a, C)$ of C has at least one point of S.

If $r = r_0$, from Definitions 3 and 4, the notions are obtained of the dentable and strongly exposed points as in [**2, 3**]. In the case of the dual space E' with $r = \sigma(E', E) = w^*$ and $r_0 = \beta(E', E)$, from Definitions 3 and 4, the notions of w^*-dentable and w^*-strongly exposed points are obtained, because $(E', w^*)' = E$.

Definition 4. Let (E, r) and (E, r_0) be two Hlcs in which $r < r_0$ and $M \subset E$. We denote with $E'_r(M)$ the subset of algebraic dual of the vector subspace $E(M)$ of E, generated by the set M, in which the restrictions to M of its elements are continuous, if the topology in M is induced by the space (E, r).

The point $x_0 \in K$ where $K \subset M$, is called $E'_r(M)$-strongly exposed of K, if there exists $x'_0 \in E'_r(M)$ such that

a) x'_0 is bounded on K and

$$\sup x'_0(K) = x'_0(x_0).$$

b) The sequence (x_n) of the points in K converges to x_0 for the topology r_0, if the real sequence $(x'_0(x_n))$ converges to $x'_0(x_0)$.

As in the case of the strongly exposed points in Hlcs, we may prove that the point $x_0 \in K$ is an $E'_r(M)$-strongly exposed point, if and only if there exists a point $x'_0 \in E'_r(M)$ which is bounded on K, and such that for every $V \in \gamma(0)$ there exists a V-small slice $T(x'_0, a, K)$ for which $x'_0(x_0) > \sup x'_0(K) - a$.

The r-strongly exposed points are $E'_r(M)$-strongly exposed ones. It is not difficult to prove that it is true.

Proposition 2. *If the set $B \subset M$ is r-closed convex and r-bounded set in E, then every $E'_r(M)$-strongly exposed point of B is r-dentable.*

In Section 4 we shall consider the tvs which are of type BM (quasi-metrizable) [**7**] for the stronger topology r_0. E. Saab [**7**] (also [**3**]) has required these spaces to be Hausdorff. Really this requirement is not needed, because in Proposition 3 we shall show that these spaces are Hausdorff.

Proposition 3. *The spaces of type* BM *are Hausdorff.*

Proof. At first we shall prove that a topological vector space (tvs) E is Hausdorff if, for every $x \neq 0$, its topological subspace which has the set $\{0, x\}$ for its support, is a Hausdorff topological subspace. Let $x \neq 0$ be a point of E. The fact that the topology induced on the set $\{0, x\}$ by the topology of E, is Hausdorff, implies the existence (in the subspace $\{0, x\}$) of a neighborhood V_0 of the origin, 0, which does not contain the point x: $x \notin V_0$. There also exists in E a neighborhood V of the origin 0 such that $V_0 = V \cap \{0, x\}$. Then $x \notin V$ because $x \notin V_0$ and $x \in \{0, x\}$. This shows that the space E is Hausdorff. The set $\{0, x\}$ is bounded and the subspace $\{0, x\}$ of a vector topological space of type BM is a Hausdorff space because it is metrizable. From this we obtain that the tvs of type BM are Hausdorff. $\square$

4. Two theorems for the dentability in the spaces with two topologies. In the two Theorems 1 and 2 below, E is a vector space endowed with two topologies r, r_0, such that $r < r_0$. We also assume that E is an Hlcs for the topology r and an lcs of the type BM for the topology r_0.

Theorem 1. *Let A be a closed bounded absolutely convex set in which every subset is an r-dentable set. If $B \subset A$ and the subset C of A is an r-closed convex set which is also complete in one of the uniform structures of the spaces (E, r) or (E, r_0), then C is the r-closed convex* *hull of its $E'_r(B \cup C)$-strongly exposed points.*

Proof. It is sufficient to prove that every slice $T_0 = T_0(x'_0, a, C)$ of C $(x'_0 \in E'_r)$ has at least one $E'_r(B \cup C)$-strongly exposed point. The set C is closed in E and $B \cup C \subset A$; consequently, if the family $(V_n)_{n \in N}$ denotes a sequence of the neighborhoods of the origin in the space (E, r_0) which generates the topology induced by r_0 in the set 2A, then from Corollary 1, for $A_n = A$, $B_n = B$, for all $n \in N$, there exists a real positive sequence $(a_n)_{n \in N}$ and a sequence $(T_n = T_n(x'_n, a_n, C))_{n \in N}$ of slices of C $(x'_n \in E'_r)$, which satisfy the conditions

a) $T_{n+1} \subset T_n \subset T_0$,

b) $T_n - T_n \subset V_n$,

c) $|x'_{n+1} - x'_n|_{B_n \cup C} \leq a_n/2^{n+1}, a_{n+1} \leq a_n/2$,

for all $n \in N$.

The slices T_n are r-closed sets and for this reason they are also closed sets in E. As $T_n \neq \varnothing$, for all $n \in N$, there exists $x_n \in T_n$ for all $n \in N$. For $n, m > \rho$ we have $x_n, x_m \in T_\rho$ or $x_n - x_m \in V_\rho$. Consequently the sequence $(x_n)_{n \in N}$ is a Cauchy sequence in each of the uniform structures induced on C by the spaces (E, r) and (E, r_0); thus, the sequence $(x_n)_{n \in N}$ converges to a point $x_0 \in C$ for the two topologies r and r_0. From the fact that $x_k \in T_n$ for all $k \geq n$, it results that $x_0 \in \cap_{n=1}^{\infty} T_n \subset T_0$. From c) we obtain that

d) $|x'_{n+m}(x) - x'_n(x)| \leq a_n/2^n$, for all $x \in B \cup C$.

This shows that the sequence $(x'_n)_{n \in N}$ in $E(B \cup C)$ converges pointwise to a point x' of the algebraic dual of $E(B \cup C)$, which is also a point of $E'_r(B \cup C)$ and bounded on C. The point x_0 is an $E'_r(B \cup C)$-strongly exposed one by the element x'. To prove it we can suppose that $V_n \supset V_{n+1}$ for all $n \in N$. Let $V \in \gamma(0)$; then there exists $\rho > 1$ such that $V_\rho \cap 2A \subset V \cap 2A \subset V$. So $T_\rho - T_\rho \subset V$.

From d) and the fact that each T_n contains the point x_0, it results that

$$x'(x_0) > \sup x'(C) - a_{\rho+1},$$

and that

$$x'_\rho(x) \geq \sup x'_\rho(C) - a_\rho, \quad \forall\, x \in T,$$

where $T = T(x', a_{\rho+1}, C)$; this shows that the slice T is a V-small slice, which contains strictly the point x_0. $\square$

From Theorem 1 we obtain the corollaries below; Corollary 1.1 generalizes, in a local form, Theorem 2.3 in [**3**], while Corollary 1.2 is in fact this theorem.

Corollary 1.1. *Let E be a locally convex space of the type* BM *and A a closed bounded absolutely convex set in E which is also a subset dentable* (*which means that every subset of A is a dentable set* [**1**]). *If $B \subset A$ and C is a convex complete subset of A, then C is the closed convex hull of its $-E'(B \cup C)$-strongly exposed points.*

Proof. It results from Theorem 1 for $r = r_0$, where $E'_r(B \cup C) = E'(B \cup C)$. $\square$

Corollary 1.2. *Let E be a dentable locally convex space of the type* BM *and B a bounded set in E. Every bounded complete convex set $C \subset E$ is the closed convex hull of its $E'(B \cup C)$ strongly exposed points. Furthermore this proposition holds true, when the locally convex space E is dentable and quasi Fréchet space and C is a closed bounded convex set in E.*

Proof. It results from Corollary 1, if one denotes by A the closed absolutely convex hull of the set $B \cup C$. $\square$

Corollary 1.3. *If E is an* lcs *of the type* BM *and C is a convex complete subset of a closed bounded absolutely convex set A, which is a subset dentable, then C is the closed convex hull of its dentable points*; *the same holds when E is an* lcs *quasi-Fréchet and dentable, and C a closed bounded convex set in E.*

Proof. The proof is based on Corollary 1.2 and on the fact that each $E'(B \cup C)$-strongly exposed point is a dentable point. $\square$

Remark 3. Corollary 1.3 contains Theorems 1.2 and 2.1 in [**3**], but it is weaker than the local form given by Theorem 2.2 in [**2**]. This local form does not result by applying the method of Phelps lemmas. To extend the first Phelps lemma for this case, we would need to prove that the sum C_1+C_2 of two closed convex sets with the RNP (the Radon Nikodym property) is a set with the RNP; but W. Schachermayer in [**8**] shows that this is not true.

Corollary 1.4. *Let E be an* Hlcs *with its dual E' a space of the type* BM. *If C is a w^*-compact convex set such that the set $B \cup C$ is contained in a closed bounded absolutely convex set of the space $(E', \beta(E', E))$ which is subset dentable, then C is the w^*-closed convex hull of its $E'_{w^*}(B \cup C)$-strongly exposed points (furthermore of its w^*-dentable points).*

Proof. We set in E' $r = w^*$ and $r_0 = \beta(E', E)$ and observe that C is complete in the uniform structure generated by the topology w^*. ◻

Corollary 1.5 below is slightly stronger than Theorem 3.1 in [**3**], because here we do not need to assume the quasi-completeness of the dual space E'. The proof is based as well only on the method resulting from the two lemmas of Phelps, and we do not need to use the lemmas of Bourgain [**4**], Namioka-Asplund [**5**] and Bishop-Phelps.

Corollary 1.5. *Let E be an* Hlcs *with its dual a space of the type* BM. *Then the following statements are equivalent*:

1. *E' is a w^*-dentable space.*

2. *Every w^*-closed equicontinuous convex set C in E' is the w^*-closed convex hull of its w^*-dentable points.*

3. *For every equicontinuous set B in E' and for every w^*-closed equicontinuous convex set C, the set C is the w^*-closed convex hull of its $E'_{w^*}(B \cup C)$-strongly exposed points.*

Proof. It is sufficient to prove that 1 implies 3. The set C is a w^*-compact convex set, while the w^*-closed absolutely convex hull $A = \overline{C}^{w^*}(B \cup C)^e$ of the set $B \cup C$ is a closed equicontinuous, i.e., also bounded, set which contains the set $B \cup C$ and such that its every subset is a w^*-dentable set (because A is an equicontinuous set and it holds true (1)). It results from Corollary 1.4 that the set C is the w^*-closed convex hull of its $E'_{w^*}(B \cup C)$-strongly exposed points. $\square$

Theorem 2. *If*

1. *the dual of the space* (E, r) *is a Fréchet space* (*with its strong topology*) *and* $(B_n)_{n\in N}$ *is an increasing sequence of* r*-closed absolutely convex sets, which is a fundamental system of bounded sets for the space* (E, r) *defining the strong topology in its dual.*

2. *For the* r*-bounded convex* C, *which is complete in one of the uniform structures of the spaces* (E, r) *or* (E, r_0), *there exists a sequence* $(A_n)_{n\in N}$ *of closed absolutely convex bounded sets in which every subset of* A_n, $n \in N$, *is an* r*-dentable set, and such that* $B_n \cup C \subset A_n$ *for all* $n \in N$,

then the set C *is the* r*-closed convex hull of its* r*-strongly exposed points.*

Proof. It is sufficient to prove that every slice $T_0 = T_0(x'_0, x_0, C)$, $x'_0 \in E'_r$ contains at least one r-strongly exposed point. Let A be the absolutely convex hull of C and $(V_n)_{n\in N}$ a decreasing sequence of neighborhoods of the origin in (E, r_0) that generates in the set $2A$ the topology induced by r_0. For the sequences $(A_n)_{n\in N}$, $(B_n)_{n\in N}$, $(V_n)_{n\in N}$ and the slice T_0, there exists a positive sequence $(a_n)_{n\in N}$ and the sequence $(T_n = T_n(x'_n, a_n, C))_{n\in N}$, $(x'_n \in E'_r)$ of slices of C, that satisfy the conditions

a) $T_{n+1} \subset T_n \subset T_0$,

b) $T_n - T_n \subset V_n$,

c$'$) $|x'_{n+1} - x'_n|_{B_n \cup C} \leq a_n/2^{n+1}$, $a_{n+1} \leq a_n/2$,

for each $n \in N$ (Corollary 1).

From these relations (as in Theorem 1 or as in [**3**, Theorem 2.5]) there exists a point $x_0 \in \cap_{n=1}^{\infty} T_n \subset T_0$. From (c$'$) there results a sequence $(x'_n)_{n\in N}$ that is a Cauchy's sequence in the space E'_r supplied with its strong topology $\beta(E'_r, E_r)$; thus it converges to a point $x' \in E'_r$ because the space $(E'_r, \beta(E'_r, E_r))$ is a Fréchet one. The sequence x'_n converges pointwise to x'; thus (from (c$'$)) we obtain that c$''$) $|x' - x'_n|_C \leq a_n/2^n$ for all $n \in N$.

Based on (c''), as in Theorem 1, we can prove that the point $x_0 \in C$ is an r-strongly exposed point by the linear form $x' \in E'_r$. □

Corollary 2.1. *If the* lcs E *is dentable and of type* BM *and its dual supplied with the strong topology is a Fréchet space, then every bounded complete convex set in* E *is the closed convex hull of its strongly exposed points.*

Proof. We have $r = r_0$ in this case. Let $(B_n)_{n \in N}$ be an increasing sequence of the closed absolutely convex sets which is a fundamental system of bounded sets in E and which also defines the strong topology in E'. For each $n \in N$ there exists an $m_n \in N$ such that $B_n \cup C \subset B_{m_n}$. After that we apply Theorem 2 where $A_n = B_{m_n}$. □

Remark 4. Corollary 2.1 is Theorem 2.5 in [**3**]. For the same reason as in Remark 3, we cannot prove by the lemmas of Phelps its local stronger version given in Theorem 3.2 [**2**].

Corollary 2.2 below is Theorem 3.2 in [**3**]. Its proof here is based on Theorem 2 and on the "symmetry" between the space E and its dual E'.

Corollary 2.2. *If* E *is a Fréchet space with its dual* E' *a space of the type* BM*, then the following two facts are equivalent*:

1. E' *is* w^**-dentable.*

2. *Every* w^**-compact convex set* C *in* E' *is the* w^**-closed convex hull of its* w^**-strongly exposed points.*

Proof. It is sufficient to prove that 1 implies 2. In this case we apply Theorem 2, where $r = w^*$ and $r_0 = \beta(E, E')$ in the dual E'. E is barreled because it is a Fréchet space. So in E' a set is a w^*-bounded set if and only if it is an equicontinuous set. From this and from the fact that the dual space $(E', w^*)' = E$ endowed with its strong topology (which is the topology of the space E) is a Fréchet space, there exists an increasing sequence $(B_n)_{n \in N}$ of w^*-closed absolutely convex sets, which is a fundamental system of bounded sets in the space (E', w^*) and which also defines the topology of E. Let C be a w^*-compact convex set in E'. For each $n \in N$ there exists $m_n \in N$ such that $B_n \cup C \subset B_{m_n}$. The sets $A_n = B_{m_n}$ are bounded (as equicontinuous sets) and subset w^*-dentable (from Condition 1) and C is complete for the uniform structure of the space (E', w^*). Then (Theorem 2) C is the w^*-closed convex hull of its w^*-strongly exposed points. □

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